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Research Article

Arnold's Projective Plane and r-Matrices

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We will explain Arnold's 2-dimensional (shortly, 2D) projective geometry (Arnold, 2005) by means of lattice theory. It will be shown that the projection of the set of nontrivial triangular r-matrices is the pencil of tangent lines of a quadratic curve on Arnold's projective plane.

1. Introduction

We briefly describe Arnold's projective geometry [1]. We recall the ordinary projective plane $\mathbb{P}(\mathbb{R}^3)$ over the real field. We replace $\mathbb{R}^3$ to the set of the quadratic functions $\widehat{Q} := ap^2 + 2bpq + cq^2$ on the canonical symplectic plane $(\mathbb{R}^2 : p, q)$. The quadratic functions form into a Lie subalgebra of the canonical Poisson algebra on the symplectic plane. This Lie algebra is isomorphic to $sl(2,\mathbb{R})$; that is, Arnold introduced a projective plane $\mathbb{P}(sl(2,\mathbb{R}))$. By the projection, a (nontrivial) quadratic function $\widehat{Q}$ corresponds to a point Q on the projective plane, and the Killing form on $sl(2,\mathbb{R})$ corresponds to a quadratic curve on the projective plane, because it is a symmetric bilinear form. Since the Killing form is nondegenerate, the associated curve defines a duality (so-called polar system) between the projective lines and the projective points. Arnold showed that the Poisson bracket $\{\widehat{Q}_1, \widehat{Q}_2\}$ corresponds to the pole point of the projective line through Q_1 and Q_2. As an application, it was shown that given a good triangle composed of three points (Q_1, Q_2, Q_3), the three altitudes intersect the same point (altitude theorem). Interestingly, the altitude theorem is shown by the Jacobi identity (see Figure 1).

The monomials of double brackets $\{\{\widehat{Q}_i, \widehat{Q}_j\}, \widehat{Q}_k\}$ correspond to the three bold lines, via the line-point duality. By the Jacobi identity, the three monomials are linearly dependent. This implies that the three lines intersect the same point.

We suppose that 2D projective geometry is encoded in the Lie algebra. However some information on the Lie algebra is lost in the process of constructing projective geometry;

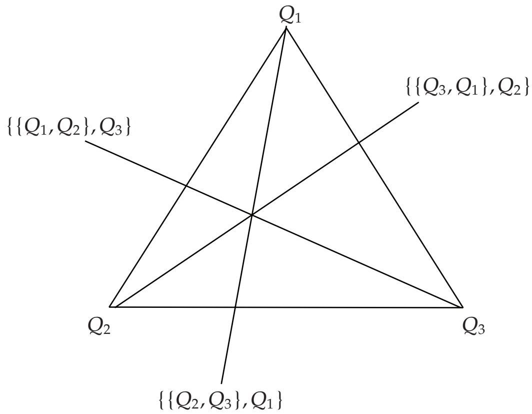

Figure 1

besides, it is not clear why it has to happen, conceptually. So we will reformulate the Arnold construction by means of lattice theory. Since the lattice is an algebra, the problem becomes more clear. We will prove, when the characteristic of the ground field is not 2, that each 3D simple Lie algebra admits a modular lattice structure. This proposition explains why 2D projective geometry can be encoded in $sl(2)$.

It is crucial to consider the *Plücker* embedding for the algebraic Arnold construction. When $\mathfrak{g}$ is 3D and simple, the Lie algebra multiplication $\mu : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$, $x \wedge y \mapsto [x, y]$ is an isomorphism, which induces an isomorphism $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g}) \cong \mathbb{P}(\mathfrak{g})$. We will show that the line-point duality on $\mathbb{P}(\mathfrak{g})$ is equivalent with the isomorphism $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g}) \cong \mathbb{P}(\mathfrak{g})$, up to the *Plücker* embedding.

As an application we study a projective geometry of triangular r-matrices, when $\mathfrak{g} = sl(2)$. We will prove that the projection of the set of nontrivial triangular r-matrices is equivalent to the pencil of tangent lines on the quadratic curve made from the Killing form. This proposition is equivalently translated as follows: the classical Yang-Baxter equation $\{r, r\} = 0$ on $sl(2)$ is equivalent to the quadratic curve on the projective plane.

2. Algebraic Arnold Construction

2.1. Subspace Lattices

A lattice is, by definition, a set equipped with two commutative associative multiplications, $\smile$ and $\frown$, satisfying the following two identities:

$$x \smile (x \frown y) = x, \quad x \frown (x \smile y) = x. \tag{2.1}$$

A lattice has a canonical order $\leq$ defined by

$$x \leq y \Longleftrightarrow x = x \frown y \quad (\Longleftrightarrow y = x \smile y). \tag{2.2}$$

A lattice is called a modular lattice when it satisfies the inequality (modular rule)

$$x \leq z \Longrightarrow x \smile (y \frown z) = (x \smile y) \frown z \tag{2.3}$$

The notion of lattice morphism is defined by the usual manner.

Example 2.1 (subspace lattices). Let V be a vector space. Consider the set of subspaces of V: $\mathrm{Latt}(V) := \{S \mid S \subset V\}$. Define two natural multiplications (cup- and cap-products) on $\mathrm{Latt}(V)$ by

$$\begin{aligned} S_1 \smile S_2 &:= S_1 + S_2, \\ S_1 \frown S_2 &:= S_1 \cap S_2, \end{aligned} \tag{2.4}$$

where $S_1, S_2 \in \mathrm{Latt}(V)$. Then $\mathrm{Latt}(V)$ becomes a modular lattice. The induced order is the natural inclusion relation $S_1 \subset S_2$. Given a linear injection $f : V_1 \to V_2$, an associated lattice morphism $\mathrm{Latt}(f)$ is naturally defined by $\mathrm{Latt}(f)(S) := f(S)$.

The subspace lattices are *complementary*; that is, the zero space $\mathbf{0}$ is the unit element with respect to $\smile$ and the total space V is $\frown$. If V is split for each S, then there exists a cosubspace $\overline{S}$ satisfying $\overline{S} \smile S = V$ and $\overline{S} \frown S = \mathbf{0}$. The subspace $\overline{S}$ (resp., S) is called a complement[1] of S (resp., $\overline{S}$). Such a lattice is called a *complemented lattice*. If V is finite dimensional, then the subspace lattice is a *complemented-modular-lattice*. A projective geometry is axiomatically defined as a complemented-modular-lattice satisfying some additional properties.

Definition 2.2. When V is $(n+1)$-dimensional, the subspace lattice $\mathrm{Latt}(V)$ is called an n-dimensional projective geometry over V.

The 1D subspaces are regarded as projective points, 2D subspaces are projective lines and so on. The zero space $\mathbf{0}$ is regarded as the empty set. For instance, given two 2D subspaces S_1 and S_2, the intersection $S_1 \frown S_2 (= S_1 \cap S_2)$ is the common point of two projective lines (if it exists).

2.2. Lie Algebra Construction of Projective Plane

Let $(\mathfrak{g}, [-,-])$ be a 3D $\mathbb{K}$-Lie algebra with a nondegenerate symmetric invariant pairing $(-,-)$, where $\mathbb{K}$ is the ground field $\mathrm{char}(\mathbb{K}) \neq 2$. We assume that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, or equivalently, the Lie bracket is an isomorphism from $\mathfrak{g} \wedge \mathfrak{g}$ to $\mathfrak{g}$. This assumption is needed in order to construct projective geometry. Such a Lie algebra is simple. In particular, when $\mathbb{K}$ is a closed field, $\mathfrak{g}$ is isomorphic to $sl(2, \mathbb{K})$.

We denote by $p(x)$ the 1D subspace generated by $x \in \mathfrak{g}$ and denote by $l(x, y)$ the 2D subspace generated by $x, y \in \mathfrak{g}$. We define new cup-and cap-products $\smile'$ and $\frown'$ on $\mathrm{Latt}(\mathfrak{g})$.

Definition 2.3 (Arnold products). (i) $p(x) \smile' p(y) := p^{\perp}([x, y]), p(x) \neq p(y)$, where $\perp$ means the orthogonal space with respect to the invariant pairing $(-,-)$.

(ii) $l(x, y) \frown' l(z, w) := p([[x, y], [z, w]]), l(x, y) \neq l(z, w)$.

(iii) $S_1 \smile' S_2 := S_1 \smile S_2$ and $S_1 \frown' S_2 := S_1 \frown S_2$, in all other cases.

Proposition 2.4. *The set* $\mathrm{Latt}(\mathfrak{g})$ *with Arnold products becomes a lattice and it is the same as the classical subspace lattice.*

Proof. By the invariancy of the pairing, we have $([x,y],x) = ([x,y],y) = 0$. This gives

$$p^{\perp}([x,y]) = p(x) + p(y) = l(x,y). \tag{2.5}$$

Hence we have $p(x)\smile' p(y) = p(x) \smile p(y)$. We prove that $l(x,y)\cap l(z,w) = p([[x,y],[z,w]])$. Since $\perp\perp = id$, we have $l^{\perp}(x,y) = p([x,y])$, $l^{\perp}(z,w) = p([z,w])$, and $p^{\perp}([[x,y],[z,w]]) = l([x,y],[z,w])$. Thus we obtain

$$\begin{aligned} p^{\perp}([[x,y],[z,w]]) &= l([x,y],[z,w]) \\ &= p([x,y]) + p([z,w]) \\ &= l^{\perp}(x,y) + l^{\perp}(z,w) \\ &= (l(x,y)\cap l(z,w))^{\perp}, \end{aligned} \tag{2.6}$$

which gives $l(x,y)\frown' l(z,w) = l(x,y) \frown l(z,w)$. □

In the following, we omit the "prime" on the Arnold products.

Remark 2.5 (Lie algebra identities versus lattice identities). We put $l([x,y]) := l(x,y)$. Then the Arnold products are coherent with the Lie bracket, namely,

$$\begin{aligned} p(x) \smile p(y) &= l([x,y]), \\ l(x) \frown l(y) &= p([x,y]). \end{aligned} \tag{2.7}$$

Tomihisa [2] discovered an interesting identity on $sl(2,\mathbb{R})$:

$$[[[x_1,y],[x_2,z]],x_3] + \text{cyclic permutation w.r.t. } (1,2,3) = 0, \tag{2.8}$$

where y, z are fixed. The Tomihisa identity induces a lattice identity

$$\begin{aligned} &((l_1 \frown l_y) \smile (l_2 \frown l_z)) \frown l_3 \\ &\quad \le (((l_2 \frown l_y) \smile (l_3 \frown l_z)) \frown l_1) \smile (((l_3 \frown l_y) \smile (l_1 \frown l_z)) \frown l_2), \end{aligned} \tag{2.9}$$

where $l_i := l(x_i)$, $l_y := l(y)$, and $l_z := l(z)$. In a study by Aicardi in [3], it was shown that the projection of Tomihisa identity is equivalent to the Pappus theorem. (The author also proved this proposition around winter 2007.) We leave it to the reader to write down the lattice identity associated with the Jacobi identity.

Given a coordinate (ξ_1, ξ_2, ξ_3) on the projective plane, the symmetric pairing is regarded as a defining equation of a nondegenerate quadratic curve (so-called polar system):

$$(-,-) = \sum a_{ij}\xi_i\xi_j. \tag{2.10}$$

We give an example of the quadratic curve made from the pairing.

Example 2.6 (see, [1, 2]). We assume that $\mathfrak{g} := sl(2,\mathbb{R})$ generated by the standard basis (X, Y, H) satisfying the following relations:

$$[H,X] = 2X, \qquad [X,Y] = H, \qquad [H,Y] = -2Y. \tag{2.11}$$

The symmetric pairing is equivalent with the Killing form. We define the scale of the form by

$$(X,Y) := \frac{1}{2}, \qquad (H,H) := 1, \tag{2.12}$$

and all others zero. We set a point $Q := \xi_1 Y + \xi_2 H + \xi_3 X$, $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$. It is on the quadratic curve, that is, $(Q,Q) = 0$ if and only if $\xi_1\xi_3 - (\xi_2)^2 = 0$, and this condition is equivalent with $(x_0)^2 = (x_1)^2 + (x_2)^2$, via the coordinate transformation

$$x_0 + x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad x_0 - x_1 = \xi_3. \tag{2.13}$$

Hence the quadratic curve is regarded as a circle on the projective plane with coordinate $[1 : x_1 : x_2]$.

Remark 2.7. The pairing induces a metric $(- + +)$ on $(\mathbb{R}^3 : x_0, x_1, x_2)$. It is well known that the inside of the circle (so-called timelike subspace) is a hyperbolic plane. The altitude theorem is strictly a theorem on the hyperbolic plane because a metric is needed to define the notion of altitude.

Given a point $p = (p_1, p_2, p_3)$, a line is defined by

$$(p,-) = \sum a_{ij}p_i\xi_j = 0. \tag{2.14}$$

This line is called the *polar line* of the point; conversely the point is called the *pole* of the line. Namely, the orthogonal space $p := l^{\perp}$ is the pole of the line l.

Corollary 2.8 (see [1]). *Given a line $l(x,y)$, $p([x,y])$ is the pole of the line.*

Figure 2 is depicting the duality defined by an ellipse.

The line l is the polar line of the point p which is inside an ellipse. The line and point are connected by the tangent lines and chords of the ellipse. This figure has been drawn in a book by Kawada in [4].

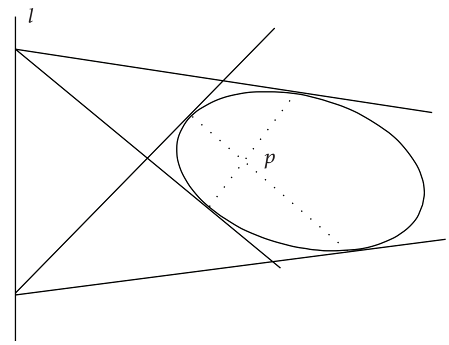

Figure 2

We did not use Jacobi identity in this section. So we will discuss the Jacobi identity on the Lie algebra in the next section.

2.3. Jacobi Identity

Proposition 2.9. *Let V be a 3D $\mathbb{K}$-vector space equipped with a skew-symmetric bracket product $[-,-]$ and a nondegenerate symmetric invariant pairing $(-,-)$. Then the bracket satisfies the Jacobi identity.*

Proof. When $x, y, z \in V$ are linearly dependent, Jacobi identity holds. So we assume that x, y, z is linear independent.

Case 1. By the invariancy of the pairing, we have

$$([[x,y],z]+[y,[x,z]],x)=([x,y],[z,x])-([x,z],[y,x])=0. \tag{2.15}$$

We assume that $[x,[y,z]]=0$. Then we obtain

$$\begin{aligned}([[x,y],z]+[y,[x,z]],y)&=([x,y],[z,y])=(y,[x,[y,z]])=0,\\ ([[x,y],z]+[y,[x,z]],z)&=-([x,z],[y,z])=(z,[x,[y,z]])=0.\end{aligned} \tag{2.16}$$

Since the pairing is nondegenerate, we have $[[x,y],z]+[y,[x,z]]=0$, which gives the Jacobi identity $[x,[y,z]]=[[x,y],z]+[y,[x,z]](=0)$.

Case 2. Assume that $[x,[y,z]]\neq 0$. We already saw that $([[x,y],z]+[y,[x,z]],x)=0$. We show that $([[x,y],z]+[y,[x,z]],[y,z])=0$. Since $\dim(V)=3$, one can put $[y,z]=ax+by+cz$, for some $a, b, c \in \mathbb{K}$. Then we have

$$\begin{aligned}
([[x,y],z] + [y,[x,z]], [y,z]) &= ([[x,y],z] + [y,[x,z]], ax + by + cz) \\
&= ([[x,y],z] + [y,[x,z]], by + cz) \\
&= ([[x,y],z], by) + ([y,[x,z]], cz) \\
&= b([x,y],[z,y]) - c([x,z],[y,z]) \\
&= b([x,y], -cz) - c([x,z], by) \\
&= -bc(x,[y,z]) - cb(x,[z,y]) = 0.
\end{aligned} \tag{2.17}$$

Therefore $[[x,y],z]+[y,[x,z]]$ is orthogonal with the independent two elements x and $[y,z]$. (If x and $[y,z]$ are linearly dependent, then $[x,[y,z]] = 0$.) The monomial $[x,[y,z]]$ is also orthogonal with x and $[y,z]$. Thus we obtain

$$[x,[y,z]] = \lambda([[x,y],z] + [y,[x,z]]), \quad \lambda(\neq 0) \in \mathbb{K}. \tag{2.18}$$

We can assume that $([x,[y,z]],y) \neq 0$ or $([x,[y,z]],z) \neq 0$, because the pairing is nondegenerate. We assume that $([x,[y,z]],y) \neq 0$ without loss of generality. Then we obtain $\lambda = 1$, because

$$([x,[y,z]],y) = -([y,z],[x,y]) = \lambda([[x,y],z],y) = \lambda([x,y],[z,y]). \tag{2.19}$$

□

The proposition above indicates that the Jacobi identity is a priori invested in the projective plane.

2.4. Duality Principle

Let μ be the Lie algebra structure on $\mathfrak{g}$; that is, $\mu : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$, $x \wedge y \mapsto [x,y]$. Since μ is an isomorphism, it induces a lattice isomorphism

$$\mathrm{Latt}(\mu) : \mathrm{Latt}(\mathfrak{g} \wedge \mathfrak{g}) \longrightarrow \mathrm{Latt}(\mathfrak{g}). \tag{2.20}$$

Let Line $:= \{p(x) \smile p(y)\}$ be the set of all lines in $\mathrm{Latt}(\mathfrak{g})$. One can define an injection pl : Line $\to \mathrm{Latt}(\mathfrak{g} \wedge \mathfrak{g})$ as

$$pl : p(x) \smile p(y) \longmapsto p(x \wedge y). \tag{2.21}$$

This mapping is called a *Plücker* embedding.

Proposition 2.10. *The diagram below is commutative:*

$$\begin{array}{ccc} \text{Line} & \xrightarrow{\perp} & \text{Point} \\ {\scriptstyle pl}\downarrow & & \downarrow{\scriptstyle i} \\ \text{Latt}(\mathfrak{g}\wedge\mathfrak{g}) & \xrightarrow{\text{Latt}(\mu)} & \text{Latt}(\mathfrak{g}) \end{array} \tag{2.22}$$

where Point is the set of points on the projective plane and $\perp$ is the duality correspondence.

Namely, the Lie algebra multiplication is equivalent to the duality principle.

3. r-Matrices

In this section, we assume that $\mathfrak{g} = sl(2,\mathbb{K})$, $\mathbb{Q} \subset \mathbb{K}$. We consider a graded commutative algebra

$$\bigwedge \mathfrak{g} := \bigwedge^3 \mathfrak{g} \oplus \bigwedge^2 \mathfrak{g} \oplus \mathfrak{g}. \tag{3.1}$$

A graded Poisson bracket $\{-,-\}$ with degree -1 is uniquely defined on $\bigwedge\mathfrak{g}$ by the axioms of graded Poisson algebra and the condition

$$\{A,B\} := [A,B], \quad \text{if } A,B \in \mathfrak{g}. \tag{3.2}$$

This bracket is called a Schouten-Nijenhuis bracket. Let r be a 2 tensor in $\bigwedge^2\mathfrak{g}$. The Maurer-Cartan (MC) equation $\{r,r\} = 0$ is called a classical Yang-Baxter equation and the solution is called a classical *triangular* r-matrix. For instance, $r := X \wedge H$ is a solution of the MC-equation.

Remark 3.1. When $\mathfrak{g} = o(3,\mathbb{R})$, there is no nontrivial triangular r-matrix.

Proposition 3.2. *The set of points $p(r)$ associated with nontrivial triangular r-matrices*

$$\mathbb{P}(r) := \{p(r) \in \text{Latt}(\mathfrak{g}\wedge\mathfrak{g}) \mid \{r,r\} = 0,\ r \neq 0\} \tag{3.3}$$

bijectively corresponds to the pencil of tangent lines of the quadratic curve made from the symmetric pairing, or, equivalently, $\mathbb{P}(r)$ bijectively corresponds to the quadratic curve.

Proof. Since $\dim\mathfrak{g} = 3$, one can write $r = r_1 \wedge r_2$ for some $r_1, r_2 \in \mathfrak{g}$. By the biderivation property of the Schouten-Nijenhuis bracket, we have

$$\{r,r\} = (\pm) r_1 \wedge [r_1,r_2] \wedge r_2. \tag{3.4}$$

Hence $\{r,r\} = 0$ if and only if $[r_1,r_2]$ is linearly dependent on r_1 and r_2.

Lemma 3.3. *The set of lines $l(x,y)$ including the pole $p([x,y])$ corresponds to $\mathbb{P}(r)$, via the Plücker embedding.*

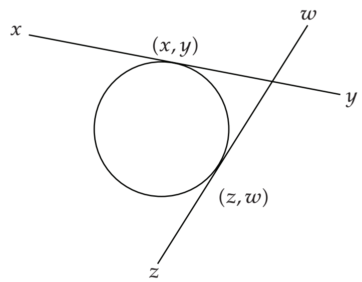

Figure 3

If $p([x, y]) \leq l(x, y)$, then the pairing $([x, y], [x, y])$ vanishes, because $([x, y], [x, y]) = ([x, y], \lambda x + \mu y) = 0$. Hence $p([x, y])$ is on the quadratic curve. It is easy to check that $l(x, y)$ is not cross over the curve. Hence the line is tangent to the curve at $p([x, y])$.

Lemma 3.4. *A line is tangent to the quadratic curve if and only if the pole is on the line, and the tangent point is the pole of the line (see Figure 3).*

Therefore $\mathbb{P}(r)$ is identified with the pencil of tangent lines of the quadratic curve. The proof of the proposition is completed. □

Acknowledgment

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Endnotes

1. The complement is not unique in general.

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