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Research Article

Renormalization, Isogenies, and Rational Symmetries of Differential Equations

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We give an example of infinite-order rational transformation that leaves a linear differential equation covariant. This example can be seen as a nontrivial but still simple illustration of an exact representation of the renormalization group.

1. Introduction

There is no need to underline the success of the renormalization group revisited by Wilson [1, 2] which is nowadays seen as a fundamental symmetry in lattice statistical mechanics or field theory. It contributed to promote 2d conformal field theories and/or scaling limits of second-order phase transition in lattice statistical mechanics.[1] If one does not take into account most of the subtleties of the renormalization group, the simplest sketch of the renormalization group corresponds to Migdal-Kadanoff decimation calculations, where the new coupling constants created at each step of the (real-space) decimation calculations are forced[2] to stay in some (slightly arbitrary) finite-dimensional parameter space. This drastic projection may be justified by the hope that the basin of attraction of the fixed points of the corresponding (renormalization) transformation in the parameter space is "large enough."

One heuristic example is always given because it is one of the very few examples of *exact* renormalization, the renormalization of the one-dimensional Ising model without a magnetic field. It is a straightforward undergraduate exercise to show that performing various decimations summing over every two, three, or N spins, one gets *exact generators of the renormalization group* reading $T_N : t \longrightarrow t^N$, where t is (with standard notations) the high temperature variable $t = \tanh(K)$. It is easy to see that these transformations T_N, depending on the integer N, commute together. Such an *exact symmetry* is associated with a covariance of the partition function per site $Z(t) = C(t) \cdot Z(t^2)$. In this particular case one recovers the (very simple) expression of the partition function per site, $2\cosh(K)$, as an infinite product of the action of (for instance) T_2 on the cofactor $C(t)$. In this very simple case, this corresponds to the using of the identity (valid for $|x| < 1$):

$$\prod_{n=0}^{\infty}\left(1 + x^{2^n}\right) = \frac{1}{1-x}. \tag{1.1}$$

For $T_3 : t \longrightarrow t^3$ one must use the identity

$$\prod_{n=0}^{\infty}\left(1 + x^{3^n} + x^{2\cdot 3^n}\right) = \prod_{n=0}^{\infty}\left(\frac{1 - x^{3^{n+1}}}{1 - x^{3^n}}\right) = \frac{1}{1-x}, \tag{1.2}$$

and for $T_N : t \longrightarrow t^N$ a similar identity where the 3 in the exponents is changed into N.

Another simple heuristic example is the one-dimensional Ising model *with a magnetic field*. Straightforward calculations enable to get an infinite number of exact generators of the corresponding renormalization group, represented as *rational* transformations[3]

$$T_N : (x, z) \longrightarrow T_N(x, z) = (x_N, z_N), \tag{1.3}$$

where the first two transformations T_2 and T_3 read in terms of the two (low-temperature well-suited and fugacity-like) variables $x = e^{4K}$ and $z = e^{2H}$:

$$\begin{gathered} x_2 = \frac{(x+z)(1+xz)}{x\cdot(1+z)^2}, \qquad z_2 = z\cdot\frac{(1+xz)}{x+z}, \\ x_3 = x\cdot\frac{\left(z^2x+2z+1\right)\left(z^2+2z+x\right)}{\left(z^2x+z+xz+x\right)^2}, \qquad z_3 = z\cdot\frac{z^2x+2z+1}{z^2+2z+x}. \end{gathered} \tag{1.4}$$

One simply verifies that these rational transformations of two (complex) variables commute. This can be checked by formal calculations for T_N and T_M for any N and M less than 30, and one can easily verify a fundamental property expected for renormalization group generators:

$$T_N \cdot T_M = T_M \cdot T_N = T_{NM}, \tag{1.5}$$

where the "dot" denotes the *composition* of two transformations. The infinite number of these rational transformations of two (complex) variables (1.3) are thus a *rational representation of the positive integers together with their product*. Such rational transformations can be studied "per se" as discrete dynamical systems, the iteration of any of these various exact generators corresponding to an orbit of the renormalization group.

Of course these two examples of exact representation of the renormalization group are extremely degenerate since they correspond to one-dimensional models.[4] Migdal-Kadanoff decimation will quite systematically yield *rational*[5] transformations similar to (1.3) in two, or more, variables.[6] Consequently, they are never (except "academical" self-similar models) exact representations of the renormalization group. The purpose of this paper is to provide simple (but *nontrivial*) examples of *exact* renormalization transformations that are not degenerate like the previous transformations on one-dimensional models.[7] In several papers [3, 4] for Yang-Baxter integrable models with a canonical genus one parametrization [5, 6] (elliptic functions of modulus k), we underlined that the *exact* generators of the renormalization group must necessarily identify with the various isogenies which amount to multiplying or dividing τ, the ratio of the two periods of the elliptic curves, by an integer. The simplest example is the *Landen transformation* [4] which corresponds to multiplying (*or dividing* because of the modular group symmetry $\tau \leftrightarrow 1/\tau$), the ratio of the two periods is

$$k \longrightarrow k_L = \frac{2\sqrt{k}}{1+k}, \quad \tau \longleftrightarrow 2\tau. \tag{1.6}$$

The other transformations[8] correspond to $\tau \leftrightarrow N \cdot \tau$, for various integers N. In the (transcendental) variable τ, it is clear that they satisfy relations like (1.5). However, in the natural variables of the model ($e^K, \tanh(K), k = \sinh(2K)$, not transcendental variables like τ), these transformations are *algebraic* transformations corresponding in fact to the *fundamental modular curves*. For instance, (1.6) corresponds to the *genus zero fundamental modular curve*

$$\begin{aligned} &j^2 \cdot j'^2 - (j + j') \cdot \left(j^2 + 1487 \cdot jj' + j'^2\right) + 3 \cdot 15^3 \cdot \left(16j^2 - 4027jj' + 16j'^2\right) \\ &\quad - 12 \cdot 30^6 \cdot (j + j') + 8 \cdot 30^9 = 0, \end{aligned} \tag{1.7}$$

or

$$\begin{aligned} &5^9 v^3 u^3 - 12 \cdot 5^6 u^2 v^2 \cdot (u+v) + 375uv \cdot \left(16u^2 + 16v^2 - 4027vu\right) \\ &\quad - 64(v+u) \cdot \left(v^2 + 1487vu + u^2\right) + 2^{12} \cdot 3^3 \cdot uv = 0, \end{aligned} \tag{1.8}$$

which relates the two Hauptmoduls $u = 12^3/j(k)$, $v = 12^3/j(k_L)$:

$$j(k) = 256 \cdot \frac{\left(1 - k^2 + k^4\right)^3}{k^4 \cdot \left(1 - k^2\right)^2}, \qquad j(k_L) = 16 \cdot \frac{\left(1 + 14k^2 + k^4\right)^3}{\left(1 - k^2\right)^4 \cdot k^2}. \tag{1.9}$$

One verifies easily that (1.7) is verified with $j = j(k)$ and $j' = j(k_L)$.

The selected values of k, the modulus of elliptic functions, $k = 0, 1$, are actually *fixed points of the Landen transformations*. The Kramers-Wannier duality $k \leftrightarrow 1/k$ maps $k = 0$ onto $k = \infty$. For the Ising (resp. Baxter) model these selected values of k correspond to the three selected subcases of the model ($T = \infty$, $T = 0$, and the critical temperature $T = T_c$), for which the elliptic parametrization of the model degenerates into a rational parametrization [4]. We have the same property for all the other algebraic modular curves corresponding to $\tau \leftrightarrow N\cdot\tau$. This is certainly the main property most physicists expect for an exact representation of a *generator of the renormalization group*, namely, that it maps a generic point of the parameter space onto the critical manifold (fixed points). Modular transformations are, in fact, the only transformations to be compatible with all the other symmetries of the Ising (resp. Baxter) model like, for instance, the gauge transformations, some extended $sl(2) \times sl(2) \times sl(2) \times sl(2)$ symmetry [7], and so forth. It has also been underlined in [3, 4] that seeing (1.6) as a transformation on *complex variables* (instead of real variables) provides two other complex fixed points which actually correspond to *complex multiplication* for the elliptic curve, and are, actually, fundamental new singularities[9] discovered on the $\chi^{(3)}$ linear ODE [8–10]. In general, this underlines the deep relation between the renormalization group and the theory of elliptic curves in a deep sense, namely, *isogenies of elliptic curves, Hauptmoduls,*[10] *modular curves and modular forms*.

Note that an algebraic transformation like (1.6) or (1.8) cannot be obtained from any *local* Migdal-Kadanoff transformation which naturally yields *rational* transformations; an exact renormalization group transformation like (1.6) can only be deduced from *nonlocal* decimations. The emergence of modular transformations as representations of exact generators of the renormalization group explains, in a quite subtle way, the difficult problem of how renormalization group transformations can be compatible with *reversibility*[11] (iteration forward and backwards). An algebraic modular transformation (1.8) corresponds to $\tau \rightarrow 2\tau$ *and* $\tau \rightarrow \tau/2$ *in the same time, as a consequence of the modular group symmetry* $\tau \leftrightarrow 1/\tau$.

A simple rational parametrization[12] of the genus zero modular curve (1.8) reads:

$$u = 1728 \frac{z}{(z+16)^3}, \qquad v = 1728 \frac{z^2}{(z+256)^3} = u\left(\frac{2^{12}}{z}\right). \tag{1.10}$$

Note that the previously mentioned reversibility is also associated with the fact that the modular curve (1.8) is invariant by $u \leftrightarrow v$, and, within the previous rational parametrization (1.10), with the fact that permuting u and v corresponds[13] to the Atkin-Lehner involution $z \leftrightarrow 2^{12}/z$.

For many Yang-Baxter integrable models of lattice statistical mechanics the physical quantities (partition function per site, correlation functions, etc.) are solutions of selected[14] linear differential equations. For instance, the partition function per site of the square (resp. triangular, etc.) Ising model is an integral of an elliptic integral of the third kind. It would be too complicated to show the precise covariance of these physical quantities with respect to (algebraic) modular transformations like (1.8). Instead, let us give, here, an illustration of the nontrivial action of the renormalization group on some elliptic function that actually occurs in the 2D Ising model: a weight-one modular form. This modular form actually, and remarkably, emerged [11] in a second-order linear differential operator factor denoted Z_2 occurring [8] for $\chi^{(3)}$, and that the reader can think as a physical quantity solution of a particular linear ODE replacing the too complicated integral of an elliptic integral of the third kind. Let us consider the second-order linear differential operator (D_z denotes d/dz):

$$\alpha = D_z^2 + \frac{(z^2 + 56z + 1024)}{z\cdot(z+16)(z+64)} \cdot D_z - \frac{240}{z\cdot(z+16)^2(z+64)}, \tag{1.11}$$

which has the (modular form) solution

$$
\begin{aligned}
&{}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; 1728\frac{z}{(z+16)^3}\right) \\
&\quad = 2 \cdot \left(\frac{z+256}{z+16}\right)^{-1/4} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; 1728\frac{z^2}{(z+256)^3}\right).
\end{aligned}
\tag{1.12}
$$

Do note that the two pull-backs in the arguments of the *same* hypergeometric function are *actually related by the modular curve relation* (1.8) (see (1.10)). The covariance (1.12) is thus the very expression of a modular form property with respect to a modular transformation ($\tau \leftrightarrow 2\tau$) corresponding to the modular transformation (1.8).

The hypergeometric function at the rhs of (1.12) is solution of the second-order linear differential operator

$$
\beta = D_z^2 + \frac{z^2 + 416z + 16384}{(z+256)(z+64)z} \cdot D_z - \frac{60}{(z+64)(z+256)^2}, \tag{1.13}
$$

which is the transformed of operator α by the Atkin-Lehner duality $z \leftrightarrow 2^{12}/z$, and, also, a conjugation of α:

$$
\beta = \left(\frac{z+16}{z+256}\right)^{-1/4} \cdot \alpha \cdot \left(\frac{z+16}{z+256}\right)^{1/4}. \tag{1.14}
$$

Along this line we can also recall that the (modular form) function[15]

$$
F(j) = j^{-1/12} \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; \frac{12^3}{j}\right), \tag{1.15}
$$

verifies:

$$
F\left(\frac{(z+16)^3}{z}\right) = 2 \cdot z^{-1/12} \cdot F\left(\frac{(z+256)^3}{z^2}\right). \tag{1.16}
$$

A relation like (1.12) is a straight generalization of the covariance we had in the one-dimensional model $Z(t) = C(t) \cdot Z(t^2)$, which basically amounts to seeing the partition function per site as some "automorphic function" with respect to the renormalization group, with the simple renormalization group transformation $t \rightarrow t^2$ being replaced by the algebraic modular transformation (1.8) corresponding to $\tau \leftrightarrow 2\tau$ (i.e., the Landen transformation (1.6)).

We have here all the ingredients for seeing the identification of exact algebraic representations of the renormalization group with the modular curves structures we tried so many times to promote (preaching in the desert) in various papers [3, 4]. However, even if there are no difficulties, just subtleties, these Ising-Baxter examples of exact algebraic representations of the renormalization group already require some serious knowledge of the modular curves, modular forms, and Hauptmoduls in the theory of elliptic curves, mixed with the subtleties naturally associated with the various branches of such algebraic (multivalued) transformations.

The purpose of this paper is to present another elliptic hypergeometric function and other much simpler (Gauss hypergeometric) second-order linear differential operators covariant by infinite-order rational transformations.

The replacement of *algebraic (modular) transformations* by simple *rational* transformations will enable us to display a complete *explicit description of an exact representation of the renormalization group* that any graduate student can completely dominate.

2. Infinite Number of Rational Symmetries on a Gauss Hypergeometric ODE

Keeping in mind modular form expressions like (1.12), let us recall a particular Gauss hypergeometric function introduced by Vidunas in [12]

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2},\frac{1}{4}\right],\left[\frac{5}{4}\right];z\right) &= \frac{1}{4}\cdot z^{-1/4}\cdot\int_0^z t^{-3/4}(1-t)^{-1/2}dt \\ &= (1-z)^{-1/2}\cdot {}_2F_1\left(\left[\frac{1}{2},\frac{1}{4}\right],\left[\frac{5}{4}\right];\frac{-4z}{(1-z)^2}\right). \end{aligned} \tag{2.1}$$

This hypergeometric function corresponds to the integral of a holomorphic form on a *genus-one* curve $P(y,t)=0$:

$$\frac{dt}{y}, \quad \text{with:}\ \ y^4 - t^3\cdot(1-t)^2 = 0. \tag{2.2}$$

Note that the function

$$\mathcal{F}(z) = z^{1/4}\cdot {}_2F_1\left(\left[\frac{1}{2},\frac{1}{4}\right],\left[\frac{5}{4}\right];z\right), \tag{2.3}$$

which is exactly an integral of an algebraic function, has an extremely simple covariance property with respect to the *infinite-order rational* transformation $z \to -4z/(1-z)^2$:

$$\mathcal{F}\left(\frac{-4z}{(1-z)^2}\right) = (-4)^{1/4}\cdot\mathcal{F}(z). \tag{2.4}$$

The occurrence of this specific infinite-order transformation is reminiscent of Kummer's quadratic relation

$${}_2F_1([a,b],[1+a-b];z) = (1-z)^{-a}\cdot {}_2F_1\left(\left[\frac{a}{2},\frac{1+a}{2}-b\right],[1+a-b];-\frac{4z}{(1-z)^2}\right), \tag{2.5}$$

but it is crucial to note that, relation (2.4) does not relate two different functions, but is an "automorphy" relation on the *same function*.

It is clear from the previous paragraph that we want to see such functions as "ideal" examples of physical functions covariant by an exact (here, rational) generator of the renormalization group. The function (2.3) is actually solution of the second-order linear differential operator:

$$\Omega = D_z^2 + \frac{1}{4}\frac{3-5z}{z\cdot(1-z)}\cdot D_z = \omega_1 \cdot D_z, \quad \text{with}$$
$$\omega_1 = D_z + \frac{1}{4}\frac{3-5z}{z\cdot(1-z)} = D_z + \frac{1}{4}\cdot\frac{d\ln\left(z^3(1-z)^2\right)}{dz}. \tag{2.6}$$

From the previous expression of ω_1 involving a log derivative of a rational function it is obvious that this second-order linear differential operator has two solutions, the constant function and an integral of an algebraic function. Since these two solutions behave very simply under the infinite-order rational transformation $z \to -4z/(1-z)^2$, it is totally and utterly natural to see how the linear differential operator Ω transforms under the rational change of variable $z \to R(z) = -4z/(1-z)^2$ (which amounts to seeing how the two-order-one operators ω_1 and D_z transform). It is a straightforward calculation to see that introducing the cofactor $C(z)$ which is the inverse of the derivative of $R(z)$

$$C(z) = -\frac{1}{4}\cdot\frac{(1-z)^3}{1+z}, \qquad \frac{1}{C(z)} = \frac{dR(z)}{dz}, \tag{2.7}$$

D_z and ω_1, respectively, transform under the rational change of variable $z \to R(z) = -4z/(1-z)^2$ as

$$D_z \longrightarrow C(z)\cdot D_z, \qquad \omega_1 \longrightarrow (\omega_1)^{(R)} = C(z)^2\cdot\omega_1\cdot\frac{1}{C(z)}, \quad \text{yielding:} \quad \Omega \longrightarrow C(z)^2\cdot\Omega. \tag{2.8}$$

Since $z \to -4z/(1-z)^2$ is of infinite-order, the second-order linear differential operator (2.6) has *an infinite number of rational symmetries* (isogenies):

$$z \longrightarrow \frac{-4z}{(1-z)^2} \longrightarrow 16\cdot\frac{(1-z)^2\cdot z}{(1+z)^4} \longrightarrow -64\cdot\frac{(1-z)^2(1+z)^4 z}{(1-6z+z^2)^4} \longrightarrow \cdots. \tag{2.9}$$

Once we have found a second-order linear differential operator (written in a unitary or monic form) Ω, covariant by the infinite-order rational transformation $z \to -4z/(1-z)^2$, it is natural to seek for higher-order linear differential operators also covariant by $z \to -4z/(1-z)^2$. One easily verifies that the successive symmetric powers of Ω are (of course) also covariant. The symmetric square of Ω,

$$D_z^3 + \frac{3}{4}\frac{3-5z}{(1-z)z}\cdot D_z^2 + \frac{3}{8}\frac{1-5z}{(1-z)z^2}\cdot D_z, \tag{2.10}$$

factorizes in simple order-one operators

$$\left(D_z + \frac{2}{4}\frac{3-5z}{(1-z)z}\right) \cdot \left(D_z + \frac{1}{4}\frac{3-5z}{(1-z)z}\right) \cdot D_z, \tag{2.11}$$

and, more generally, the symmetric Nth power[16] of Ω reads

$$\left(D_z + \frac{N}{4}\frac{3-5z}{z(1-z)}\right) \cdot \left(D_z + \frac{N-1}{4}\frac{3-5z}{z(1-z)}\right) \cdots \left(D_z + \frac{1}{4}\frac{3-5z}{z(1-z)}\right) \cdot D_z. \tag{2.12}$$

The covariance of such expressions is the straight consequence of the fact that the order-one factors

$$\omega_k = D_z + \frac{k}{4}\frac{3-5z}{z\cdot(1-z)}, \quad k = 0, 1, \ldots, N, \tag{2.13}$$

transform very simply under $z \rightarrow -4z/(1-z)^2$:

$$\omega_k \longrightarrow (\omega_k)^{(R)} = (C(z))^{k+1} \cdot \omega_k \cdot (C(z))^{-k}. \tag{2.14}$$

More generally, let us consider a rational transformation $z \rightarrow R(z)$, the corresponding cofactor $C(z) = 1/R'(z)$, and the order-one operator $\omega_1 = D_z + A(z)$. We have the identity

$$C(z) \cdot D_z \cdot \left(\frac{1}{C(z)}\right) = D_z - \frac{d\ln(C(z))}{dz}. \tag{2.15}$$

The change of variable $z \rightarrow R(z)$ on ω_1 reads

$$D_z + A(z) \longrightarrow C(z) \cdot D_z + A(R(z)) = C(z) \cdot (D_z + B(z)). \tag{2.16}$$

We want to impose that this rhs expression can be written (see (2.8)) as

$$C(z)^2 \cdot (D_z + A(z)) \cdot \frac{1}{C(z)}, \tag{2.17}$$

which, because of (2.15), occurs if

$$B(z) = A(z) - \frac{d\ln(C(z))}{dz}, \tag{2.18}$$

yielding a "Rota-Baxter-like" [13, 14] functional equation on $A(z)$ and $R(z)$

$$\left(\frac{dR(z)}{dz}\right)^2 \cdot A(R(z)) = \frac{dR(z)}{dz} \cdot A(z) + \frac{d^2R(z)}{dz^2}. \tag{2.19}$$

Remark 2.1. Coming back to the initial Gauss hypergeometric differential operator the covariance of Ω becomes a conjugation. Let us start with the Gauss hypergeometric differential operator for (2.1):

$$H_0 = 8z \cdot (1-z) \cdot D_z^2 + 2 \cdot (5-7z) \cdot D_z - 1. \tag{2.20}$$

It is transformed by $z \to R(z) = -4z/(1-z)^2$ into

$$H_1 = 8z \cdot (1-z) \cdot D_z^2 - 2(3z-5) \cdot D_z + \frac{4}{1-z} = (1-z)^{1/2} \cdot H_0 \cdot (1-z)^{-1/2}, \tag{2.21}$$

then by $z \to R(R(z)) = R_2(z) = 16z(1-z)^2/(1+z)^4$ into

$$\begin{aligned} H_2 &= 8z \cdot (1-z) \cdot D_z^2 - 2\frac{(3z-1)(z+5)}{z+1} \cdot D_z + 16\frac{z-1}{(z+1)^2} \\ &= \left(\frac{z+1}{\sqrt{z-1}}\right) \cdot H_0 \cdot \left(\frac{z+1}{\sqrt{z-1}}\right)^{-1}, \end{aligned} \tag{2.22}$$

and more generally for $z \to R_N = R(R(R \cdots (R(z) \cdots)$

$$H_N = C_N \cdot H_0 \cdot C_N^{-1}, \quad \text{where: } C_N = z^{1/4} \cdot R_N^{-1/4}. \tag{2.23}$$

2.1. A Few Remarks on the "Rota-Baxter-Like" Functional Equation

The functional equation[17](2.19) is the (necessary and sufficient) condition for $\Omega = (D_z + A(z)) \cdot D_z$ to be covariant by $z \to R(z)$.

Using the chain rule formula of derivatives of composed functions:

$$\begin{aligned} \frac{dR(R(z))}{dz} &= \frac{dR(z)}{dz} \cdot \left[\frac{dR(z)}{dz}(R(z))\right], \\ \frac{d^2R(R(z))}{dz^2} &= \frac{d^2R(z)}{dz^2} \cdot \left[\frac{dR(z)}{dz}(R(z))\right] + \left(\frac{dR(z)}{dz}\right)^2 \cdot \left[\frac{d^2R(z)}{dz^2}(R(z))\right], \end{aligned} \tag{2.24}$$

one can show that, for $A(z)$ fixed, the "Rota-Baxter-like" functional equation (2.19) is invariant by the composition of $R(z)$ by itself $R(z) \to R(R(z)), R(R(R(z))), \ldots$. This result can be generalized to any composition of various $R(z)$'s satisfying (2.19). This is in agreement with the fact that (2.19) is the condition for $\Omega = (D_z + A(z)) \cdot D_z$ to be covariant by $z \to R(z)$ it must be invariant by composition of $R(z)$'s (for $A(z)$ fixed).

Note that we have not used here the fact that for globally nilpotent [11] operators, $A(z)$ and $B(z)$ are necessarily log derivatives of Nth roots of rational functions.

For $R(z) = -4z/(1-z)^2$:

$$A(z) = \frac{1}{4} \cdot \frac{d\ln(a(z))}{dz}, \qquad B(z) = \frac{1}{4} \cdot \frac{d\ln(b(z))}{dz},$$
$$a(z) = (1-z)^2 \cdot z^3, \qquad b(z) = z^3 \cdot \frac{(1+z)^4}{(1-z)^{10}}. \tag{2.25}$$

The existence of the underlying $a(z)$ in (2.25) consequence of a global nilpotence of the order-one differential operator, can however be seen in the following remark on the zeros of the lhs and rhs terms in the functional equation (2.19). When $R(z)$ is a rational function (e.g., $-4z/(1-z)^2$ or any of its iterates $R^{(n)}(z)$), the lhs and rhs of (2.19) are rational expressions. The zeros are roots of the numerators of these rational expressions. Because of (2.25) the functional equation (2.19) can be rewritten (after dividing by $R'(z)$) as

$$\left(\frac{dR(z)}{dz}\right) \cdot A(R(z)) = A(z) + \frac{d}{dz}\left(\ln\left(\frac{dR(z)}{dz}\right)\right) = \frac{1}{4} \cdot \frac{d}{dz}\left(\ln\left(a(z) \cdot \left(\frac{dR(z)}{dz}\right)^4\right)\right). \tag{2.26}$$

One easily verifies, in our example, that the zeros of the rhs of (2.26) come from the zeros of $A(R(z))$ (and not from the zeros of $R'(z)$ in the lhs of (2.26)). The zeros of the log-derivative rhs of (2.26) correspond to $a(z) \cdot R'(z)^4 = \rho$, where ρ is a constant to be found. Let us consider for $R(z)$ the nth iterates of $-4z/(1-z)^2$ that we denote $R^{(n)}(z)$. A straightforward calculation shows that the zeros of $A(R^{(n)}(z))$ or $a'(R^{(n)}(z))$ (where $a'(z)$ denotes the derivative of $a(z)$ namely, $(z-1)(5z-3) \cdot z^2$) actually correspond to the general closed formula:

$$5^5 \cdot a(z) \cdot \left(\frac{dR^{(n)}(z)}{dz}\right)^4 - 4 \cdot 3^3 \cdot (-4)^n = 0. \tag{2.27}$$

More precisely the zeros of $5 \cdot R^{(n)}(z) - 3$ verify (2.27), or, in other words, the numerator of $5R^{(n)}(z) - 3$ divides the numerator of the lhs of (2.27).

In another case for $T(z)$ given by (2.45), which also verifies (2.19) (see below), the relation (2.27) is replaced by

$$5^5 \cdot a(z) \cdot \left(\frac{dT^{(n)}(z)}{dz}\right)^4 - 4 \cdot 3^3 \cdot (-7-24i)^n = 0. \tag{2.28}$$

More generally for a rational function $\rho(x)$, obtained by an arbitrary composition of $-4z/(1-z)^2$ and $T(z)$, we would have

$$5^5 \cdot a(z) \cdot \left(\frac{d\rho(z)}{dz}\right)^4 - 4 \cdot 3^3 \cdot \lambda^n = 0. \tag{2.29}$$

where λ corresponds to

$$\rho(x) = \lambda \cdot z + \cdots, \quad \lambda = \left[\frac{d\rho(z)}{dz}\right]_{z=0}. \tag{2.30}$$

2.2. Symmetries of Ω, Solutions to the "Rota-Baxter-Like" Functional Equation

Let us now analyse all the symmetries of the linear differential operator $\Omega = (D_z + A(z)) \cdot D_z$ by analyzing all the solutions of (2.19) for a given $A(z)$. For simplicity we will restrict to $A(z) = (3-5z)/z/(1-z)/4$ which corresponds to $R(z) = -4z/(z-1)^2$ and all its iterates (2.9). Let us first seek for other (more general) solutions that are *analytic at* $z = 0$:

$$R(z) = a_1 \cdot z + a_2 \cdot z^2 + a_3 \cdot z^3 + \cdots . \tag{2.31}$$

It is a straightforward calculation to get, order by order from (2.19), the successive coefficients a_n in (2.31) as polynomial expressions (with rational coefficients) of the first coefficient a_1 with

$$\begin{gathered} a_2 = -\frac{2}{5} \cdot a_1 \cdot (a_1 - 1), \qquad a_3 = \frac{1}{75} \cdot a_1 \cdot (a_1 - 1) \cdot (7a_1 - 17), \\ a_4 = -\frac{2}{4875} \cdot a_1 \cdot (a_1 - 1) \cdot \left(41a_1^2 - 232a_1 + 366\right), \ldots, \\ a_n = -\frac{n}{5} \cdot a_1 \cdot (a_1 - 1) \cdot \frac{P_n(a_1)}{P_n(-4)}, \end{gathered} \tag{2.32}$$

where $P_n(a_1)$ is a polynomial with integer coefficients of degree $n-2$. Since we have here a series depending on one parameter a_1 we will denote it $R_{a_1}(z)$. This is a quite remarkable series depending on one parameter.[18] One can easily verify that this series actually reduces (as it should!) to the successive iterates (2.9) of $-4z/(1-z)^2$ for $a_1 = (-4)^n$. In other words this one-parameter family of "functions" actually reduces to rational functions for an infinite number of integer values $a_1 = (-4)^n$.

Furthermore, one can also verify a quite essential property we expect for a representation of the renormalization group, namely, that two $R_{a_1}(z)$ for different values of a_1 commute, the result corresponding to the product of these two a_1:

$$R_{a_1}(R_{b_1}(z)) = R_{b_1}(R_{a_1}(z)) = R_{a_1 \cdot b_1}(z). \tag{2.33}$$

The neutral element must necessarily correspond to $a_1 = 1$ which is actually the identity transformation $R_1(z) = z$. We have an "absorbing" element corresponding to $a_1 = 0$, namely, $R_0(z) = 0$. Performing the inverse of $R_{a_1}(z)$ (with respect to the composition of functions) amounts to changing a_1 into its inverse $1/a_1$. Let us explore some "reversibility" property of our exact representation of a renormalization group with the inverse of the rational transformations (2.9). The inverse of $R_{-4}(z) = -4z/(1-z)^2$ must correspond to $a_1 = -1/4$:

$$R_{-1/4}(z) = -\frac{1}{4} \cdot z - \frac{1}{8}z^2 - \frac{5}{64}z^3 - \frac{7}{128}z^4 - \frac{21}{512}z^5 + \cdots . \tag{2.34}$$

However, a straight calculation of the inverse of $R_{-4}(z) = -4z/(1-z)^2$ gives a multivalued function, or if one prefers, two functions

$$\begin{aligned} S^{(1)}_{-1/4}(z) &= \frac{z - 2 + 2\sqrt{1-z}}{z} = -\frac{1}{4} \cdot z - \frac{1}{8}z^2 + \cdots , \\ S^{(2)}_{-1/4}(z) &= \frac{z - 2 - 2\sqrt{1-z}}{z} = -\frac{4}{z} + 2 + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots , \end{aligned} \tag{2.35}$$

which are the two roots of the simple quadratic relation ($R_{-4}(z') = z$):

$$z'^2 - 2\cdot\left(1 - \frac{2}{z}\right)\cdot z' + 1 = 0, \tag{2.36}$$

where it is clear that the product of these two functions is equal to +1. The radius of convergence of $S^{(1)}_{-1/4}(z)$ is 1.

Because of our choice to seek for functions analytical at $z = 0$ our renormalization group representation "chooses" the unique root that is analytical at $z = 0$, namely, $S^{(1)}_{-1/4}(z)$. For the next iterate of $R_{-4}(z) = -4z/(1-z)^2$ in (2.9) the inverse transformation corresponds to the roots of the polynomial equation of degree four ($R_{16}(z') = z$):

$$z'^4 + \left(4 - \frac{16}{z}\right)\cdot z'^3 + \left(6 + \frac{32}{z}\right)\cdot z'^2 + \left(4 - \frac{16}{z}\right)\cdot z' + 1 = 0, \tag{2.37}$$

which yields four roots, one of which is analytical at $z = 0$ and corresponds to $a_1 = 1/(-4)^2$ in our one-parameter family of (renormalization) transformations:

$$S^{(1)}_{1/16}(z) = \frac{1}{16}z + \frac{3}{128}z^2 + \frac{53}{4096}z^3 + \frac{277}{32768}z^4 + \frac{3181}{524288}z^5 + \cdots, \tag{2.38}$$

its (multiplicative) inverse $S^{(2)}_{1/16}(z) = 1/S^{(1)}_{1/16}(z)$:

$$S^{(2)}_{1/16}(z) = \frac{16}{z} - 6 - \frac{17}{16}z - \frac{67}{128}z^2 - \frac{1333}{4096}z^3 - \frac{7445}{32768}z^4 + \cdots, \tag{2.39}$$

and two (formal) Puiseux series ($u = \pm\sqrt{z}$):

$$S^{(3)}_{1/16}(z) = 1 + u + \frac{1}{2}u^2 + \frac{3}{8}u^3 + \frac{1}{4}u^4 + \frac{27}{128}u^5 + \frac{5}{32}u^6 + \cdots. \tag{2.40}$$

Many of these results are better understood when one keeps in mind that there is a special transformation $J : z \leftrightarrow 1/z$ which is *also a R-solution of* (2.19) and verifies many compatibility relations with these transformations (Id denotes the identity transformation $R_0(z)$):

$$\begin{gathered} R_{-4}\cdot J = R_{-4}, \qquad S^{(2)}_{-1/4}\cdot R_{-4} = J, \qquad R_{-4}\cdot S^{(1)}_{-1/4} = S^{(1)}_{-1/4}\cdot R_{-4} = Id, \\ S^{(1)}_{1/16}(z) = S^{(1)}_{-1/4}\cdot S^{(1)}_{-1/4}, \qquad S^{(2)}_{1/16}(z) = S^{(1)}_{-1/4}\cdot S^{(2)}_{-1/4}, \\ J\cdot S^{(1)}_{-1/4} = S^{(2)}_{-1/4}, \qquad J\cdot S^{(2)}_{-1/4} = S^{(1)}_{-1/4}, \ldots, \end{gathered} \tag{2.41}$$

where the dot corresponds, here, to the composition of functions. These symmetries of the linear differential operator Ω correspond to isogenies of the elliptic curve (2.2).

It is clear that we have another one-parameter family corresponding to $J \cdot R_{a_1}$ with an expansion of the form

$$\begin{aligned} J \cdot R_{a_1} = {} & \frac{b_1}{z} - \frac{2}{5} \cdot (b_1 - 1) - \frac{1}{15} \cdot \frac{b_1^2 - 1}{b_1} \cdot z - \frac{2}{975} \cdot \frac{(b_1 - 1)(4b_1 + 1)(4b_1 + 3)}{b_1^2} \cdot z^2 \\ & - \frac{1}{248625} \cdot \frac{(b_1 - 1)(4b_1 + 1)\left(1268b_1^2 + 951b_1 + 91\right)}{b_1^3} \cdot z^3 \\ & - \frac{2}{2071875} \cdot \frac{(b_1 - 1)(4b_1 + 1)\left(3688b_1^3 + 2766b_1^2 + 404b_1 + 17\right)}{b_1^4} \cdot z^4 + \cdots . \end{aligned} \tag{2.42}$$

For $b_1 = -1/4$, $b_1 = (-1/4)^2$, $b_1 = (-1/4)^3$, this family reduces to the (multiplicative) inverse of the successive rational functions displayed in (2.9)

$$-\frac{1}{4} \cdot \frac{(1 - z)^2}{z} \longrightarrow \frac{1}{16} \cdot \frac{(1 + z)^4}{(1 - z)^2 \cdot z} \longrightarrow -\frac{1}{64} \cdot \frac{\left(1 - 6z + z^2\right)^4}{(1 - z)^2 (1 + z)^4 \cdot z} \longrightarrow \cdots , \tag{2.43}$$

which can also be written as:

$$\begin{aligned} & -\frac{1}{4} \cdot \left(z + \frac{1}{z}\right) + \frac{1}{2}, \qquad \frac{1}{16} \cdot \left(z + \frac{1}{z}\right) + \frac{3}{8} + \frac{z}{(1 - z)^2}, \\ & -\frac{1}{64} \cdot \left(z + \frac{1}{z}\right) + \frac{13}{32} - \frac{z}{4} \cdot \frac{17 - 60z + 102z^2 - 60z^3 + 17z^4}{(1 - z)^2 (1 + z)^4}, \\ & \frac{1}{256} \cdot \left(z + \frac{1}{z}\right) + \frac{51}{128} + \frac{z}{16} \cdot \frac{17 - 60z + 102z^2 - 60z^3 + 17z^4}{(1 - z)^2 (1 + z)^4} + 16 \frac{z \cdot (1 - z)^2 (1 + z)^4}{\left(z^2 - 6z + 1\right)^4}, \\ & -\frac{1}{1024} \cdot \left(z + \frac{1}{z}\right) + \frac{205}{512} - \frac{z}{164} \cdot \frac{17 - 60z + 102z^2 - 60z^3 + 17z^4}{(1 - z)^2 (1 + z)^4} \\ & \quad - 4 \frac{z \cdot (1 - z)^2 (1 + z)^4}{\left(z^2 - 6z + 1\right)^4} - 64 \frac{z \cdot (1 - z)^2 (1 + z)^4 \left(z^2 - 6z + 1\right)^4}{\left(1 + 20z - 26z^2 + 20z^3 + z^4\right)^4}, \ldots, \\ & \frac{1}{(-4)^n} \cdot \left(z + \frac{1}{z}\right) + \frac{2}{54^n} \left(4^n - (-1)^n\right) \\ & \quad + \frac{z}{(-4)^{n-2}} \cdot \frac{17 - 60z + 102z^2 - 60z^3 + 17z^4}{(1 - z)^2 (1 + z)^4} + \frac{z}{(-4)^{n-6}} \cdot \frac{(1 - z)^2 (1 + z)^4}{\left(z^2 - 6z + 1\right)^4} \\ & \quad + \frac{z}{(-4)^{n-8}} \cdot \frac{(1 - z)^2 (1 + z)^4 \left(z^2 - 6z + 1\right)^4}{\left(1 + 20z - 26z^2 + 20z^3 + z^4\right)^4} + \cdots , \end{aligned} \tag{2.44}$$

where we discover some "additive structure" of these successive rational functions.

In fact, due to the specificity of this elliptic curve (occurrence of complex multiplication), we have another remarkable rational transformation solution of (2.19), preserving covariantly Ω. Let us introduce the rational transformation (i denotes $\sqrt{-1}$):

$$T(z) = z \cdot \left(\frac{z - (1+2i)}{1 - (1+2i) \cdot z} \right)^4, \tag{2.45}$$

we also have the remarkable covariance [12]:

$${}_2F_1\left(\left[\frac{1}{2}, \frac{1}{4} \right], \left[\frac{5}{4} \right]; z \right) = \frac{1 - z/(1+2i)}{1 - (1+2i)z} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{4} \right], \left[\frac{5}{4} \right]; T(z) \right), \tag{2.46}$$

which can be rewritten in a simpler way on (2.3) (see (2.4)).

It is a straightforward matter to see that $T(z)$ actually belongs to the $R_{a_1}(z)$ one-parameter family:

$$\begin{gathered} T(z) = R_{a_1}(z) = -(7+24i) \cdot z + \cdots, \quad a_1 = -25 \cdot \rho, \\ \rho = \frac{(7+24i)}{25}, \qquad |\rho| = 1. \end{gathered} \tag{2.47}$$

As far as the reduction of (2.32) to a rational function is concerned, it is straightforward to see that:

$$\begin{aligned} &(1-z)^2 \cdot (1+z)^4 \cdot R_{a_1}(z) \\ &\quad = a_1 \cdot z + \cdots - \frac{2}{175746796875} \cdot a_1 \cdot (a_1 - 1) \cdot (a_1 + 4) \cdot (a_1 - 16) \cdot P_8(a_1) \cdot z^8 + \cdots \\ &\qquad - \frac{1}{N(n)} \cdot a_1 \cdot (a_1 - 1) \cdot (a_1 + 4) \cdot (a_1 - 16) \cdot P_n(a_1) \cdot z^n + \cdots, \end{aligned} \tag{2.48}$$

where $N(n)$ is a large integer growing with n, and P_n is a polynomial with integer coefficients of degree $n-4$, or

$$\begin{aligned} &(1 - (1+2i) \cdot z)^4 \cdot R_{a_1}(z) \\ &\quad = a_1 \cdot z + \cdots - \frac{4}{1243125} \cdot a_1 \cdot (a_1 - 1) \cdot (a_1 + 7 + 24i) \cdot (P_6(a_1) + iQ_6(a_1)) \cdot z^6 + \cdots \\ &\qquad + \frac{1}{N(n)} \cdot a_1 \cdot (a_1 - 1) \cdot (a_1 + 7 + 24i) \cdot (P_n(a_1) + iQ_n(a_1)) \cdot z^n + \cdots, \end{aligned} \tag{2.49}$$

where P_n and Q_n are two polynomials with integer coefficients of degree, respectively, $n-3$ and $n-4$.

Similar calculations can be performed for $T^*(z)$ defined by

$$T^*(z) = z \cdot \left(\frac{z - (1-2i)}{(1-2i)z - 1} \right)^4, \tag{2.50}$$

for which we also have the covariance

$$ {}_2F_1\Big(\Big[\frac{1}{2},\frac{1}{4}\Big],\Big[\frac{5}{4}\Big];z\Big) = \frac{1-z/(1-2i)}{1-(1-2i)z}\cdot {}_2F_1\Big(\Big[\frac{1}{2},\frac{1}{4}\Big],\Big[\frac{5}{4}\Big];T^*(z)\Big). \tag{2.51} $$

It is a simple calculation to check that any iterate of $T(z)$ (resp. $T^*(z)$) is actually a solution of (2.19) and corresponds to $R_{a_1}(z)$ for the infinite number of values $a_1 = (-7-24i)^N$ (resp. $(-7+24i)^N$). Furthermore, one verifies, as it should (see (2.33)), that the three rational functions $R_{-4}(z)$, $T(z)$, and $T^*(z)$ commute. It is also a straightforward calculation to see that the rational function built from any composition of $R_{-4}(z)$, $T(z)$, and $T^*(z)$ is actually a solution of (2.19). We thus have a *triple infinity* of values of a_1, namely $a_1 = (-4)^M\cdot(-7-24i)^N\cdot(-7+24i)^P$ *for any integer* M, N *and* P, for which $R_{a_1}(z)$ reduces to rational functions. We are in fact describing (some subset of) the isogenies of the elliptic curve (2.2), and identifying these isogenies with a discrete subset of the renormalization group. Conversely, a functional equation like (2.19) can be seen as a way to extend the n-fold composition of a rational function $R(z)$ (namely $R(R(\cdots R(z)\cdots)))$ to *n any complex number*.

2.3. Revisiting the One-Parameter Family of Solutions of the "Rota-Baxter-Like" Functional Equation

This extension can be revisited as follows. Keeping in mind the well-known example of the parametrization of the standard map $z \to 4z\cdot(1-z)$ with $z=\sin^2(\theta)$, yielding $\theta \to 2\theta$, let us seek for a (transcendental) parametrization $z = P(u)$ such that

$$ R_{-4}(P(u)) = P(-4u) \quad \text{or}: \ R_{-4} = P\cdot H_{-4}\cdot P^{-1}, \tag{2.52} $$

where H_{a_1} denotes the scaling transformation $z \to a_1\cdot z$ (here $H_{-4} : z \to -4\cdot z$) and P^{-1} denotes the inverse transformation of P (for the composition). One can easily find such a (transcendental) parametrization order by order

$$ \begin{aligned} P(z) = z &- \frac{2}{5}z^2 + \frac{7}{75}z^3 - \frac{82}{4875}z^4 + \frac{1078}{414375}z^5 \\ &- \frac{452}{1243125}z^6 + \frac{57311}{1212046875}z^7 - \frac{1023946}{175746796875}z^8 + \cdots, \end{aligned} \tag{2.53} $$

and similarly for its inverse (for the composition) transformation

$$ \begin{aligned} Q(z) = P^{-1}(z) = z &+ \frac{2}{5}z^2 + \frac{17}{75}z^3 + \frac{244}{1625}z^4 + \frac{45043}{414375}z^5 \\ &+ \frac{2302}{27625}z^6 + \frac{128941}{1939275}z^7 + \frac{15365176}{281194875}z^8 + \cdots. \end{aligned} \tag{2.54} $$

This approach is reminiscent of the conjugation introduced in Siegel's theorem [15–17]. It is a straightforward matter to see (order by order) that one actually has

$$ R_{a_1}(P(u)) = P(a_1\cdot u) \quad \text{or}: \ R_{a_1} = P\cdot H_{a_1}\cdot P^{-1}. \tag{2.55} $$

The structure of the (one-parameter) renormalization group and the extension of the composition of n times a rational function $R(z)$ (namely, $R(R(\cdots R(z)\cdots)))$ to n *any complex number* become a straight consequence of this relation. Along this line one can define some "infinitesimal composition" ($\epsilon \simeq 0$):

$$R_{1+\epsilon}(z) = P \cdot H_{1+\epsilon} \cdot P^{-1}(z) = z + \epsilon \cdot F(z) + \cdots, \tag{2.56}$$

where one can find, order by order, the "infinitesimal composition" function $F(z)$:

$$\begin{aligned} F(z) = z &- \frac{2}{5}z^2 - \frac{2}{15}z^3 - \frac{14}{195}z^4 - \frac{154}{3315}z^5 - \frac{22}{663}z^6 \\ &- \frac{418}{16575}z^7 - \frac{9614}{480675}z^8 - \frac{2622}{160225}z^9 + \cdots. \end{aligned} \tag{2.57}$$

It is straightforward to see, from (2.33), that the function $F(z)$ satisfies the following functional equations involving a rational function $R(z)$ (in the one-parameter family $R_{a_1}(z)$):

$$\begin{gathered} \frac{dR(z)}{dz} \cdot F(z) = F(R(z)), \qquad \frac{dR^{(n)}(z)}{dz} \cdot F(z) = F\left(R^{(n)}(z)\right), \quad \text{where:} \\ R^{(n)}(z) = R(R(\cdots R(z))\cdots). \end{gathered} \tag{2.58}$$

$F(z)$ cannot be a rational or algebraic function. Let us consider the fixed points of $R^{(n)}(z)$. Generically $dR^{(n)}(z)/dz$ is not equal to 0 or ∞ at any of these fixed points. Therefore one must have $F(z) = 0$ or $F(z) = \infty$ for the infinite set of these fixed points: $F(z)$ cannot be a rational or algebraic function, it is a transcendental function, and similarly for the parametrization function $P(z)$. In fact, let us introduce the function

$$\begin{aligned} G(z) &= (1 - z) \cdot F(z), \\ G(z) &= z - \frac{7}{5}z^2 + \frac{4}{15}z^3 + \frac{4}{65}z^4 + \frac{28}{1105}z^5 + \frac{44}{3315}z^6 + \frac{44}{5525}z^7 \\ &\quad + \frac{836}{160225}z^8 + \frac{1748}{480675}z^9 + \cdots + g_n \cdot z^n + \cdots. \end{aligned} \tag{2.59}$$

One actually finds that the successive g_n satisfies the very simple (hypergeometric function) relation:

$$\frac{g_{n+1}}{g_n} = \frac{4n - 9}{4n + 1}. \tag{2.60}$$

The function $G(z)$ is actually the hypergeometric function solution of the homogeneous operator

$$D_z^2 + \frac{1}{4}\frac{13z - 3}{z \cdot (1 - z)} \cdot D_z + \frac{3}{4}\frac{6z^2 - 3z + 1}{(1 - z)^2 \cdot z^2}, \tag{2.61}$$

or of the inhomogeneous ODE

$$4z \cdot (1-z) \cdot \frac{dG(z)}{dz} + (9z-3) \cdot G(z) - z \cdot (1-z)^2 = 0. \tag{2.62}$$

One deduces the expression of $F(z)$ as a hypergeometric function

$$F(z) = z \cdot (1-z)^{1/2} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; z\right) = \left.\frac{\partial R_{a_1}}{\partial a_1}\right|_{a_1=1}. \tag{2.63}$$

Finally we get the linear differential operator annihilating $F(z)$

$$\Omega_F = D_z^2 + \frac{1}{4} \cdot \frac{5z-3}{z(1-z)} \cdot D_z + \frac{1}{4} \cdot \frac{3-6z+5z^2}{(1-z)^2 z^2} = D_z \cdot \left(D_z - \frac{1}{4} \cdot \frac{3-5z}{z \cdot (1-z)}\right), \tag{2.64}$$

which is, in fact, nothing but Ω^* the *adjoint of* linear differential operator Ω (see (2.6)). One easily checks[19] that the second-order differential equation $\Omega_F(y(z)) = 0$ transforms under the change of variable $z \rightarrow -4z/(1-z)^2$ into the second-order differential equation $\Omega_F^{(R)}(y(z)) = 0$ with $\Omega_F^{(R)} = C(z)^2 \cdot \omega_F^{(R)}$ where the unitary (monic) operator $\omega_F^{(R)}$ is the conjugate of Ω_F:

$$\begin{aligned}
\omega_F^{(R)} &= D_z^2 - \frac{1}{4} \cdot \frac{11z^2+30z+3}{z \cdot (1-z)(1+z)} \cdot D_z + \frac{1}{4} \cdot \frac{3+12z+50z^2+12z^3+3z^4}{z^2 \cdot (1-z)^2(1+z)^2} \\
&= \left(\frac{1}{C(z)}\right) \cdot D_z \cdot \left(D_z - \frac{1}{4} \cdot \frac{3-5z}{z \cdot (1-z)}\right) \cdot C(z) \\
&= \left(\frac{1}{C(z)}\right) \cdot \Omega_F \cdot C(z) = \left(\frac{1}{C(z)}\right) \cdot \Omega^* \cdot C(z)
\end{aligned} \tag{2.65}$$

with $C(z) = 1/R'(z)$ and the "dot" denotes the composition of operators. Actually, the factors in the adjoint Ω^* transform under the change of variable $z \rightarrow -4z/(1-z)^2$ as follows[20]:

$$D_z \longrightarrow C(z) \cdot D_z, \qquad \omega_1^* = \longrightarrow \left(\omega_1^*\right)^{(R)} = \omega_1^* \cdot C(z), \qquad \Omega^* \longrightarrow \Omega_F^{(R)} = C(z) \cdot \Omega^* \cdot C(z) \tag{2.66}$$

which is precisely the transformation we need to match with (2.58) and see the ODE $\Omega^*(F(z)) = 0$ compatible with the change of variable $z \rightarrow -4z/(1-z)^2$:

$$\begin{aligned}
\Omega^*(F(z)) = 0 &\longrightarrow (C(z) \cdot \Omega^* \cdot C(z))(F(R(z))) \\
&= (C(z) \cdot \Omega^* \cdot C(z))\left(R'(z) \cdot F(z)\right) = C(z) \cdot \Omega^*(F(z)) = 0.
\end{aligned} \tag{2.67}$$

This is, in fact, a quite general result that will be seen to be valid in a more general (higher genus) framework (see (2.148), (2.150) in what follows).

Not surprisingly one can deduce from (2.33) and the previous results, in particular (2.63), the following results for $R_{a_1}(z)$:

$$-4 \cdot \frac{\partial R_{a_1}}{\partial a_1}\bigg|_{a_1=-4} = F(R(z)), \qquad (-4)^n \cdot \frac{\partial R_{a_1}}{\partial a_1}\bigg|_{a_1=(-4)^n} = F\Big(R^{(n)}(z)\Big), \tag{2.68}$$

where $R(z) = -4z/(1-z)^2$ and $R^{(n)}(z)$ denotes $R(R(\cdots R(R(z))))$. Of course we have similar relation for $T(z)$, -4 being replaced by $-7-24i$. Therefore the partial derivative $\partial R_{a_1}/\partial a_1$ that can be expressed in terms of hypergeometric functions for for a *double infinity* of values of a_1, namely, $a_1 = (-4)^M \times (-7-24i)^N$.

One can, of course, check, order by order, that (2.58) is actually verified for any function in the one-parameter family $R_{a_1}(z)$:

$$\frac{dR_{a_1}(z)}{dz} \cdot F(z) = F(R_{a_1}(z)), \tag{2.69}$$

which corresponds to an infinitesimal version of (2.33).

From (2.56) one simply deduces

$$z \cdot \frac{dP(z)}{dz} = F(P(z)), \tag{2.70}$$

that we can check, order by order from (2.53), the series expansion of $P(z)$, and from (2.57) the series expansion of $F(z)$, but also

$$\frac{dQ(z)}{dz} \cdot F(z) = Q(z) \tag{2.71}$$

that we can, check order by order, from (2.54), the series expansion of $Q(z) = P^{-1}(z)$ and from (2.57). We now deduce that the log-derivative of the "well-suited change of variable" $Q(z)$ is nothing but the (multiplicative) inverse of a hypergeometric function $F(z)$:

$$\frac{d\ln(Q(z))}{dz} = \frac{1}{F(z)}, \qquad Q(z) = \lambda \cdot \exp\left(\int^z \frac{dz}{F(z)}\right). \tag{2.72}$$

The function $Q(z)$ is solution of the *nonlinear* differential equation

$$\begin{aligned} &-4z^2 \cdot (1-z)^2 \cdot \left(Q \cdot Q^{(1)} \cdot Q^{(3)} + \left(Q^{(1)}\right)^2 \cdot Q^{(2)} - 2Q \cdot \left(Q^{(2)}\right)^2\right) \\ &\quad + z \cdot (3-5z)(1-z) \cdot Q^{(1)} \cdot \left(Q \cdot Q^{(2)} - \left(Q^{(1)}\right)^2\right) \\ &\quad + \left(5z^2 - 6z + 3\right) \cdot Q \cdot \left(Q^{(1)}\right)^2 = 0, \end{aligned} \tag{2.73}$$

where the $Q^{(n)}$'s denote the nth derivative of $Q(z)$. At first sight $Q(z)$ would be a *nonholonomic* function, however, remarkably, it is a *holonomic* function solution of an order-five operator which factorizes as follows:

$$\Omega_Q = \left(D_z + \frac{3-5z}{(1-z)\cdot z}\right)\cdot\left(D_z + \frac{3}{4}\cdot\frac{3-5z}{(1-z)\cdot z}\right)\cdot \\ \times\left(D_z + \frac{2}{4}\cdot\frac{3-5z}{(1-z)\cdot z}\right)\cdot\left(D_z + \frac{1}{4}\cdot\frac{3-5z}{(1-z)\cdot z}\right)\cdot D_z, \tag{2.74}$$

yielding the exact expression of $Q(z)$ in terms of hypergeometric functions:

$$Q(z) = z\cdot\left({}_2F_1\left(\left[\frac{1}{2},\frac{1}{4}\right],\left[\frac{5}{4}\right];z\right)\right)^4 = \frac{z}{1-z}\cdot\left({}_2F_1\left(\left[\frac{1}{4},\frac{3}{4}\right],\left[\frac{5}{4}\right];-\frac{z}{1-z}\right)\right)^4, \tag{2.75}$$

that is, the fourth power of (2.3), with the differential operator (2.74) being the symmetric fourth power of Ω. From (2.3) we immediately get the covariance of $Q(z)$:

$$Q\left(-\frac{4z}{(1-z)^2}\right) = -4\cdot Q(z), \tag{2.76}$$

and, more generally, $Q(R_{a_1}) = a_1\cdot Q(z)$. Since $Q(z)$ and $F(z)$ are expressed in terms of the same hypergeometric function, the relation (2.71) must be an identity on that hypergeometric function. This is actually the case. This hypergeometric function verifies the inhomogeneous equation:

$$4\cdot z\cdot\frac{d\mathcal{H}(z)}{dz} + \mathcal{H}(z) - (1-z)^{-1/2} = 0, \tag{2.77}$$

where

$$\mathcal{H}(z) = {}_2F_1\left(\left[\frac{1}{2},\frac{1}{4}\right],\left[\frac{5}{4}\right];z\right). \tag{2.78}$$

Recalling $Q(P(z)) = z$, one has the following functional relation on $P(z)$:

$$P(z)\cdot{}_2F_1\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{5}{4}\right];P(z)\right)^4 = z. \tag{2.79}$$

Noting that $Q(z^4)^{1/4} = \mathcal{F}(z^4)$ (see (2.3)) can be expressed in term of an incomplete elliptic integral of the first kind of argument $\sqrt{-1}$

$$z\cdot{}_2F_1\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{5}{4}\right];z^4\right) = \mathrm{Elliptic}F\left(z,\sqrt{-1}\right), \tag{2.80}$$

one can find that (2.79) rewrites on $P(z)$ as

$$\text{Elliptic}\, F\left(P(z)^{1/4}, \sqrt{-1}\right) = z^{1/4}, \tag{2.81}$$

from which we deduce that the function $P(z)$ is nothing but a *Jacobi elliptic function* [21]

$$P(z) = \left(\text{sn}(z^{1/4}, \sqrt{-1})\right)^4. \tag{2.82}$$

In Appendix B we display a set of "Painlevé-like" ODEs[22] verified by $P(z)$. From the simple nonlinear ODE on the Jacobi elliptic sinus, namely, $S'' + 2 \cdot S^3 = 0$, and the exact expression of $P(z)$ in term of Jacobi elliptic sinus, one can deduce other *nonlinear* ODEs verified by the *nonholonomic* function $P(z)$ ($P^{(1)} = dP(z)/dz$, $P^{(2)} = d^2P(z)/dz^2$):

$$z^{3/2} \cdot \left(P^{(1)}\right)^2 - (1 - P) \cdot P^{3/2} = 0, \tag{2.83}$$

$$P^{(2)} - \frac{3}{4} \cdot \frac{\left(P^{(1)}\right)^2}{P} + \frac{3}{4} \cdot \frac{P^{(1)}}{z} + \frac{1}{2} \cdot \frac{P^{3/2}}{z^{3/2}} = 0. \tag{2.84}$$

2.4. Singularities of the Jacobi Elliptic Function $P(z)$

Most of the results of this section, and to some extent, of the next one, are straight consequences of the exact closed expression of $P(z)$ in terms of an elliptic function. Following the pedagogical approach of this paper we will rather follow a heuristic approach not taking into account the exact result (2.82), to display simple methods and ideas that can be used beyond exact results on a specific example.

From a diff-Padé analysis of the series expansion of $P(z)$, we got the sixty (closest to $z = 0$) singularities. In particular we got that $P(z)$ has a radius of convergence $R \simeq 11.81704500807\cdots$ corresponding to the following (closest to $z = 0$) singularity $z = z_s$ of $P(z)$:

$$\begin{aligned} z_s &= -11.817045008077115768316337283432582087420697\cdots \\ &= (-4) \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; 1\right)^4 = -\frac{1}{16} \cdot \frac{\pi^6}{\Gamma(3/4)^8}. \end{aligned} \tag{2.85}$$

This singularity corresponds to a pole of order four: $P(z) \simeq (z - z_s)^{-4}$. The function $P(z)$ has many other singularities:

$$\begin{aligned} &3^4 \cdot z_s, (161 \pm 240i) \cdot z_s, (-7 \pm 24i) \cdot z_s, (-119 \pm 120i) \cdot z_s, \cdots \\ &5^4 \cdot z_s, (41 \pm 840i) \cdot z_s, (-527 \pm 336i) \cdot z_s, (-1519 \pm 720i) \cdot z_s, \cdots \\ &7^4 \cdot z_s, (1241 \pm 2520i) \cdot z_s, (-567 \pm 1944i) \cdot z_s, (-3479 \pm 1320i) \cdot z_s, \cdots \end{aligned} \tag{2.86}$$

In fact, introducing x and y the real and imaginary part of these singularities in z_s units, one finds out that they correspond to the double infinity of points

$$x = \left(m_1^2 - 2m_1m_2 - m_2^2\right) \cdot \left(m_1^2 + 2m_1m_2 - m_2^2\right),$$
$$y = 4 \cdot m_1m_2 \cdot (m_2 - m_1) \cdot (m_2 + m_1), \tag{2.87}$$

where m_1 and m_2 are two integers, and they all lie on the intersection of an infinite number of genus zero curves indexed by the fourth power of an integer $M = m^4$ ($m = m_1$ or $m = m_2$):

$$2^{12}M^4 - 2^{11} \cdot x \cdot M^3 - 2^7 \cdot \left(17y^2 + 14x^2\right) \cdot M^2 - 2^5 \cdot x \cdot \left(8x^2 + 7y^2\right) \cdot M + y^4 = 0. \tag{2.88}$$

The parametrization (2.87) describes not only the poles of $P(z)$ when $m_1 + m_2$ is odd, but also the zeros of $P(z)$ when $m_1 + m_2$ is even. This (infinite) proliferation of singularities confirms the nonholonomic character of $P(z)$.

These results are simply inherited from (2.82). The zeros and poles of the elliptic sinus $sn(z, i)$ correspond to two lattice of periods. Denoting $\mathcal{K}_1$ and $\mathcal{K}_2$ the two periods of the elliptic curve, the location of the poles and zeros reads, respectively,

$$P_{n_1,n_2} = 2n_1 \cdot \mathcal{K}_1 + (2n_2 + 1) \cdot \mathcal{K}_2,$$
$$Z_{n_1,n_2} = 2n_1 \cdot \mathcal{K}_1 + 2n_2 \cdot \mathcal{K}_2,$$
$$\mathcal{K}_1 = \frac{\pi^{3/2}}{2^{3/2}} \cdot \frac{1}{\Gamma(3/4)^2}, \quad \mathcal{K}_2 = \left(1 - \sqrt{-1}\right) \cdot K_1 \tag{2.89}$$

making crystal clear the fact that we have complex multiplication for this elliptic curve. The formula (2.87) just amount to saying that the poles and zeros of $sn(z^{1/4}, i)$ are located at $P^4_{n_1,n_2}$ and $Z^4_{n_1,n_2}$:

$$P^4_{n_1,n_2} = -\frac{z_s}{4} \cdot ((2n_1 + 2n_2 + 1) + i \cdot (2n_2 + 1))^4,$$
$$Z^4_{n_1,n_2} = -\frac{z_s}{4} \cdot ((2n_1 + 2n_2 + 1) + i \cdot 2n_2)^4. \tag{2.90}$$

The correspondence with (2.87) is $m_1 = n_1 + 2n_2 + 1$, $m_2 = -n_1$ for the poles and $m_1 = n_1 + 2n_2$, $m_2 = -n_1$ for the zeros.

Remark 2.2. let us consider the $a_1 \rightarrow \infty$ limit of the one-parameter series R_{a_1} (see (2.31), (2.32)) rewriting $R_{a_1}(z)$ as $\widetilde{R}_{b_1}(u)$

$$\widetilde{R}_{b_1}(u) = R_{a_1}(z), \quad \text{with}: \ b_1 = \frac{1}{a_1}, \ u = \frac{z}{b_1}. \tag{2.91}$$

In the $a_1 \rightarrow \infty$ limit, that is the $b_1 \rightarrow 0$ limit, one easily verifies, order by order in u, that $\widetilde{R}_{b_1}(u)$ becomes *exactly* the transcendental parametrization function (2.53):

$$\widetilde{R}_{b_1}(u) \longrightarrow P(u) \quad \text{when } b_1 \longrightarrow 0. \tag{2.92}$$

For $a_1 = (-4)^n$ ($n \rightarrow \infty$), one finds that the radius of convergence of the $R_{a_1}(z)$ series becomes in the $n \rightarrow \infty$ limit $R_n \simeq z_s/4^n$, in agreement[23] with (2.91).

2.5. $P(z)$ *and an Infinite Number of Rational Transformations: The Sky Is the Limit*

Note that some nonlinear ODEs associated with $P(z)$ and displayed in Appendix B, namely (B.3) and (B.10), and the functional equation (2.79), are invariant by the change of variable $(P(z), z) \rightarrow (-4P(z)/(1-P(z))^2, -4z)$. In fact (B.3), (2.79), and (B.10) are invariant by $(P(z), z) \rightarrow (-4P(z)/(1-P(z))^2, -4z)$, but also $(P(z), z) \rightarrow (-(1-P(z))^2/4/P(z), -z/4)$, and also by $(P(z), z) \rightarrow (1/P(z), z)$.

The function $P(z)$ satisfies the functional equation:

$$P(-4 \cdot z) = -\frac{4P(z)}{1-P(z)^2}, \tag{2.93}$$

but also

$$\begin{aligned} P((-7-24i)\cdot z) &= T(P(z)), \\ P((-7+24i)\cdot z) &= T^*(P(z)), \end{aligned} \tag{2.94}$$

and, more generally, as can be checked order by order on series expansions

$$P(a_1 \cdot z) = R_{a_1}(P(z)). \tag{2.95}$$

For example, considering the "good" branch (2.35) for the inverse of $-4z/(1-z)^2$, namely $S^{(1)}_{-1/4}(z)$, we can even check, order by order, on the series expansions of $P(z)$ and $S^{(1)}_{-1/4}(z)$ the functional relation

$$S^{(1)}_{-1/4}(P(z)) = P\left(-\frac{z}{4}\right), \tag{2.96}$$

valid for $|P(z)| < 1$ *since the radius of convergence of* $S^{(1)}_{-1/4}(z)$ *is* 1.

Recalling the functional equations (2.94) it is natural to say that if $P(z)$ is singular at $z = z_s$, then, for almost all the rational functions, in particular $T(z)$ (resp. $T^*(z)$) the $T(P(z))$ is also singular $z = z_s$, and thus, from (2.94), $P(z)$ is also singular at $z = (-7 \pm 24i) \cdot z_s$. It is thus extremely natural to see the emergence of the infinite number of singularities in (2.87) of the form $z = (N_1 + i \cdot N_2) \cdot z_s$, as a consequence of (2.95) together with a reduction of the one-parameter series $R_{a_1}(z)$ to a rational function for an infinite number of selected values of a_1, namely the $N_1 + i \cdot N_2$ in (2.87). This is actually the case for all the values displayed in (2.87). For instance, for $a_1 = 3^4 = 81$ we get the following simple rational function:

$$R_{81}(z) = z \cdot \left(\frac{z^2 + 6z - 3}{3z^2 - 6z - 1} \right)^4, \tag{2.97}$$

for which it is straightforward to verify that this rational transformation commutes with $T(z)$, $T^*(z)$, $-4z/(1-z)^2$, and is a solution of the Rota-Baxter-like functional equation (2.19). The case $a_1 = 5^4 = 625$ in (2.87), is even simpler, since it just requires to compose $T(z)$ and $T^*(z)$

$$\begin{aligned} R_{625}(z) &= T(T^*(z)) = T^*(T(z)) \\ &= z \cdot \left(\frac{z^2 - 2z + 5}{5z^2 - 2z + 1} \right)^4 \cdot \left(\frac{1 - 12z - 26z^2 + 52z^3 + z^4}{1 + 52z - 26z^2 - 12z^3 + z^4} \right)^4, \end{aligned} \tag{2.98}$$

which, again verifies (2.19) and commutes with all the other rational functions, in particular (2.97). We also obtained the rational function corresponding to $a_1 = 7^4 = 2401$, namely:

$$R_{2401}(z) = z \cdot \left(\frac{N_{2401}(z)}{D_{2401}(z)} \right)^4, \tag{2.99}$$

with:

$$N_{2401}(z) = z^{12} \cdot D_{2401}\left(\frac{1}{z}\right), \tag{2.100}$$

$$\begin{aligned} D_{2401}(z) = 1 &+ 196z - 1302z^2 + 14756z^3 - 15673z^4 - 42168z^5 \\ &+ 111916z^6 - 82264z^7 + 35231z^8 - 19852z^9 \\ &+ 2954z^{10} + 308z^{11} - 7z^{12}. \end{aligned} \tag{2.101}$$

The polynomial $N_{2401}(z)$ satisfies many functional equations, like, for instance (with $R_{-4}(z) = -4z/(1-z^2)$):

$$4^{12} \cdot D_{2401}\left(\frac{1}{R_{-4}(z)}\right) = D_{2401}(z) \cdot D_{2401}\left(\frac{1}{z}\right) \tag{2.102}$$

and also

$$(1-z)^{49} \cdot D_{2401}(R_{-4}(z))^2 = D_{2401}(z)^4 - z^{49} \cdot D_{2401}\left(\frac{1}{z}\right)^4. \tag{2.103}$$

We also obtained the rational function corresponding to $a_1 = 11^4 = 14641$, namely,

$$R_{14641}(z) = z \cdot \left(\frac{N_{14641}(z)}{D_{14641}(z)}\right)^4, \tag{2.104}$$

with:

$$N_{14641}(z) = z^{30} \cdot D_{14641}\left(\frac{1}{z}\right), \tag{2.105}$$

$$\begin{aligned} D_{14641}(z) = {} & 1 + 1210z - 33033z^2 + 2923492z^3 + 5093605z^4 \\ & - 385382514z^5 + 3974726283z^6 - 14323974808z^7 \\ & + 57392757037z^8 - 291359180310z^9 + 948497199067z^{10} \\ & - 1642552094436z^{11} + 1084042069649z^{12} + 1890240552750z^{13} \\ & - 6610669151537z^{14} + 9712525647792z^{15} - 8608181312269z^{16} \\ & + 5384207244702z^{17} - 3223489742187z^{18} + 2175830922716z^{19} \\ & - 1197743580033z^{20} + 387221579866z^{21} - 50897017743z^{22} \\ & - 7864445336z^{23} + 5391243935z^{24} - 815789634z^{25} \\ & + 28366041z^{26} - 5092956z^{27} + 207691z^{28} + 2794z^{29} - 11z^{30}, \end{aligned} \tag{2.106}$$

and, of course, one can verify that $R_{14641}(z)$ actually commutes with R_{-4}, R_{81}, R_{625}, $R_{2401}(z)$, and is a solution of the Rota-Baxter-like functional equation (2.19). Similarly to $R_{2401}(z)$ (see (2.102), (2.103)), we also have the functional equations:

$$4^{30} \cdot D_{14641}\left(\frac{1}{R_{-4}(z)}\right) = D_{14641}(z) \cdot D_{14641}\left(\frac{1}{z}\right), \tag{2.107}$$

and also

$$(1-z)^{(4\cdot 30+1)} \cdot D_{14641}(R_{-4}(z))^2 = D_{14641}(z)^4 - z^{(4\cdot 30+1)} \cdot D_{14641}\left(\frac{1}{z}\right)^4. \tag{2.108}$$

Next we obtained the rational function corresponding to $a_1 = 13^4 = 28561$, which verifies (2.19) namely,

$$\begin{aligned} & R_{28561}(z) = z \cdot \left(\frac{N_{28561}(z)}{D_{28561}(z)}\right)^4, \quad \text{with } N_{28561}(z) = z^{42} \cdot D_{28561}\left(\frac{1}{z}\right), \\ & N_{28561}(z) = z^{42} \cdot D_{28561}\left(\frac{1}{z}\right), \end{aligned} \tag{2.109}$$

$$
\begin{aligned}
D_{28561}(z) &= \left(1 - 22z + 235z^2 - 228z^3 + 39z^4 + 26z^5 + 13z^6\right) \cdot D_{28561}^{(36)}(z) \\
D_{28561}^{(36)}(z) &= 1 + 2388z - 61098z^2 + 19225300z^3 + 606593049z^4 \\
&\quad - 1543922656z^5 + 7856476560z^6 - 221753896032z^7 + 1621753072244z^8 \\
&\quad - 4542779886736z^9 + 2731418674664z^{10} + 36717669656304z^{11} \\
&\quad - 200879613202428z^{12} + 547249607666784z^{13} - 934179604482832z^{14} \\
&\quad + 1235038888776160z^{15} - 1788854212778642z^{16} + 3018407750933816z^{17} \\
&\quad - 4349780716415868z^{18} + 4419228090228152z^{19} - 2899766501472914z^{20} \\
&\quad + 931940880451552z^{21} + 413258559018224z^{22} - 857795672629664z^{23} \\
&\quad + 659989056851972z^{24} - 304241349909008z^{25} + 87636987790824z^{26} \\
&\quad - 14593362219920z^{27} + 1073204980340z^{28} + 45138167200z^{29} \\
&\quad - 23660433008z^{30} + 2028597792z^{31} - 29540327z^{32} + 3238420z^{33} \\
&\quad - 73386z^{34} - 492z^{35} + z^{36}.
\end{aligned}
\tag{2.110}
$$

We get similar results, mutatis mutandis, than the ones previously obtained (commutation, functional equations like (2.107), (2.108), etc.), namely,

$$
4^{42} \cdot D_{28561}\left(\frac{1}{R_{-4}(z)}\right) = D_{28561}(z) \cdot D_{28561}\left(\frac{1}{z}\right), \ldots. \tag{2.111}
$$

The "palindromic" nature of (2.97) (2.98), (2.99), (2.104) and (2.109) (see (2.100), (2.105)), (2.109)) corresponds to the fact that these rational transformations commute with J:

$$
\frac{1}{R_{81}(z)} = R_{81}\left(\frac{1}{z}\right), \quad \frac{1}{R_{625}(z)} = R_{625}\left(\frac{1}{z}\right), \ldots. \tag{2.112}
$$

In fact, more generally, we have $R_{N^4}(1/z) = 1/R_{N^4}(z)$ for N any odd integer ($N = 9, 21, \ldots$) and $R_{N^4}(1/z) = R_{N^4}(z)$ for any even integer N.

From (2.87) one can reasonnably conjecture that the fourth power of *any integer* will provide a new example of $R_{a_1}(z)$ being a rational function. The simple nontrivial example corresponds to the already found rational function

$$
R_{16}(z) = 16 \cdot \frac{z \cdot (1-z)^2}{(z+1)^4}. \tag{2.113}
$$

We already have explicit rational functions for all values of a_1 of the form N^4 for $N = 2, 3, \ldots, 16$ and of course, we can in principle, build explicit rational functions for all the N's product of the previous integers. Along this line it is worth noticing that the coefficients of the series $R_{a_1}(z)$ are all integers when a_1 is the fourth power of any integer.

We are thus starting to build an infinite number of (elementary) *commuting* rational transformations, *any* composition of these (infinite number of) rational transformations giving rational transformations satisfying (2.19) and preserving the linear differential operator Ω. This set of rational transformations is a pretty large set! Actually this set of rational transformations corresponds to the isogenies of the underlying elliptic function.

The proliferation of the singularities of $P(z)$ corresponds to this (pretty large) set of rational transformations. Recalling (2.96), the previous singularity argument is *not valid* [24] for the (well-suited) inverse transformations ($S^{(1)}_{-1/4}(z), \ldots$) of these rational transformations because (2.96) requires $|P(z)| < 1$ (corresponding to the radius of convergence of $S^{(1)}_{-1/4}(z)$) and the singularity $z = z_s$ corresponds precisely to "hit" the value $P(z) = 1$.

2.6. Other Examples of Selected Gauss Hypergeometric ODEs

For heuristic reasons we have focused on $A(z) = (3-5z)/z/(1-z)/4$, but of course, one can find many other examples and try to generalize these examples.

For instance, introducing

$$A(z) = \frac{1}{6} \cdot \frac{d \ln\left((1-z)^3 z^5\right)}{dz} = \frac{1}{6} \cdot \frac{5-8z}{(1-z)z} \tag{2.114}$$

the rational transformation

$$R(z) = -27 \cdot \frac{z}{(1-4z)^3}, \tag{2.115}$$

verifies the "Rota-Baxter-like" functional relation (2.19). This example corresponds to the following covariance [12] on a Gauss hypergeometric integral (of the $c = 1 + b$ type, see below):

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{6}\right], \left[\frac{7}{6}\right]; z\right) &= \frac{z^{-1/6}}{6} \cdot \int_0^z t^{-5/6}(1-t)^{-1/2} \cdot dt \\ &= (1-4z)^{-1/2} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{6}\right], \left[\frac{7}{6}\right]; -27\frac{z}{(1-4z)^3}\right) \end{aligned} \tag{2.116}$$

which is associated with the elliptic curve

$$y^6 - (1-t)^3 \cdot t^5 = 0. \tag{2.117}$$

Another example (of the $c = 1 + a$ type, see below) is

$$A(z) = \frac{1}{3} \cdot \frac{d \ln\left((1-z)^2 z^2\right)}{dz} = \frac{2}{3} \cdot \frac{1-2z}{(1-z)z}, \tag{2.118}$$

where the rational transformation

$$R(z) = \frac{z \cdot (z-2)^3}{(1-2z)^3} = -8z - 36z^2 - 126z^3 - 387z^4 + \cdots \tag{2.119}$$

verifies the "Rota-Baxter-like" functional relation (2.19). This example corresponds to the following covariance [12] on a Gauss hypergeometric integral:

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{4}{3}\right]; z\right) &= \frac{z^{-1/3}}{3} \cdot \int_0^z t^{-2/3}(1-t)^{-2/3} \cdot dt \\ &= \frac{1}{2} \cdot \frac{2-z}{1-2z} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{4}{3}\right]; \frac{z \cdot (z-2)^3}{(1-2z)^3}\right), \end{aligned} \tag{2.120}$$

which is associated with the elliptic curve

$$y^3 - (1-t)^2 \cdot t^2 = 0. \tag{2.121}$$

Note that, similarly to the main example of the paper, there exist many rational transformations[25] satisfying (2.19) that cannot be reduced to iterates of (2.119), for instance,

$$\begin{aligned} T(z) = -27 \cdot \frac{z \cdot (1-z)\left(z^2 - z + 1\right)^3}{\left(z^3 + 3z^2 - 6z + 1\right)^3} &= -27z - 378z^2 \\ &- 3888z^3 - 34074z^4 - 271620z^5 - 2032209z^6 + \cdots . \end{aligned} \tag{2.122}$$

One verifies immediately that (2.122) actually verifies (2.19) with (2.118). Not surprisingly, the two rational transformations (2.119) and (2.122) commute.

Another simple example with rational symmetries corresponds to $\Omega = (D_z + A(z)) \cdot D_z$ with

$$A(z) = -\frac{1}{2} \cdot \frac{3z-1}{z(1-z)} = \frac{1}{2} \cdot \frac{d \ln\left(z \cdot (1-z)^2\right)}{dz}. \tag{2.123}$$

It has the simple (genus zero) hypergeometric solution[26]:

$$\mathcal{F}(z) = z^{1/2} \cdot {}_2F_1\left(\left[1, \frac{1}{2}\right], \left[\frac{3}{2}\right]; z\right) = \operatorname{arctanh}\left(z^{1/2}\right). \tag{2.124}$$

The linear differential operator Ω is covariant under the change of variable $z \rightarrow 1/z$ and $z \rightarrow R(z)$, where[27]

$$R(z) = \frac{4z}{(1+z)^2}. \tag{2.125}$$

One can easily check that (2.123) and (2.125) satisfy the functional equation (2.19). One also verifies that (2.123) and $z \to 1/z$ or the iterates of (2.125) satisfy the functional equation (2.19). The solution of the adjoint operator are $(1-z)\cdot z^{1/2}$ and

$$\begin{aligned} F(z) &= z\cdot(1-z)\cdot {}_2F_1\left(\left[1,\frac{1}{2}\right],\left[\frac{3}{2}\right];z\right) \\ &= z^{1/2}\cdot(1-z)\cdot \operatorname{arctanh}\left(z^{1/2}\right) = z-\frac{2}{3}z^2-\frac{2}{15}z^3 \\ &\quad -\frac{2}{35}z^4-\frac{2}{63}z^5-\frac{2}{99}z^6-\frac{2}{143}z^7+\cdots. \end{aligned} \tag{2.126}$$

One verifies, again, that (2.126) and (2.125) commute, (2.126) corresponding to the "infinitesimal composition" of (2.125) (see (2.56)).

A first natural generalization amounts to keeping the remarkable factorization (2.6) which will, in fact, reduce the covariance of a second-order operator to the covariance of a first-order operator.[28] Such a situation occurs for Gauss hypergeometric functions ${}_2F_1([a,b],[1+a];z)$ solution of the (a,b)-symmetric linear differential operator

$$z\cdot(1-z)\cdot D_z^2+(c-(a+b+1)\cdot z)\cdot D_z-a\cdot b, \tag{2.127}$$

as soon as[29] $c=1+a$. For instance

$$\mathcal{F}(z) = z^a\cdot {}_2F_1([a,b],[1+a];z), \tag{2.128}$$

is an integral of a simple algebraic function and is solution with the constant function of the second-order operator

$$\begin{aligned} \Omega &= \left(D_z+\frac{(a-b-1)z+1-a}{z\cdot(1-z)}\right)\cdot D_z \\ &= \left(D_z+\frac{d\ln\left((1-z)^b\cdot z^{1-a}\right)}{dz}\right)\cdot D_z, \end{aligned} \tag{2.129}$$

yielding a new $A(z)$:

$$A(z) = \frac{(1-a)+(a-b-1)z}{(1-z)\cdot z} = \frac{1-a}{z}-\frac{b}{1-z}. \tag{2.130}$$

The adjoint of (2.129) has the simple solution $z^{1-a}\cdot(1-z)^b$:

$$F(z) = z\cdot(1-z)^b\cdot {}_2F_1([a,b],[1+a];z). \tag{2.131}$$

Due to the (a, b)-symmetry of (2.127) we have a similar result for $c = 1+b$. The function $\mathcal{F}(z) = z^b \cdot {}_2F_1([a, b], [1 + b]; z)$ is solution of (2.129) where a and b have been permuted:

$$\left(D_z + \frac{(b - a - 1)z + 1 - b}{z \cdot (1 - z)}\right) \cdot D_z, \tag{2.132}$$

yielding another $A(z)$

$$A(z) = \frac{(1 - b) + (b - a - 1)z}{(1 - z) \cdot z}, \tag{2.133}$$

The adjoint of (2.132) has the solution $(1-z)^a \cdot z^{1-b}$ together with the hypergeometric function:

$$F(z) = z \cdot (1 - z)^a \cdot {}_2F_1([a, b], [1 + b]; z), \tag{2.134}$$

where one recovers the previous result (2.126).

We are seeking for (Gauss hypergeometric) second-order differential equations[30] with an *infinite number* of (hopefully rational, if not algebraic) symmetries: this is another way to say that we are not looking for generic Gauss hypergeometric differential equations, but Gauss hypergeometric differential equations *related to elliptic curves*, and thus having an infinite set of such isogenies. We are necessarily in the framework where the two parameters a and b of the Gauss hypergeometric are *rational numbers* in order to have *integral of algebraic functions* (yielding globally nilpotent [11] second-order differential operators). Let us denote by D the common denominator of the two rational numbers $a = N_a/D$ and $b = N_b/D$, function (2.128) is associated to a period of the algebraic curve

$$y^D = (1 - t)^{N_b} \cdot t^{D-N_a}. \tag{2.135}$$

We just need to restrict to triplets of integers (N_a, N_b, D) such that the previous curve is an elliptic curve.

Let us give an example (of the $c = 1 + b$ type) that *does not* correspond to a genus one curve, with

$${}_2F_1\left(\left[\frac{1}{3}, \frac{1}{6}\right], \left[\frac{7}{6}\right]; z\right) = \frac{1}{6} \cdot z^{-1/6} \cdot \int_0^z t^{-5/6}(1 - t)^{-1/3} \cdot dt, \tag{2.136}$$

which corresponds to the *genus two* curve:

$$y^6 - (1 - t)^2 \cdot t^5 = 0. \tag{2.137}$$

Again one introduces $A(z)$

$$A(z) = \frac{1}{6} \cdot \frac{d \ln\left((1 - z)^2 z^5\right)}{dz} = \frac{1}{6} \cdot \frac{5 - 7z}{z \cdot (1 - z)} \tag{2.138}$$

and seeks for $R(z)$ as series expansions analytical at $z = 0$. One gets actually, order by order, a one-parameter family

$$\begin{aligned}
R_{a_1}(z) = {} & a_1 \cdot z - \frac{2}{7} a_1 \cdot (a_1 - 1) \cdot z^2 \\
& + \frac{1}{637} a_1 \cdot (a_1 - 1) \cdot (17 a_1 - 87) \cdot z^3 \\
& - \frac{2}{84721} a_1 \cdot (a_1 - 1) \cdot \left(113 a_1^2 - 856 a_1 + 3438\right) \cdot z^4 \\
& - \frac{1}{38548055} a_1 \cdot (a_1 - 1) \cdot \left(3674 a_1^3 + 121194 a_1^2 - 552261 a_1 + 2095059\right) \cdot z^5 + \cdots \\
& + \frac{1 + \epsilon_n}{N(n)} \cdot a_1 \cdot (a_1 - 1) \cdot P_n(a_1) \cdot z^n + \cdots,
\end{aligned} \tag{2.139}$$

where $\epsilon_n = 0$ for n odd and $\epsilon_n = 1$ for n even, and $N(n)$ is a (large) integer depending on n, and $P_n(a_1)$ is a polynomial with integer coefficients of degree $n - 2$. One easily verifies, order by order, that one gets a one-parameter family of transformations commuting for different values of the parameter:

$$R_{a_1}(R_{b_1}(z)) = R_{b_1}(R_{a_1}(z)) = R_{a_1 b_1}(z). \tag{2.140}$$

As far as the "algorithmic complexity" of this series (2.139) is concerned it is worth noticing that the degree growth [18] of the series coefficients is actually *linear* and not exponential as we could expect [19] at first sight. Even if this series is transcendental, it is not a "wild" series.

Seeking for selected values of a_1 such that the previous series (2.139) reduces to a rational function one can try to reproduce the simple calculations (2.48), (2.49), but unfortunately "shooting in the dark" because we have no hint of a well-suited denominator (if any!) like the polynomials in the lhs of (2.48), (2.49).

It is also worth noticing that if we slightly change $A(z)$ into

$$A(z) = \frac{1}{N} \cdot \frac{d \ln\left((1 - z)^2 z^5\right)}{dz} = \frac{1}{N} \cdot \frac{5 - 7z}{z \cdot (1 - z)}, \tag{2.141}$$

the algebraic curve (2.137) becomes $y^N - (1 - t)^2 \cdot t^5 = 0$ which has, for instance genus five for $N = 11$, but genus zero for $N = 7$. For any of these cases of (2.141) one can easily get, order by order, a one-parameter series R_{a_1} totally similar to (2.139) with, again, polynomials $P_n(a_1)$ of degree $n - 2$.

The first coefficient a_2 is in general

$$a_2 = -\frac{2}{2N - 5} \cdot a_1 \cdot (a_1 - 1). \tag{2.142}$$

For the genus zero case, $N = 7$

$$a_2 = -\frac{2}{9} \cdot a_1 \cdot (a_1 - 1), \quad a_3 = -\frac{1}{1296} \cdot a_1 \cdot (a_1 - 1) \cdot (127 - a_1),$$
$$a_4 = -\frac{1}{134136} \cdot a_1 \cdot (a_1 - 1) \cdot \left(254a_1^2 + 185a_1 + 7499\right), \ldots, \tag{2.143}$$
$$a_n = -\frac{1}{N(n)} \cdot a_1 \cdot (a_1 - 1) \cdot P_n(a_1),$$

which corresponds to the solution

$$\frac{2}{7} \cdot \int_0^z z^{-5/7} \cdot (1-z)^{-2/7} \cdot dt = z^{2/7} \cdot {}_2F_1\left(\left[\frac{2}{7}, \frac{2}{7}\right], \left[\frac{9}{7}\right]; z\right). \tag{2.144}$$

Using the parametrization of the *genus zero curve*

$$y = -\frac{(u+1)^2 \cdot u^5}{(u+1)^7 - u^7}, \qquad t = -\frac{u^7}{(u+1)^7 - u^7}, \tag{2.145}$$

one can actually perform the integration (2.144) of dt/y and get an alternative form of the hypergeometric function (2.144):

$$\int_0^z z^{-5/7} \cdot (1-z)^{-2/7} \cdot dt = \int_0^u \rho(u) \cdot du = \int_0^v \frac{v}{1-v^7} \cdot dv,$$
$$\text{where: } z = -\frac{u^7}{(u+1)^7 - u^7}, \ \rho(u) = \frac{(u+1)^4 \cdot u}{(u+1)^7 - u^7}, \tag{2.146}$$
$$\text{and: } v = \frac{u}{1+u}, \ z = \frac{v^7}{v^7 - 1}.$$

Except transformations like $v \to \omega \cdot v$ (with $\omega^7 = 1$) which have no impact on z, it seems difficult to find rational symmetries in this genus zero case.

For $N = 11$ (genus five) the first successive coefficients read:

$$a_2 = -\frac{2}{17} \cdot a_1 \cdot (a_1 - 1),$$
$$a_3 = -\frac{1}{8092} \cdot a_1 \cdot (a_1 - 1) \cdot (143a_1 + 367),$$
$$a_4 = -\frac{1}{206346} \cdot a_1 \cdot (a_1 - 1) \cdot \left(1186a_1^2 + 2473a_1 + 5011\right), \ldots, \tag{2.147}$$
$$a_n = -\frac{1}{N(n)} \cdot a_1 \cdot (a_1 - 1) \cdot P_n(a_1).$$

The "infinitesimal composition" function $F(z)$ (see (2.56), (2.57), and (2.58)) reads,

$$F(z) = \left.\frac{\partial R_{a_1}}{\partial a_1}\right|_{a_1=1} = z - \frac{2}{17}z^2 - \frac{15}{238}z^3 - \frac{5}{119}z^4 - \frac{37}{1190}z^5 - \frac{888}{36295}z^6 - \frac{2183}{108885}z^7 - \frac{4366}{258213}z^8 - \frac{58941}{4045337}z^9 - \frac{1807524}{141586795}z^{10} - \frac{46543743}{4106017055}z^{11} - \frac{5305986702}{521464165985}z^{12} + \cdots \tag{2.148}$$

and again we can actually check that this is actually the series expansion of the hypergeometric function

$$z \cdot (1-z) \cdot {}_2F_1\left(\left[1, \frac{15}{11}\right], \left[\frac{17}{11}\right]; z\right), \tag{2.149}$$

solution of Ω^* the adjoint of the Ω linear differential operator corresponding to this (genus 5) $N = 11$ case:

$$\Omega^* = D_z \cdot \left(D_z + \frac{1}{11} \cdot \frac{7z-5}{z \cdot (1-z)}\right). \tag{2.150}$$

We have similar results for (2.128), (2.129), (2.130). As far as these one-parameter families of transformations R_{a_1} are concerned, the only difference between the generic cases corresponding to *arbitrary genus* and *genus one* cases like (2.118) is that in the generic higher genus case, only a *finite number* of values of the parameter a_1 can correspond to rational functions. Note that this higher genus result generalizes to the arbitrary genus Gauss hypergeometric functions (2.128) and associated operators (2.129) and function (2.130). In this general case one can also get order by order a one-parameter family of transformations R_{a_1} satisfying a commutation relation (2.141).

Note that $R(z) = 1/z$ is *actually a solution of* (2.19) *for this genus-two example* (2.139). Along this line of selected $R(z)$ solutions of (2.19) many interesting subcases of this general case (2.128), (2.129), (2.130) are given in Appendix C.1.

In our previous genus-one examples, with this close identification between the *renormalization group and the isogenies of elliptic curves*, we saw that, in order to obtain linear differential operators covariant by an infinite number of transformations (rational or algebraic), we must restrict our second-order Gauss hypergeometric differential operator to Gauss hypergeometric *associated to elliptic curves* (see Appendices C and D). Beyond this framework we still have one-parameter families (see (2.141)) but we cannot expect an infinite number of rational (and probably algebraic) transformations to be particular cases of such families of transcendental transformations.

3. Conclusion

We have shown that several selected Gauss hypergeometric linear differential operators associated to elliptic curves, and factorised into order-one linear differential operators, actually present an *infinite number of rational symmetries* that actually identify with the *isogenies of the associated elliptic curves* that are perfect illustrations of *exact representations of the renormalization group*. We actually displayed all these calculations, results, and structures because they are perfect examples of exact renormalization transformations. For more realistic models (corresponding to Yang-Baxter models with elliptic parametrizations), the previous calculations and structures become more involved and subtle, the previous rational transformations being replaced by algebraic transformations corresponding to *modular curves*. For instance, in our models of lattice statistical mechanics (or enumerative combinatorics, etc.), we are often getting globally nilpotent linear differential operators [11] of quite high orders [20–24] that, in fact, factor into globally nilpotent operators of smaller orders[31] which, for Yang-Baxter integrable models with a *canonical elliptic parametrization*, must necessarily " *be associated with elliptic curves.*" Appendix D provides some calculations showing that the integral for $\chi^{(2)}$, the two-particle contribution of the susceptibility of the Ising model [25–27], is clearly and straightforwardly associated with an elliptic curve.

We wanted to highlight the importance of *explicit constructions* in answering difficult or subtle questions.

All the calculations displayed in this paper are elementary calculations given explicitly for heuristic reasons. The simple calculations (in particular with the introduction of a simple Rota-Baxter like functional equation) should be seen as some undergraduate training to more realistic renormalization calculations that will require a serious knowledge of fundamental modular curves, *modular forms*, Hauptmoduls, Gauss-Manin or Picard-Fuchs structures [28, 29] and, beyond, some knowledge of mirror symmetries [30–34] of Calabi-Yau manifolds, these *mirror symmetries* generalizing[32] the Hauptmodul structure for elliptic curves.

Appendices

A. Comment on the Rota-Baxter-Like Functional Equation (2.19)

We saw, several times, that the Rota-Baxter-like functional equation (2.19) is such that for a given $A(z)$ one gets a one-parameter family of analytical functions $R(z)$ obtained order by order by series expansion (see (2.32), (2.139)). Conversely for a given $R(z)$, for instance, $R(z) = -4z/(1-z)^2$, let us see if $R(z)$ can come from a unique $A(z)$. Assume that there are two $A(z)$ satisfying (2.26) with the same $R(z) = -4z/(1-z)^2$. We will denote $\delta(z)$ the difference of these two $A(z)$, and we will also introduce $\Delta(z) = z \cdot \delta(z)$. It is a straightforward calculation to see that $\Delta(z)$ verifies

$$\Delta(z) = \frac{1+z}{1-z} \cdot \Delta\left(\frac{-4z}{(1-z)^2}\right), \tag{A.1}$$

which has, beyond $\Delta(z) = 0$, at least one solution analytical at $z = 0$ that we can get order by order:

$$\Delta(z) = 1 + \frac{2}{5}z + \frac{22}{75}z^2 + \frac{394}{1625}z^3 + \frac{262634}{1243125}z^4 + \cdots . \tag{A.2}$$

It is straightforward to show from (A.1), from similar arguments we introduced for (2.57) on the functional equations (2.58) that $\Delta(z)$ is a transcendental function.

B. Miscellaneous Nonlinear ODEs on $P(z)$

From (2.70) one can get

$$F'(P(z)) = 1 + z \cdot \frac{P^{(2)}}{P^{(1)}},$$
$$F''(P(z)) = \frac{P^{(2)}}{\left(P^{(1)}\right)^2} + z \cdot \frac{P^{(3)}}{\left(P^{(1)}\right)^2} - z \cdot \frac{\left(P^{(2)}\right)^2}{\left(P^{(1)}\right)^3}, \tag{B.1}$$

and from (2.64), the linear second-order ODE on $F(z)$, one deduces the third-order nonlinear ODE[33] on the (at first sight *nonholonomic*) function $P(z)$:

$$\begin{aligned} &z \cdot \left(5P^2 - 6P + 3\right) \cdot \left(P^{(1)}\right)^4 - P \cdot (5P-3) \cdot (P-1) \cdot \left(P^{(1)}\right)^3 \\ &\quad - z \cdot (P-1) \cdot P \cdot (5P-3) \cdot P^{(2)} \cdot \left(P^{(1)}\right)^2 \\ &\quad + 4P^2 \cdot (P-1)^2 \cdot \left(P^{(2)} + z \cdot P^{(3)}\right) \cdot P^{(1)} \\ &\quad - 4z \cdot \left(P^{(2)}\right)^2 \cdot P^2 \cdot (P-1)^2 = 0, \end{aligned} \tag{B.2}$$

where the $P^{(n)}$'s denote the nth derivative of $P(z)$. This third order nonlinear ODE has a rescaling symmetry $z \longrightarrow \rho \cdot z$, for any ρ, and, also, an interesting symmetry, namely an invariance by $z \longrightarrow z^{\alpha}$, for any[34] value of α.

In a second step, using differential algebra tools, and, more specifically, the fact that $P(Q(z)) = Q(P(z)) = z$ together with the linear ODE for $Q(z)$, one finds the simpler second-order nonlinear ODE for $P(z)$:

$$P^{(2)} - \frac{1}{4} \cdot \frac{5P-3}{(P-1) \cdot P} \cdot \left(P^{(1)}\right)^2 + \frac{3}{4} \cdot \frac{1}{z} \cdot P^{(1)} = 0 \tag{B.3}$$

or

$$P^{(2)} - \left(\frac{3}{4} \cdot \frac{1}{P} + \frac{1}{2} \cdot \frac{1}{P-1}\right) \cdot \left(P^{(1)}\right)^2 + \frac{3}{4} \cdot \frac{1}{z} \cdot P^{(1)} = 0. \tag{B.4}$$

Note that, more generally, the second-order nonlinear ODE

$$P^{(2)} - \left(\frac{3}{4} \cdot \frac{1}{P} + \frac{1}{2} \cdot \frac{1}{P-1}\right) \cdot \left(P^{(1)}\right)^2 + \frac{\eta}{z} \cdot P^{(1)} = 0, \tag{B.5}$$

yields (B.2) for any value of the constant η. The change of variable $z \longrightarrow z^{\alpha}$, changes the parameter η into $1 + \alpha \cdot (\eta - 1)$. In particular the involution $z \leftrightarrow 1/z$ changes $\eta = 3/4$ into $\eta = 5/4$.

This nonlinear ODE, looking like *Painlevé V*, is actually *invariant* by the change of variable $P \rightarrow -4P/(1-P)^2$. It is, also, invariant by any rescaling $z \rightarrow \lambda z$, like the particular degenerate[35] subcase of *Painlevé V*

$$y'' - \left(\frac{1}{2y} + \frac{1}{y-1}\right) \cdot y'^2 + \frac{1}{z} \cdot y' = 0. \tag{B.6}$$

With (2.70) we recover the "Gauss-Manin" idea of Painlevé functions being seen as deformations of elliptic functions:

$$z \cdot \frac{dP(z)}{dz} = P(z) \cdot (1 - P(z))^{1/2} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; P(z)\right). \tag{B.7}$$

or

$$-2z \cdot \frac{d \operatorname{arctanh}\left((1 - P(z))^{1/2}\right)}{dz} = {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; P(z)\right). \tag{B.8}$$

In fact, recalling $Q(P(z)) = z$, one also has the relation

$$P(z) \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; P(z)\right)^4 = z, \tag{B.9}$$

yielding with (B.7) the simple *nonlinear order-one* differential equation

$$z^3 \cdot (P')^4 - (1-P)^2 \cdot P^3 = 0, \tag{B.10}$$

already seen with (2.83), and that we can write in a separate way:

$$\frac{dP}{(1-P)^{1/2} \cdot P^{3/4}} = \frac{dz}{z^{3/4}}. \tag{B.11}$$

Note that $P(z^{4\cdot(1-\eta)})$ is actually solution of (B.5).

Equation (B.10) has (B.9) as a solution but in general the Puiseux series solutions $P_\mu(z)$ of the functional equation (μ is a constant):

$$\begin{gathered} P_\mu(z)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; P_A(z)\right) = \mu + z^{1/4} \quad \text{or} \\ P_\mu(z) = P\left(\left(\mu + z^{1/4}\right)^4\right), \\ P_\mu(z) = P\left(\mu^4\right) + 4 \cdot \mu^3 \cdot P'\left(\mu^4\right) \cdot z^{1/4} + \cdots. \end{gathered} \tag{B.12}$$

It is a straightforward exercise of differential algebra to see that the order-one nonlinear differential equation (B.10) implies (B.3). In particular not only (B.9) is solution of (B.3) but also all the Puiseux series solutions (B.12) of (B.10). More generally the solutions of the functional equation:

$$P_{\mu,\lambda}(z)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; P_{\mu,\lambda}(z)\right) = \mu + \lambda \cdot z^{1/4} \tag{B.13}$$

verify (B.3). This corresponds to the fact that

$$z^3 \cdot (P')^4 - \lambda^4 \cdot (1-P)^2 \cdot P^3 = 0, \tag{B.14}$$

yields (B.2) which is scaling symmetric ($z \rightarrow \rho \cdot z$) when (B.10) is not. More generally

$$z^{4\eta} \cdot (P')^4 - \lambda^4 \cdot (1-P)^2 \cdot P^3 = 0 \tag{B.15}$$

yields (B.2) for any value of the parameters η and λ. Finally, one also has that the solution of the functional equation

$$P_\eta(z)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{5}{4}\right]; P_\eta(z)\right) = \mu + \lambda \cdot z^{1-\eta} \tag{B.16}$$

is solution of (B.2), but also of (B.5) and even of (B.15).

Equation (B.5) with $\eta = 1/2$ (instead of $\eta = 3/2$ in (B.3)) has a solution, analytical at $z = 0$:

$$1 + x + \frac{1}{2}x^2 + \frac{7}{40}x^3 + \frac{1}{20}x^4 + \frac{121}{9600}x^5 + \frac{7}{2400}x^6 + \frac{211}{332800}x^7 + \frac{41}{312000}x^8 + \cdots . \tag{B.17}$$

This series has a singularity at $-1/4 \cdot z_s^2$, where z_s is given by (2.85). The radius of convergence of (B.17) corresponds to this singularity, namely, $R = 1/4 \cdot z_s^2$. This singularity result can be understood from the fact that, at $\eta = 1/2$, $P(z^2)$ is actually solution of (B.5).

In fact, we have the following solutions of (B.5) for various selected values of η. For $\eta = 0$, $P(z^4)$ is solution of (B.5). For $\eta = 2/3$, $P(z^{4/3})$ is solution of (B.5), and, more generally, $P(z^{4\cdot(1-\eta)})$ is solution of (B.5).

C. Gauss Hypergeometric ODEs Related to Elliptic Curves

It is not necessary to recall the close connection between Gauss hypergeometric functions and elliptic curves, or even modular curves [35, 36] and Hauptmoduls. This is very clear on the Goursat-type relation

$$\begin{aligned} &{}_2F_1\left(\left[2a, \frac{2a+1}{3}\right], \left[\frac{4a+2}{3}\right]; x\right) \\ &= \left(1 - x + x^2\right)^{-a} \cdot {}_2F_1\left(\left[\frac{a}{3}, \frac{a+1}{3}\right], \left[\frac{4a+5}{6}\right]; \frac{27}{4} \cdot \frac{(x-1)^2 \cdot x^2}{(1-x+x^2)^3}\right), \end{aligned} \tag{C.1}$$

which generalizes the simpler quadratic Gauss relation:

$$ {}_2F_1\left([a,b], \left[\frac{a+b+1}{2}\right]; x\right) = {}_2F_1\left(\left[\frac{a}{2}, \frac{b}{2}\right], \left[\frac{a+b+1}{2}\right]; 4x(1-x)\right). \tag{C.2} $$

On (C.1) one recognizes (the inverse of) the *Klein modular invariant* [36] for the pull-back of the hypergeometric function on the rhs.

Many values of $[[a,b],[c]]$ are known to correspond to elliptic curves like $[[1/2,1/2],[1]]$ (complete elliptic integrals of the first and second kind) or modular forms: $[[1/12,5/12],[1]]$, $[[2/3,2/3],[1]]$, $[[2/3,2/3],[3/2]]$, and they can even be simply related:

$$ \left(\frac{z+27}{27}\right)^{1/3} \cdot {}_2F_1\left(\left[\frac{2}{3}, \frac{2}{3}\right], [1]; -\frac{1}{27}z\right) = \mu(z) \cdot {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1]; 1728\frac{z}{(z+27)(z+3)^3}\right), \tag{C.3} $$

where:

$$ \mu(z) = \left(\frac{(z+27)(z+3)^3}{729}\right)^{-1/12}. \tag{C.4} $$

Once we have a hypergeometric function corresponding to an elliptic curve for some values of (a,b,c), one can find other values of (a,b,c) also corresponding to elliptic curves

$$ {}_2F_1([a,b],[c];x) \longrightarrow x^{1-c} \cdot {}_2F_1([1+a-c, 1+b-c], [2-c]; x). \tag{C.5} $$

In order to provide simple examples of linear differential ODEs we will restrict ourselves (just for heuristic reasons) to Gauss hypergeometric second-order differential equations.

Let us recall the *Euler integral representation* of the Gauss hypergeometric functions:

$$ \begin{aligned} {}_2F_1([a,b],[c];z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \int_0^1 \frac{dw}{w} w^b \cdot (1-w)^{c-1-b} \cdot (1-zw)^{-a} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \cdot \int_0^1 \frac{dw}{w} w^a \cdot (1-w)^{c-1-a} \cdot (1-zw)^{-b} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \cdot z^{-a} \int_0^z \frac{du}{u} u^a \cdot \left(1-\frac{u}{z}\right)^{c-1-a} \cdot (1-u)^{-b}. \end{aligned} \tag{C.6} $$

On the last line of (C.6), the selected role of $c = 1 + a$ is quite clear.

Recall that the corresponding second-order differential operator is *invariant under the permutation of* a *and* b which is not obvious[37] on the Euler integral representations of the hypergeometric functions (this amounts to permuting 0 and ∞). The permutation of a and b is always floating around in this paper.

When the three parameters a, b and c of the Gauss hypergeometric functions are rational numbers we have integrals of *algebraic functions* and, therefore, we know [11, 37–40] that the corresponding second-order differential operator is necessarily *globally nilpotent* [11, 37–40]. Let us restrict to a, b, and c being *rational numbers* $a = N_a/D$, $b = N_b/D$ and $c = N_c/D$, where D is the common denominator of these three rational numbers. The Gauss hypergeometric functions are naturally associated to the *pencil of algebraic curves*

$$y^D = (1-u)^{N_b} \cdot u^{D-N_a} \cdot \left(1 - \frac{u}{z}\right)^{-N_c+D+N_a}. \tag{C.7}$$

Recalling the main example of the paper, one associates with ${}_2F_1([1/4, 1/2], [5/4]; z)$

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{4}\right], \left[\frac{5}{4}\right]; z\right) &= \frac{\Gamma(5/4)}{\Gamma(1/2)\Gamma(3/4)} \cdot \int_0^1 \frac{dw}{w} \cdot w^{1/2} \cdot (1-w)^{-1/4} \cdot (1-zw)^{-1/4} \\ &= \frac{\Gamma(5/4)}{\Gamma(1/2)\Gamma(3/4)} \cdot z^{-1/2} \cdot \int_0^z u^{-1/2} \cdot \left(1 - \frac{u}{z}\right)^{-1/4} \cdot (1-u)^{-1/4} \cdot du \end{aligned} \tag{C.8}$$

the z-pencil of elliptic curves[38]

$$y^4 - u^2 \cdot (1-u) \cdot \left(1 - \frac{u}{z}\right) = 0, \tag{C.9}$$

where we associated (see (2.2)) to ${}_2F_1([1/2, 1/4], [5/4]; z)$ the elliptic curve

$$y^4 - u^3 \cdot (1-u)^2 = 0. \tag{C.10}$$

C.1. Miscellaneous Examples

In the more general (2.128), (2.129), (2.130), (resp. (2.132), (2.133)) framework, one can find many interesting subcases.

(i) The previous $R(z) = 1/z$ involution is solution of the functional relation (2.19) when $a = 2b$ if $c = 1 + b$, or $b = 2a$ if $c = 1 + a$.

(ii) The involution $R(z) = 1 - z$ is solution of the functional relation (2.19) when $a + b = 1$ if $c = 1 + b$, or $c = 1 + a$.

(iii) The infinite-order transformation:

$$R(z) = t \cdot \frac{z}{1 + (t-1) \cdot z}, \qquad R^{(n)}(z) = t^n \cdot \frac{z}{1 + (t^n - 1) \cdot z}, \tag{C.11}$$

is solution of the functional relation (2.19) when $a = 1 + b$ if $c = 1 + b$, or $b = 1 + ac = 1 + a$.

(iv) The scaling transformation $R(z) = t \cdot z$ is solution of the functional relation (2.19) when $a = 0$ and $c = 1 + b$ (resp., $b = 0$ and $c = 1 + a$).

(v) We also have a quite degenerate situation for $b = 1$ or $a = 1$ when $c = 2$ with the infinite-order transformation

$$R(z) = 1 - t \cdot (1 - z), \qquad R^{(n)}(z) = 1 - t^n \cdot (1 - z), \tag{C.12}$$

solution of (2.19).

(vi) The two order-three transformations

$$R(z) = \frac{z-1}{z}, \qquad R(R(z)) = \frac{1}{1-z}, \tag{C.13}$$

are solutions of the functional relation (2.19) for $a = 2/3, b = 1/3, c = 4/3$, or $a = 1/3, b = 1/3, c = 4/3$.

D. Ising Model Susceptibility: $\widetilde{\chi}^{(2)}$ and Elliptic Curves

The two-particle contribution of the susceptibility of the Ising model [25–27] is given by a double integral. This double integral on two angles $\widetilde{\chi}^{(2)}$ reduces to a simple integral[39] (because the two angles are opposite):

$$\widetilde{\chi}^{(2)} = \int_0^{\pi} d\theta \cdot y^2 \cdot \frac{1+x^2}{1-x^2} \cdot \left(\frac{x \cdot \sin(\theta)}{1-x^2}\right)^2, \tag{D.1}$$

where

$$x = A - B, \qquad A = \frac{1}{2w} - \cos(\theta), \qquad B^2 = A^2 - 1, \qquad y^2 = \frac{1}{A^2-1}. \tag{D.2}$$

Denoting $C = \cos(\theta)$ we can rewrite the integral $\chi^{(2)}$ as

$$\widetilde{\chi}^{(2)} = \int_0^1 \frac{dC}{(1-C^2)^{1/2}} \cdot x^2 \cdot y^2 \cdot \frac{1+x^2}{(1-x^2)^3}, \tag{D.3}$$

that we want to see as:

$$\int_0^1 \frac{dC}{z} = \int_0^w \frac{dq}{Z}. \tag{D.4}$$

The variable z reads:

$$\frac{1}{z} - \frac{1}{(1-C^2)^{1/2}} \cdot x^2 \cdot y^2 \cdot \frac{1+x^2}{(1-x^2)^3} = 0, \tag{D.5}$$

which after simplifications gives

$$A^2\left(C^2-1\right) \cdot z^2 + \left(A^2-1\right)^5 = 0, \tag{D.6}$$

that is

$$\left(\frac{1}{2w} - C\right)^2 \cdot \left(C^2-1\right) \cdot z^2 + \left(\left(\frac{1}{2w} - C\right)^2 - 1\right)^5 = 0. \tag{D.7}$$

In terms of the variable $q = w \cdot C$ one can rewrite (see (D.4)) the integral (D.3) as an incomplete integral:

$$256 \cdot (1-2q)^2\left(q^2-w^2\right) \cdot Z^2 w^4 + (2q-1+2w)^5(2q-1-2w)^5 = 0. \tag{D.8}$$

This w-pencil of algebraic curves is actually a *w-pencil of genus one curves*, seen as algebraic curves in Z and q.

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Endnotes

1. The renormalization group approach of important problems like first-order phase transitions, commensurate-incommensurate phase transitions, or off-critical problems is more problematic.
2. In contrast with functional renormalization group [41–43].
3. One simply verifies that these transformations reduce to the previous $T_N : t \rightarrow t^N$ in the $z = 1$ limit (no magnetic field).
4. For instance the fixed points of (1.3) are not isolated fixed points but lie on (an infinite number) of genus zero curves.
5. In well-suited Boltzmann weight variables like x and z in (1.3), and not in (bad) variables like K, the coupling constants or the temperature.
6. Such representations of the renormalization group are not *exact* representations (the exact transformation acts in an infinite number of parameters) but some authors tried to define "improved" renormalization transformations imposing the compatibility (commutation) of the renormalization transformations with some known exact symmetries of the model (Kramers-Wannier duality, gauge symmetries...).
7. For which the partition function or other physical quantities are algebraic functions.
8. See for instance (2.18) in [44].
9. Suggesting an understanding [4, 45] of the quite rich structure of infinite number of the singularities of the $\chi^{(n)}$'s in the complex plane from a Hauptmodul approach [4, 45]. Furthermore the notion of *Heegner numbers* is closely linked to the isogenies mentioned here [4]. An exact value of the j-function $j(\tau)$ corresponding one of the first Heegner number is, e.g., $j(1 + i) = 12^3$.
10. It should be recalled that the mirror symmetry found with Calabi-Yau manifolds [30–34] can be seen as higher-order generalizations of Hauptmoduls. We thus have already generalizations of this identification of the renormalization and modular structure when one is not restricted to elliptic curves anymore.
11. The fact that the renormalization group must be reversible has apparently been totally forgotten by most of the authors who just see a semigroup corresponding to forward iterations converging to the critical points (resp. manifolds).
12. Corresponding to Atkin-Lehner polynomials and Weber's functions.

13. Conversely, and more precisely, writing $1728z^2/(z+256)^3 = 1728z'/(z'+16)^3$ gives the Atkin-Lehner [46] involution $z \cdot z' = 2^{12}$, together with the quadratic relation $z^2 - zz'^2 - 48zz' - 4096z' = 0$
14. They are not only Fuchsian, the corresponding linear differential operators are globally nilpotent or$G$-operators [11].
15. Where j is typically the j-function [44, 47].
16. Such formula is actually valid for $\Omega_A = (D_z + A(z)) \cdot D_z$ for any $A(z)$. Denoting $\mathcal{S}_N$ symmetric Nth power of Ω_A one has $\mathcal{S}_N = (D_z + A(z)) \cdot \mathcal{S}_{N-1}$.
17. The Rota-Baxter relation of weight Θ reads: $R(x)R(y) + \Theta R(xy) \quad R(R(x)y + xR(y))$.
18. For $A(z)$ given we get a one-parameter family of $R(z)$ solution of (2.19). Conversely, for $R(z)$ given one can ask if there are several $A(z)$ such that (2.19) is verified. This is sketched in Appendix A.
19. Using the command "dchange" with PDEtools in Maple.
20. Note that the result for ω_1^* is nothing but transformation (2.14) on ω_k for $k = -1$. Also note that the two transformations, performing the change of variable $z \to -4z/(1-z)^2$ and taking the adjoint, *do not commute*: $(\omega_1^*)^{(R)} \neq ((\omega_1)^{(R)})^*$.
21. Denoted $JacobiSN$ in Maple: $P(z) = (JacobiSN(z^{1/4}, I))^4$.
22. As a (nonholonomic) elliptic function $P(z)$ provides elementary examples [48] of nonlinear ODEs with the Painlevé property (like the Weierstrass P-function).
23. It is the absolute value of the inverse of the image of the n-th iterate of $S^{(1)}_{-1/4}$ of -1.
24. If this previous singularity argument was valid we would have had singularities as close as possible to $z = 0$ (namely, $z_s/(-4)^n$), yielding a zero radius of convergence. Similarly combining $T^*(z)$ and the inverse of $T(z)$ we would have obtained an infinite number of singularities on the circle of radius $|z_s|$.
25. Note a (small) misprint in formula (64) page 174 of Vidunas [12].
26. Of the $c = 1 + b$ type (see below).
27. The change of variable (2.125) can be parametrized with hyperbolic tangents: $z \to z'$ with $z = \tanh(u)^2$,$z' = \tanh(2u)^2$. Note that $z \to 4 \cdot z/(1-z)^2$ is parametrized by $z = \tan(u)^2 z' = \tan(2u)^2$ but $z \to -4 \cdot z/(1-z)^2$ is not parametrized by trigonometric functions.
28. Thus avoiding the full complexity (and subtleties) of the covariance of ODEs by algebraic transformations like modular transformations (1.8).
29. See for instance (C.6) in Appendix C.
30. More generally in our models of lattice statistical mechanics (or enumerative combinatorics etc.) we are seeking for (high order) globally nilpotent [11] operators that, in fact, factor into globally nilpotent operators of smaller order, which, for Yang-Baxter integrable models with a canonical elliptic parametrization, must necessarily "be associated with elliptic curves". Appendix D provides some calculations showing that the integral for $\chi^{(2)}$, the two-particle contribution of the susceptibility of the Ising model [25–27] is clearly, and straightforwardly, associated with an elliptic curve.
31. Experimentally [21] and as could be expected from Dwork's conjecture [11], one often finds for these small order factors hypergeometric second-order operators and sometimes selected Heun functions [49] (or their symmetric products).

32. For instance equation (1.9) of [31]. Do note that the periods of certain K3 families (and hence the original Calabi-Yau family) can be described by the squares of the periods of the elliptic curves [31]. The mirror maps of some K3 surface families are always reciprocals of some McKay-Thompson series associated to the Monstruous Moonshine list of Conway and Norton, with the mirror maps of these examples being always automorphic functions for *genus zero* [32, 33].
33. Using differential algebra tools one can verify that (2.84) implies (B.2).
34. Beyond diffeomorphisms of the circle: the parameter α can be a complex number.
35. Having the movable-poles solutions: $(\alpha^\beta + z^\beta)^2/(\alpha^\beta - z^\beta)^2$.
36. Taking for$x$ the elliptic lambda function.
37. For instance for $2F_1([1/4, 1/2], [5/4]; z)$ it changes an Euler integral with $\Gamma(5/4)/\Gamma(1/4)\Gamma(1) = 1/4$ into an Euler integral with $\Gamma(5/4)/\Gamma(3/4)\Gamma(1/2) = (1/4) \cdot ((2\pi)^{1/2}/\Gamma(3/4)^2)$.
38. The algebraic curves (C.9) are genus one curves for any value of z, except $z = 1$, where the curve becomes the union of two rational curves $(u^2 - u + y^2)(u^2 - u - y^2) = 0$.
39. The prefactors in front of the integrals are not relevant for our discussion here.

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