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Research Article

Dimensional Enhancement via Supersymmetry

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We explain how the representation theory associated with supersymmetry in diverse dimensions is encoded within the representation theory of supersymmetry in one time-like dimension. This is enabled by algebraic criteria, derived, exhibited, and utilized in this paper, which indicate which subset of one-dimensional supersymmetric models describes "shadows" of higher-dimensional models. This formalism delineates that minority of one-dimensional supersymmetric models which can "enhance" to accommodate extra dimensions. As a consistency test, we use our formalism to reproduce well-known conclusions about supersymmetric field theories using one-dimensional reasoning exclusively. And we introduce the notion of "phantoms" which usefully accommodate higher-dimensional gauge invariance in the context of shadow multiplets in supersymmetric quantum mechanics.

1. Introduction

Supersymmetry [1–5] imposes increasingly rigid constraints on the construction of quantum field theories [6–11] as the number of space-time dimensions increases. Thus, there are fewer supersymmetric models in six dimensions than there are in four, and yet fewer in ten dimensions [12]. In eleven dimensions there seems to be a unique possibility [13], at least on-shell. (Anomaly freedom imposes seemingly distinct algebraic constraints which make this situation even more interesting.) However, the off-shell representation theory for supersymmetry is well understood only for relatively few supersymmetries, and remains a mysterious subject in contexts of special interest, such as $N = 4$ Super Yang Mills theory, and the four ten-dimensional supergravity theories [14].

Many lower-dimensional models can be obtained from higher-dimensional models by dimensional reduction [15, 16]. Thus, a subset of lower-dimensional supersymmetric theories derives from the landscape of possible ways that extra dimensions can be removed. But most lower-dimensional theories do not seem to be obtainable from higher-dimensional theories by such a process; they seem to exist only in lower-dimensions. We refer to a lower-dimensional model obtained by dimensional reduction of a higher-dimensional model as the "shadow" of the higher-dimensional model. So we could rephrase our comment above by saying that not all lower dimensional supersymmetric theories may be interpreted as shadows.

It is a straightforward process to construct a shadow theory from a given higher-dimensional theory. But it is a more subtle proposition to construct a higher-dimensional supersymmetric model from a lower-dimensional model, or to determine whether a lower dimensional model actually does describe a shadow, especially of a higher-dimensional theory which is also supersymmetric. We have found resident within lower-dimensional supersymmetry an algebraic key which provides access to this information. A primary purpose of this paper is to explain this.

It is especially interesting to consider reduction to one time-like dimension, by switching off the dependence of all fields on all of the spatial coordinates. Such a process reduces quantum field theory to quantum mechanics. Upon making such a reduction, information regarding the spin representation content of the component fields is replaced with $\mathcal{R}$-charge assignments. But it is not obvious whether the full higher-dimensional field content, or the fact that the one-dimensional model can be obtained in this way, is accessible information given the one-dimensional theory alone. As it turns out, this information lies encoded within the extended one-dimensional supersymmetry transformation rules.

We refer to the process of restructuring a one-dimensional theory so that fields depend also on extra dimensions in a way consistent with covariant $\mathfrak{spin}(1, D-1)$ assignments and other structures, such as higher-dimensional supersymmetries, as "dimensional enhancement". This process describes the reverse of dimensional reduction. We like to envision this in terms of the relationship between a higher-dimensional "ambient" theory, and the restriction to a zero-brane embedded in the larger space. A supersymmetric quantum mechanics then describes the "worldline" physics on the zero-brane. And the question as to whether this worldline physics "enhances" to an ambient space-time field theory is the reverse of viewing the worldline physics as the restriction of a target-space theory to the zero-brane.

If the particular supersymmetric quantum mechanics obtained by restriction of a given theory to a zero-brane depended on the particular $\mathfrak{spin}(1, D-1)$-frame described by that zero-brane, the higher-dimensional theory would not respect $\mathfrak{spin}(1, D-1)$-invariance. Thus, if a one-dimensional theory enhances into a $\mathfrak{spin}(1, D-1)$-invariant higher-dimensional theory, then the higher-dimensional theory obtained in this way should be agnostic regarding the presence or absence of an actual zero-brane on which such a one-dimensional theory might live. This observation, in conjunction with the requirement of higher-dimensional supersymmetry, provides the requisite constraint needed to resolve the enhancement question. In particular, by imposing $\mathfrak{spin}(1, D-1)$-invariance on the enhanced supercharge operator, we are able to complete the ambient field-theoretic supercharge operator given merely the "time-like" restrictions of this operator. We find this interesting and surprising.

The proposition that one can systematically delineate those one-dimensional theories which can enhance to higher-dimensions, and also discern how the higher-dimensional spin structures may be switched back on, is empowered by the fact that the representation theory of one-dimensional supersymmetry is relatively tame when compared with the representation theory of higher-dimensional supersymmetry, for a variety of reasons. This enables the prospect of disconnecting the problem of spin assignments from the problem of classifying and enumerating supersymmetry representations, allowing these concerns to be addressed separately, and then merged together afterwards. With this motivation in mind, we have been developing a mathematical context for the representation theory of one-dimensional supersymmetry, also with other collaborators.

The primary purpose of this paper is to demonstrate that the landscape of supersymmetry representation theory (in any number of space-time dimensions) resides fully-encoded within the seemingly-restricted regime of one-dimensional worldline supersymmetry representation theory. We find this result remarkable, compelling, and noteworthy, regardless of how complicated it may prove to algorithmically "extract" this information. But we demonstrate below that algorithms to perform such extractions do exist. In fact, we present explicit examples of algorithms which delineate one-dimensional models which are shadows of higher-dimensional models from those which are not. We do not purport that our algorithms are optimized. And we view this paper as a plateau from which more efficient algorithms could be developed. A cursory accounting of the complexity of the general problem is addressed in Section 5.

In a sequence of papers [17–23], we have explored the connection between representations of supersymmetry and aspects of graph theory. We have shown that elements of a wide and physically relevant class of one-dimensional supermultiplets with vanishing central charge are equivalent to specific bipartite graphs which we call Adinkras; all of the salient algebraic features of the multiplets translate into restrictive and defining features of these objects. A systematic enumeration of those graphs meeting the requisite criteria would thereby supply means for a corresponding enumeration of representations of supersymmetry.

In [24, 25], we have developed the paradigm further, explaining how, in the case of N-extended supersymmetry, the topology of all connected Adinkras are specified by quotients of N-dimensional cubes, and how the quotient groups are equivalent to doubly-even linear binary block codes. Thus, the classification of connected Adinkras is related to the classification of such codes. In this way we have discovered an interesting connection between supersymmetry representation theory and coding theory [26–28]. All of this is part of an active endeavor aimed at delineating a mathematically-rigorous representation theory in one-dimension.

In this paper we use the language of Adinkras, in a way which does not presuppose a deep familiarity with this topic. We have included Appendix B as a brief and superficial primer, which should enable the reader to appreciate the entirety of this paper self-consistently. Further information can be had by consulting our earlier papers on the subject.

In this paper we focus on the special case of enhancement of one-dimensional $N = 4$ supersymmetric theories into four-dimensional $N = 1$ theories. This is done to keep our discussion concise and concrete. Another motivating reason is because the supersymmetry representation theory for 4D $N = 1$ theories is well known. Thus, part and parcel of our discussion amounts to a consistency check on the very formalism we are developing. From this point of view, this paper provides a first step in what we hope is a continuing process by which yet-unknown aspects of off-shell supersymmetry can be discerned. In the context of 4D theories, we use standard physics nomenclature, and refer to $\mathfrak{spin}(1,3)$-invariance as "Lorentz" invariance.

We should mention that the prospect that aspects of higher-dimensional supersymmetry might be encoded in one-dimensional theories was suggested years ago in unpublished work [29] by Gates et al. Accordingly, we had used that attractive proposition as a prime motivator for developing the Adinkra technology in our earlier work. This paper represents a tangible realization of that conjecture. Complementary approaches towards resolving a supersymmetry representation theory have been developed in [30–36]. Other ideas concerning the relevance of one-dimensional models to higher-dimensional physics were explored in [37, 38].

This paper is structured as follows. In Section 2 we describe an algebraic context for discussing supersymmetry tailored to the process of dimensional reduction to zero-branes and, vice-versa, to enhancing one-dimensional theories. We explain how higher-dimensional spin structures can be accommodated into vector spaces spanned by the boson and fermion fields, and how the supercharges can be written as first-order linear differential operators which are also matrices which act on these vector spaces. This is done by codifying the supercharge in terms of diophantine "linkage matrices", which describe the central algebraic entities for analyzing the enhancement question.

In Section 3 we explain how Lorentz invariance allows one to determine "space-like" linking matrices from the "time-like" linking matrices associated with one-dimensional supermultiplets, and thereby construct a postulate enhancement. We then use this to derive nongauge enhancement conditions, which provide an important sieve which identifies those one-dimensional multiplets which cannot enhance to four-dimensional nongauge matter multiplets.

In Section 4 we apply our formalism in a methodical and pedestrian manner to the context of minimal one-dimensional $N = 4$ supermultiplets, and show explicitly how the known structure of 4D $N = 1$ nongauge matter may be systematically determined using one-dimensional reasoning coupled only with a choice of 4D spin structure. We explain also how our nongauge enhancement condition provides the algebraic context which properly delineates the chiral multiplet shadow from its 1D "twisted" analog, explaining why the latter cannot enhance.

In Section 6 we generalize our discussion to include 2-form field-strengths subject to Bianchi identities. This allows access to the question of enhancement to vector multiplets. In the process we introduce the notion of one-dimensional "phantom" fields which prove useful in understanding how gauge invariance manifests on shadow theories. We use the context of the 4D $N = 1$ Abelian vector multiplet as an archetype for future generalizations.

We also include five appendices which are an important part of this paper. Appendix A is especially important, as this provides the mathematical proof that imposing Lorentz invariance of the postulate linkage matrices allows one to correlate the entire higher-dimensional supercharge with its time-like restriction. We also derive in this appendix algebraic identities related to the spin structure of enhanced component fields, which should provide for interesting study in the future generalizations of this work.

Appendix B is a brief summary of our Adinkra conventions, explaining technicalities, such as sign conventions, appearing in the bulk of the paper. Appendix C explains the dimensional reduction of the 4D $N = 1$ chiral multiplet, complementary to the nongauge enhancement program described in Section 4. Appendix D explains the dimensional reduction of the 4D $N = 1$ Maxwell field-strength multiplet, complementary to Section 5. This shows in detail how phantom sectors correlate with gauge aspects of the higher-dimensional theory. Appendix E is a discussion of four-dimensional spinors useful for understanding details of our calculations.

We use below some specialized terminology. Accordingly, we finish this introduction section by providing the following three-term glossary, for reference purposes.

Shadow

We refer to the one-dimensional multiplet which results from dimensional reduction of a higher-dimensional multiplet as the “shadow” of that higher-dimensional construction.

Adinkra

The term Adinkra refers to 1D supermultiplets represented graphically, as explained in Appendix B. We sometimes use the terms Adinkra, supermultiplet, and multiplet synonymously.

Valise

A Valise supermultiplet, or a Valise Adinkra, is one in which the component fields span exactly two distinct engineering dimensions. These multiplets form representative elements of larger “families” of supermultiplets derived from these using vertex-raising operations, as explained below. Thus, larger families of multiplets may be unpacked, as from a suitcase (or a valise), starting from one of these multiplets.

2. Ambient versus Shadow Supersymmetry

It is easy to derive a one-dimensional theory by dimensionally reducing any given higher-dimensional supersymmetric theory. Practically, this is done by switching off the dependence of all fields on the spatial coordinates, by setting $\partial_a \to 0$. One way to envision this process is as a compactification, whereby the spatial dimensions are rendered compact and then shrunk to zero size. Alternatively, we may envision this process as describing a restriction of a theory onto a zero-brane, which is a time-like one-dimensional submanifold embedded in a larger, ambient, space-time. Using this latter metaphor, we refer to the restricted theory as the “shadow” of the ambient theory, motivated by the fact that physical shadows are constrained to move upon a wall or a wire upon which the shadow is cast.

2.1. Ambient Supersymmetry

Supersymmetry transformation rules can be written in terms of off-shell degrees of freedom, by expressing all fields and parameters in terms of individual tensor or spinor components. Thus, without loss of generality, we can write the set of boson components as ϕ_i and the set of fermion components as $\psi_{\hat{\imath}}$, without being explicitly committal as the the $\mathfrak{spin}(1, D-1)$-representation implied by these index structures. Generically, a $\mathfrak{spin}(1, D-1)$-transformation acts on these components as

$$\begin{aligned} \delta_L \phi_i &= \frac{1}{2}\theta^{\mu\nu}\left(T_{\mu\nu}\right)_i{}^j \phi_j, \\ \delta_L \psi_{\hat{\imath}} &= \frac{1}{2}\theta^{\mu\nu}\left(\widetilde{T}_{\mu\nu}\right)_{\hat{\imath}}{}^{\hat{\jmath}} \psi_{\hat{\jmath}}, \end{aligned} \tag{2.1}$$

where the label L is a mnemonic which specifies these as “Lorentz” transformations. Here, $(T_{\mu\nu})_i{}^j$ represents the spin algebra as realized on the boson fields and $(\widetilde{T}_{\mu\nu})_{\hat{\imath}}{}^{\hat{\jmath}}$ represents the spin algebra as realized on the fermion fields, while θ^{0a} parameterizes a boost in the ath spatial direction and θ^{ab} parameterizes a rotation in the ab-plane. According to the spin-statistics theorem, $(\widetilde{T}_{\mu\nu})_{\hat{\imath}}{}^{\hat{\jmath}}$ should describe a spinor representation and $(T_{\mu\nu})_i{}^j$ should describe a direct product of tensor representations. The spin representations may also involve constraints. For example, boson components may configure as closed p-forms.

In four-dimensions the $N = 1$ supersymmetry algebra is generated by a Majorana spinor supercharge with components Q_A subject to the anticommutator relationship $\{Q_A, Q_B\} = 2iG^{\mu}_{AB}\partial_\mu$ where $G^{\mu}_{AB} = -(\Gamma^\mu C^{-1})_{AB}$. A parameter-dependent supersymmetry transformation is generated by $\delta_Q(\epsilon) = -i\overline{\epsilon}^A Q_A$, where ϵ_A describes an infinitesimal Majorana spinor parameter, and $\overline{\epsilon}^A = (\epsilon^\dagger\Gamma_0)^A$ is the corresponding barred spinor. It proves helpful, for our express purpose of restricting to a zero-brane, to use a Majorana basis where all spinor components, and all four gamma matrices, are real. (See Appendix E for specifics related to this basis.) Furthermore, in this basis, we have the nice result $G^0_{AB} = \delta_{AB}$. With this choice, we can rewrite our supersymmetry transformation as $\delta_Q(\epsilon) = -i\epsilon^A Q_A$, where $Q_A = (\Gamma_0)_A^B Q_B$. (This merely technical reorganization facilitates dimensional reduction of 4D multiplets, as done in Appendices C and D.) The four-dimensional $N = 1$ supersymmetry algebra, written in terms of the operators Q_A, is then given by

$$\{Q_A, Q_B\} = 2i\delta_{AB}\partial_\tau - 2iG^a_{AB}\partial_a, \tag{2.2}$$

where $x^0 := \tau$ is the time-like coordinate parameterizing the the zero-brane to which we intend to restrict, and $x^a := (x^1, x^2, x^3)$ are the three space-like coordinates transverse to the zero-brane. To dimensionally reduce a four-dimensional field theory to a one-dimensional field theory, we set $\partial_a = 0$. In this way, the second term on the right-hand side of (2.2) disappears, and we obtain the one-dimensional $N = 4$ supersymmetry algebra.

It proves helpful to add a notational distinction, by writing $\delta_Q\phi_i = -i\epsilon^A(Q_A)_i{}^{\hat{\imath}}\psi_{\hat{\imath}}$ and $\delta_Q\psi_{\hat{\imath}} = -i\epsilon^A(\widetilde{Q}_A)_{\hat{\imath}}{}^i\phi_i$, appending a tilde to $\widetilde{Q}_A$ when this describes a fermion transformation rule. The supercharges may be represented as first-order linear differential operators, as

$$\begin{aligned}(Q_A)_i{}^{\hat{\imath}} &= (u_A)_i{}^{\hat{\imath}} + \left(\Delta^\mu_A\right)_i{}^{\hat{\imath}}\partial_\mu,\\ \left(\widetilde{Q}_A\right)_{\hat{\imath}}{}^i &= i(\widetilde{u}_A)_{\hat{\imath}}{}^i + i\left(\widetilde{\Delta}^\mu_A\right)_{\hat{\imath}}{}^i\partial_\mu,\end{aligned} \tag{2.3}$$

where u_A, $\widetilde{u}_A$, Δ^μ_A, and $\widetilde{\Delta}^\mu_A$ are real valued "linkage matrices" which play a central role in our discussion below.

The matrices $(u_A)_i{}^{\hat{\imath}}$ describe "links" corresponding to supersymmetry maps from the bosons ϕ_i to fermions $\psi_{\hat{\imath}}$ having engineering dimension one-half unit greater than the bosons. Therefore, these codify "upward" maps connecting lower-weight fermions to higher-weight bosons. (The term "weight" refers to the engineering dimension of the field. We sometimes use the term weight in lieu of dimension, to avoid confusion with space-time dimension. The weight of a field correlates with the vertex "height" on an Adinkra diagram.) Similarly, the matrices $(\widetilde{u}_A)_{\hat{\imath}}^i$ codify "upward" maps connecting lower-weight fermions to higher-weight bosons. The matrices $(\Delta^\mu_A)_i{}^{\hat{\imath}}$ and $(\widetilde{\Delta}^\mu_A)_{\hat{\imath}}{}^i$ codify "downward" maps accompanied by their respective derivatives ∂_μ.

The component fields may be construed so that the linkage matrices conform to a special structure, known as the Adinkraic structure. This says that there is at most one nonvanishing entry in each column and at most one nonvanishing entry in each row. Moreover, the nonvanishing entries take the values ±1. All known higher-dimensional off-shell representations in the standard literature satisfy this condition. (The only counter examples that we know of were contrived by us in [20], as special deformations of one-dimensional Adinkraic representations. And we suspect that these do not enhance. Further scrutiny will be needed to ascertain any relevance of nonAdinkraic multiplets to physics. We find it sensible for now to focus on Adinkraic representations, especially since all known field theoretic multiplets are in this class.)

The supersymmetry algebra (2.2) implies

$$\begin{aligned} \left(u_{(A}\widetilde{u}_{B)}\right)_i{}^j &= 0, \qquad \left(\widetilde{u}_{(A}u_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} = 0, \\ \left(\Delta^{(\mu}_{(A}\widetilde{\Delta}^{\nu)}_{B)}\right)_i{}^j &= 0, \qquad \left(\widetilde{\Delta}^{(\mu}_{(A}\Delta^{\nu)}_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} = 0, \end{aligned} \tag{2.4}$$

which describes a higher-dimensional analog of the Adinkra loop parity rule described in [17] and below, and also implies

$$\begin{aligned} \left(u_{(A}\widetilde{\Delta}^{\mu}_{B)} + \Delta^{\mu}_{(A}\widetilde{u}_{B)}\right)_i{}^j &= \Lambda^{\mu}_{AB}\delta_i{}^j, \\ \left(\widetilde{u}_{(A}\Delta^{\mu}_{B)} + \widetilde{\Delta}^{\mu}_{(A}u_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} &= \Lambda^{\mu}_{AB}\delta_{\hat{\imath}}{}^{\hat{\jmath}}, \end{aligned} \tag{2.5}$$

where $\Lambda^{\mu}_{AB} = (\Gamma_0 G^{\mu}\Gamma_0)_{AB} = (\Gamma_0\Gamma^{\mu}\Gamma_0 C^{-1})_{AB}$, whereby $\Lambda^0_{AB} = G^0_{AB}$ and $\Lambda^a_{AB} = -G^a_{AB}$. Equations (2.5) play a central role in this paper.

The classification of representations of supersymmetry in diverse dimensions is equivalent to the question of classifying and enumerating the possible sets of real linkage matrices which can satisfy the algebraic requirements in (2.4) and (2.5), and identifying the corresponding spin representation matrices $(T_{\mu\nu})_i{}^j$ and $(\widetilde{T}_{\mu\nu})_{\hat{\imath}}{}^{\hat{\jmath}}$.

2.2. Shadow Supersymmetry

The one-dimensional $N = 4$ superalgebra is specified by $\{Q_A, Q_B\} = 2i\delta_{AB}\partial_0$, which corresponds to (2.2) in the limit $\partial_a \to 0$. In this case, the supercharges are represented as

$$\begin{aligned} (Q_A)_i{}^{\hat{\imath}} &= (u_A)_i{}^{\hat{\imath}} + (d_A)_i{}^{\hat{\imath}}\partial_\tau, \\ (\widetilde{Q}_A)_{\hat{\imath}}{}^i &= i(\widetilde{u}_A)_{\hat{\imath}}{}^i + i(\widetilde{d}_A)_{\hat{\imath}}{}^i\partial_\tau. \end{aligned} \tag{2.6}$$

This is identical to (2.3) except the index μ is restricted to the sole value $\mu = 0$, and the down matrices have been renamed by writing Δ^0_A as d_A and $\widetilde{\Delta}^0_A$ as $\widetilde{d}_A$. As mentioned above, the fields may be configured so that each linkage matrix has not more than one nonvanishing entry in each row and likewise in each column, and the nonvanishing entries are ±1. This specialized structuring enables the faithful translation of 1D supercharges in terms of helpful and interesting graphs known as Adinkras, as mentioned in Section 1. The reader should consult Appendix B for a simple-but-practical overview of this concept.

The algebra obeyed by 1D linkage matrices may be obtained from (2.4) and (2.5) by allowing only the value 0 for the space-time indices μ and ν. Thus, the linkage matrices are constrained by

$$\begin{aligned} \left(u_{(A}\widetilde{u}_{B)}\right)_i{}^j &= 0, \qquad \left(\widetilde{u}_{(A}u_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} = 0, \\ \left(d_{(A}\widetilde{d}_{B)}\right)_i{}^j &= 0, \qquad \left(\widetilde{d}_{(A}d_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} = 0. \end{aligned} \tag{2.7}$$

These relationships imply a "loop parity" rule, described in our earlier papers, which says that any closed bicolor loop on an Adinkra diagram must involve an odd number of edges with odd parity. The linkage matrices are further constrained by

$$\begin{aligned} \left(u_{(A}\widetilde{d}_{B)} + d_{(A}\widetilde{u}_{B)}\right)_i{}^j &= \delta_{AB}\delta_i{}^j, \\ \left(\widetilde{u}_{(A}d_{B)} + \widetilde{d}_{(A}u_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} &= \delta_{AB}\delta_{\hat{\imath}}{}^{\hat{\jmath}}. \end{aligned} \tag{2.8}$$

In this context, the algebra defined by (2.8) was called a "Garden algebra" by Gates and Rana in [39, 40], and the the matrices u_A and d_A were called Garden matrices. The larger algebra given in (2.4) and (2.5) generalizes this concept to diverse space-time dimensions, and accordingly subsumes these smaller algebras.

A one-dimensional supermultiplet is specified by the set of linkage matrices u_A, $\widetilde{u}_A$, d_A, and $\widetilde{d}_A$ or equivalently by the Adinkra diagram representing these matrices. Given a set of linkage matrices one can construct the equivalent Adinkra. Alternatively, given an Adinkra, one can use this to "read off" the equivalent set of linkage matrices. Given either of these, one can ascertain supersymmetry transformation rules and invariant action functionals from which one can study one-dimensional physics. The linkage matrices associated with any Adinkra satisfy the algebra (2.7) and (2.8) by definition.

3. Enhancement Criteria for Shadow Supermultiplets

The requirement that the linkage matrices appearing in the supercharges (2.3) are $\mathfrak{spin}(1, D-1)$-invariant has remarkable implications. One of these is the fact that the "space-like" linkage matrices Δ_A^a are completely determined by the "time-like" linkage matrices Δ_A^0. The proof of this assertion is given as Appendix A, with the result

$$\begin{aligned} \left(\Delta_A^a\right)_i{}^{\hat{\imath}} &= -\left(\Gamma^0\Gamma^a\right)_A{}^B\left(\Delta_B^0\right)_i{}^{\hat{\imath}}, \\ \left(\widetilde{\Delta}_A^a\right)_{\hat{\imath}}{}^i &= -\left(\Gamma^0\Gamma^a\right)_A{}^B\left(\widetilde{\Delta}_B^0\right)_{\hat{\imath}}{}^i. \end{aligned} \tag{3.1}$$

It is interesting that the matrix $\left(\Gamma^0\Gamma^a\right)_A{}^B$ is precisely twice a boost operator in the ath spatial direction, in the spinor representation. It is also interesting that the result (3.1) holds irrespective of the $\mathfrak{spin}(1, D-1)$-representations described by the component fields. That is, the assignment of the matrices $(T_{\mu\nu})_i{}^j$ and $(\widetilde{T}_{\mu\nu})_{\hat{\imath}}{}^{\hat{\jmath}}$ defined in (2.1) does not influence (3.1). These nontrivial consequences are derived explicitly in Appendix A. (The Lorentz invariance of the linkage matrices does imply interesting and interlocking constraints on the allowable choices of $(T_{\mu\nu})_i{}^j$ and $(\widetilde{T}_{\mu\nu})_{\hat{\imath}}{}^{\hat{\jmath}}$. These are exhibited in Appendix A. Such correlations are certainly expected, and we suspect that (A.4) and (A.5) have deep and useful implications, which we hope to explore in future work.)

It is worth mentioning that the form of (3.1) agrees precisely with the linkage matrices derived from Salam-Strathdee superfields. Also, the appearance of Γ^a on the right-hand side is tied closely to the appearance of the Γ^a in the defining supersymmetry algebra.

The result (3.1) is the crux ingredient which allows one to determine whether a given one-dimensional supermultiplet describes the shadow of a higher-dimensional supermultiplet. This follows because any one-dimensional multiplet organizes as (2.3) where the index μ assumes only the value 0. To probe whether that multiplet describes a shadow, one creates "provisional" off-brane linkages using the powerful expression (3.1). Since there is no algebraic guarantee that the transformation rules so-extended will properly close the higher-dimensional superalgebra, nor that the boson and fermion vector spaces will properly assemble into representations of the higher-dimensional spin group, the higher-dimensional superalgebra itself, applied to this construction, provides the requisite analytic probe of that possibility: if the one-dimensional multiplet is a shadow then the provisional construction will close the higher-dimensional superalgebra; if it is not possible, then it will not.

The supersymmety algebra in D-dimensions closes only if $(\Omega^\mu_{AB})_i{}^j\partial_\mu\phi_j = 0$ and $(\widetilde{\Omega}^\mu_{AB})_{\hat{\imath}}{}^{\hat{\jmath}}\partial_\mu\psi_{\hat{\jmath}} = 0$, where we define the following useful matrices:

$$
\begin{aligned}
\left(\Omega^\mu_{AB}\right)_i{}^j &= \left(u_{(A}\,\widetilde{\Delta}^\mu_{B)} + \Delta^\mu_{(A}\widetilde{u}_{B)}\right)_i{}^j - \Lambda^\mu_{AB}\delta_i{}^j,\\
\left(\widetilde{\Omega}^\mu_{AB}\right)_{\hat{\imath}}{}^{\hat{\jmath}} &= \left(\widetilde{u}_{(A}\,\Delta^\mu_{B)} + \widetilde{\Delta}^\mu_{(A}u_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} - \Lambda^\mu_{AB}\delta_{\hat{\imath}}{}^{\hat{\jmath}}.
\end{aligned}
\tag{3.2}
$$

This requirement is a minor restructuring of (2.5). In this way, we have written the supersymmetry algebra as a linear algebra problem, cast as matrix equations.

Many important supemultiplets exhibit gauge invariances, manifest as physical redundancies inherent in the vector spaces spanned by the component fields. In these cases, the matrices $(\Omega^\mu_{AB})_i{}^j$ and $(\widetilde{\Omega}^\mu_{AB})_{\hat{\imath}}{}^{\tilde{\jmath}}$, are not unique. Instead these describe classes of matrices interrelated by operations faithful to the gauge structure. We describe this interesting situation, in Section 6. It is useful, however, to begin our discussion with what we call nongauge matter multiplets, which do not exhibit redundancies of this sort. For this smaller but nevertheless interesting and relevant class of supermultiplets, the higher-dimensional supersymmetry algebra is satisfied only if

$$
\begin{aligned}
\left(\Omega^\mu_{AB}\right)_i{}^j &= 0,\\
\left(\widetilde{\Omega}^\mu_{AB}\right)_{\hat{\imath}}{}^{\hat{\jmath}} &= 0.
\end{aligned}
\tag{3.3}
$$

We refer to these equations as our nongauge enhancement criteria. These enable a practical algorithm for testing whether a given 1D supermultiplet represents the shadow of a 4D nongauge matter multiplet.

We use the linkage matrices for a given 1D supermultiplet in conjuction with the 4D gamma matrices to compute all of the $d \times d$ matrices Ω^{μ}_{AB} and $\widetilde{\Omega}^{\mu}_{AB}$, defined in (3.2). If (3.3) is satisfied, that is, if all of these matrices are identically null, then the 1D multiplet passes an important, nontrivial, and necessary requirement for enhancement to a 4D nongauge matter multiplet. If these matrices do not vanish, then the 1D multiplet cannot enhance to a 4D nongauge matter multiplet. In the latter case, further analysis must be done to probe whether this multiplet can enhance to a gauge multiplets. Equation (3.3) represents a useful "sieve" in the separation of 1D multiplets into groups as shadows versus nonshadows.

A second important sieve derives from the spin-statistics theorem. As it turns out, a minority of 1D multiplets actually pass the test (3.3). But those that do come in pairs related in-part by a Klein flip, which is an involution under which the statistics of the fields are reversed—boson fields are replaced with fermion fields and vice versa. Thus, we can organize those multiplets which pass the test (3.3) into such pairs. We then ascertain which elements of each pair satisfy the requirement that fermions assemble as spinors and the bosons as tensors. Those multiplets that do not pass this test describe another class of multiplets which do not describe ordinary shadows. Typically, one multiplet out of each pairing satisfies the spin-statistics test while the other multiplet fails this test. (The important role of the Klein flip in the representation theory of superalgebras was addressed by one of the authors (G.L.) in previous work [41].)

In the explicit examples analyzed below in this paper, it is obvious when certain multiplets which pass the first enhancement test (3.3) fail the spin-statistics test. This occurs when the multiplicity of fermions with a common engineering dimension is not a multiple of four, thereby obviating assemblage into 4D spinors. In fact our analysis below is remarkably clean. In more general cases, we suspect that more careful attention to the implications of the Lorentz invariance of the provisional supercharge, codified by equations such as (A.4), will provide the requisite sophistication needed to address enhancement at higher N and higher D. We think this will be a most interesting undertaking.

4. Nongauge Matter Multiplets

In this section we impose our enhancement equation (3.3) on the linkage matrices associated with all of the minimal $N = 4$ Adinkras, of which there are 60 in total, to ascertain which of these represent shadows of 4D $N = 1$ nongauge matter multiplets. Since the represention theory for minimal irreducible multiplets in 4D $N = 1$ supersymmetry is well known, this setting provides a natural laboratory for testing our technology. The principal result of this section is that our enhancement equation properly corroborates what is known about nongauge matter in 4D $N = 1$ supersymmetry, thereby passing an important consistency test. Another principal result of this section identifies our enhancement equation as the natural algebraic sieve which distinguishes the chiral multiplet shadow from its "twisted" analog.

By the term "nongauge matter multiplets" we refer to 4D supermultiplets which involve component fields neither subject to gauge transformations nor subject to differential constraints, such as Bianchi identities. This excludes the vector and the tensor multiplets, as well as the the corresponding field-strength multiplets. We postpone a discussion of these interesting cases until the next section. In fact, the only nongauge matter multiplet in 4D $N = 1$ supersymmetry is the chiral multiplet. (An antichiral multiplet, which can be formed as the Hermitian conjugate of a chiral multiplet, is not distinct from the latter as representation of the 4D $N = 1$ supersymmetry algebra separate from inherent complex structures; the assignment of possible $U(1)$ charge assignments represents "extra" data not considered overtly in this paper. Ignoring the complex structure, the chiral and the antichiral multiplets have indistinguishable shadows.) As we will see, among the 60 different minimal $N = 4$ Adinkras there are exactly four which satisfy our primary enhancement condition (3.3). For two of these, the fermions configure as a spinor and the bosons configure as Lorentz scalars. For the other two the fermions configure as Lorentz scalars while the bosons configure as a spinor. The latter case fails the spin-statistics test, which says that fermions must assemble as spinors, and bosons must assemble as tensors. Thus, our method identifies the two Adinkras which can provide shadows of 4D minimal nongauge matter multiplets. (Conceivably, the fact that there are two such enhanceable $N = 4$ minimal Adinkras may relate to the fact that there are two complementary choices of complex structure, related to the chiral and antichiral multiplets, as mentioned in the previous footnote.)

It is noteworthy that there are two separate minimal $N = 4$ Adinkra families, related by a so-called twist, implemented by toggling the parity of one of the four edge colors. Thus, the shadow of the chiral multiplet has a twisted analog which cannot enhance to 4D. That multiplet, which has been called the twisted chiral multiplet, describes 1D physics which cannot be obtained by restriction from four-dimensions. We have long wondered what algebraic feature distinguishes these two. (We learned about this interesting curiosity from Jim Gates, in the context of a former collaboration.) As it turns out, the linkage matrices for the chiral multiplet shadow satisfy the enhancement equations (3.3) whereas the linkage matrices for the twisted chiral multiplet do not. This answers this long-puzzling question. Details are presented below in this section.

In order to ascertain whether a given Adinkra enhances to 4D we need to subject the corresponding linkage matrices to the space-like subset of the equations in (3.3). (The time-like equations are satisfied automatically, since an Adinkra is a representation of 1D supersymmetry by construction.) In the case of testing enhancement to 4D $N = 1$ supersymmetry, each of the two conditions in (3.3) describes 30 matrix equations for each of the 60 Adinkras to be tested, since for each of the three choices for a, the corresponding symmetric matrices $G^a_{AB} = G^a_{(AB)}$ have ten independent components. Thus, according to the crudest counting argument, in order to test both the bosonic and fermionic conditions in (3.3) for all the minimal $N = 4$ Adinkras, we need to check $60 \times 30 \times 2 = 3600$ matrix equations, each involving products of 4×4 matrices. This is a simple matter which we have managed expediently using rudimentary Mathematica programming. (The complexity of the more general problem is addresed in the following section.)

The smallest Adinkras which can possibly enhance to describe 4D supersymmetry are $N = 4$ Adinkras describing $4 + 4$ off-shell degrees of freedom. We therefore start by considering $N = 4$ bosonic 4-4 Valise Adinkras. There are exactly two of these not interrelated by cosmetic field redefinitions. These are exhibited in Figure 1. In this paper we correlate the four edge colors with choices of the index A so that purple, blue, green, and red correspond respectively to the operators $Q_{1,2,3,4}$. For purposes of setting a convention for ordering the rows and columns of our linkage matrices, we sequence the boson fields ϕ_i and the fermion fields $\psi_{\hat{\imath}}$ using the obvious faithful correspondence with the index choices. Furthermore, in the Adinkras exhibited in this section, the white vertices labeled 1, 2, 3, 4 represent the boson fields ϕ_i with corresponding index choices, while the black vertices labeled 1, 2, 3, 4 represent

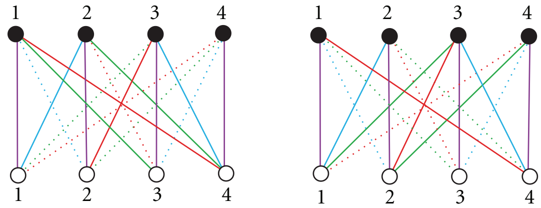


Figure 1: The two $N = 4$ Valise Adinkras. The Adinkra on the right is obtained from the Adinkra on the left by implementing a "twist", toggling the parity of the green edges.

the fermion fields $\psi_{\hat{\imath}}$ with corresponding index choices. This allows us to readily translate each Adinkra into precise linkage matrices, using the technology explained in Appendix B.

The linkage matrices $(u_A)_i{}^{\hat{\imath}}$ corresponding to the first Adinkra in Figure 1 are exhibited in Table 1. (The diligent reader should verify the correspondence between Figure 1 and Table 1 using the simple technology explained in Appendix B.) These codify the "upward" links connecting the bosons ϕ_i to the fermions $\psi_{\hat{\imath}}$ having greater engineering dimension. Since there are no edges linking downward from any of the boson vertices, it follows that $(d_A)_i{}^{\hat{\imath}} = 0$ in this case. Similarly, we have $(\tilde{u}_A)_{\hat{\imath}}{}^{i} = 0$, reflecting the fact that none of the fermions have upward directed edges. Finally, we have $(\tilde{d}^A)_{\hat{\imath}}{}^{j} = \delta^{AB}\delta^{jk}(u_B)_k{}^{\hat{k}}\delta_{\hat{k}\hat{\imath}}$ and $(d^A)_i{}^{\hat{\jmath}} = \delta^{AB}\delta^{\hat{\jmath}\hat{k}}(\tilde{u}_B)_{\hat{k}}{}^{k}\delta_{ki}$, schematically $\tilde{d}_A = u_A^T$ and $d_A = \tilde{u}_A^T$, reflecting the fact that every edge describes a pairing of an upward directed term and a corresponding downward directed term. The relationships $\tilde{d}_A = u_A^T$ and $d_A = \tilde{u}_A^T$ are characteristic of "standard Adinkras". Nonstandard Adinkras, which can include "one-way" upward Adinkra edges, appear in gauge multiplet shadows, as explained below, and in also in other contexts of interest. (Some considerations involving "one-way" Adinkra edges were described in both [21, 23].)

4.1. The $N = 4$ Bosonic 4-4 Adinkras

Using the features $\tilde{u}_A = d^B = 0$ and $\tilde{d}^A = u_A^T$, and using the matrices G^a_{AB} in (E.10), we can begin to analyze the enhancement equations associated with the left Adinkra in Figure 1. Consider the first equation in (3.3) for the index choices $(a \mid A, B) = (a \mid 1, 1)$. Since $\Lambda^1_{11} = -G^1_{11} = 0$, that 4×4 matrix equation reads $u_1\tilde{\Delta}^1_1 = 0$. We then use (3.1), along with the gamma matrices in (E.3) to determine $\tilde{\Delta}^1_1 = -\tilde{\Delta}^0_3$, which is equivalent to $\tilde{\Delta}^1_1 = -\tilde{d}_3$ using the nomenclature $\tilde{\Delta}^0_A \equiv \tilde{d}_A$. Thus, the first equation in (3.3) reduces for the left Adinkra in Figure 1 and the index selections $(a \mid A, B) = (1 \mid 1, 1)$ to the simple matrix equation $u_1 u_3^T = 0$, where we have also used $\tilde{d}_3 = u_3^T$. Using Table 1, it is easy to check that this simple requirement is not satisfied. This tells us that the left Adinkra in Figure 1 cannot enhance to a 4D nongauge matter multiplet. Since the linkage matrices associated with the right Adinkra in Figure 1 are obtained from Table 1 by toggling the overall sign on u_3 only, and since the enhancement equation $u_1 u_3^T = 0$ is unchanged by such an operation, it follows that neither Adinkra in Figure 1 can enhance to a 4D nongauge matter multiplet.

The methodology explained in the last paragraph can be applied systematically for each possible index choice $(a \mid A, B)$ for any selected Adinkra. In each case the time-like linkage matrices Δ^a_{AB} are determined using (3.1), so that the enhancement equation can be translated to a matrix statement involving the linkage matrices specific to the 1D multiplet

Table 1: The boson "up" linkage matrices for the left 4-4 Valise Adinkra shown in Figure 1. The up matrices for the right Adinkra in that figure are obtained from these by changing the sign of u_3.

$u_1 = \left(\begin{array}{cc\|cc} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$	$u_2 = \left(\begin{array}{cc\|cc} & 1 & & \\ -1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right)$
$u_3 = \left(\begin{array}{cc\|cc} & & -1 & \\ & & & -1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right)$	$u_4 = \left(\begin{array}{cc\|cc} & & & -1 \\ & & 1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right)$

directly corresponding to the Adinkra. In the following discussion we do not repeat most of these steps. But the reader should be aware that (3.1) is used in each example which we discuss, and the use of this equation is what allows us to cast the enhancement equation in terms of the matrices u_A, $d_A = \Delta^0_A$, and their transposes.

4.2. The $N = 4$ Bosonic 3-4-1 Adinkras

Consider next those Adinkras obtained by raising one vertex starting with each Adinkra in Figure 1. There are four possibilities starting from each of the two Valises, namely one possibility associated with raising any one of the four bosons. For example, if we raise the boson vertex labeled "4" starting from each Valise, what results are the two Adinkras in Figure 2. In these cases, we end up with three bosons at the lowest level, four fermions at the next level, and a single boson at the next level. We refer to Adinkras with this distribution of vertex multiplicities as bosonic 3-4-1 Adinkras, where the sequence of numerals faithfully enumerates the sequence of vertex multiplicities at successively higher levels. (These alternate between boson and fermion multiplicities, naturally.) It is easy to see that there are exactly eight bosonic 3-4-1 $N = 4$ Adinkras, and that these split evenly into two groups interrelated by a twist operation.

We should point out that two Adinkras are equivalent if they are mapped into each other by cosmetic renaming of vertices, equivalent to linear automorphisms on the vector spaces spanned by the bosons ϕ_i or fermions $\psi_{\hat{\imath}}$, in cases where these maps preserve all vertex height assignments. Such transformations have been called "inner automorphisms". The simplest examples correspond to rescaling any component field by a factor of -1. This manifests on an Adinkra by simultaneously toggling the parity of every edge connected to the vertex representing that field, that is, by changing dashed edges into solid edges and vice-versa. (This is referred to as "flipping the vertex", and was described already in [17].) Our observation that there are two distinct families of minimal $N = 4$ Adinkras interrelated by a twist operation refers to the readily-verifiable fact that one cannot "undo" a twist by any inner automorphism. (The curious reader might find it amusing to draw Adinkra diagrams, and investigate this statement for his or her self.) It is also true that there are only two twist classes of minimal $N = 4$ Adinkras, despite the fact that there are four different colors which can be used to implement a twist. This is so because a given twist applied using any chosen

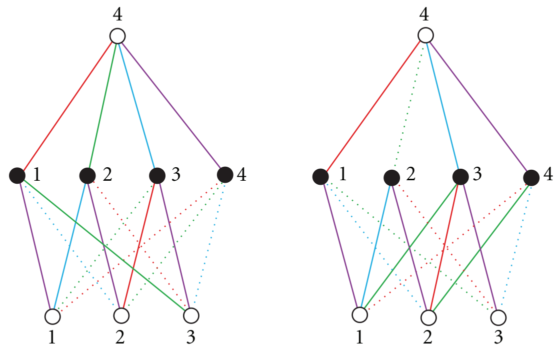


Figure 2: The two 3-4-1 Adinkras obtained from the Valise Adinkras in Figure 1 by raising one vertex. Here we have raised the boson vertex labeled 4.

edge color can be equivalently implemented as a twist applied using any other edge color augmented by a suitable inner automorphism.

When we raise an Adinkra vertex, the up and down linkage matrices accordingly modify. For example, consider the the 3-4-1 Adinkra on the left in Figure 2, obtained from the Adinkra on the left in Figure 1 by raising the ϕ_4 vertex. The corresponding boson up and down matrices, which are straightforward to read off of the Adinkra, are shown in Table 2. Note that in this case the boson down matrices d_A no longer vanish as they did in the case of the Valise. This is because the ϕ_4 vertex obtains downward links after being raised. The fermion up matrices, which are determined for this standard Adinkra using $\tilde{u}_A = d_A^T$, are also nonvanishing after this vertex raise, since each of the fermions obtains an upward link to the boson ϕ_4.

Given a standard Adinkra, it is possible to raise the nth boson vertex if and only if the nth row of each boson down matrix is null, that is, provided $(d_A)_n{}^{\hat{\imath}} = 0$ for all values of $\hat{\imath}$. This criterion ensures that the nth boson vertex does not have any downward links which would preclude the vertex from being raised. (Since for standard Adinkras we have $\tilde{u}_A = d_A^T$, this criterion also implies that there are no lower fermions which link upward to the boson in question.) Absent such a tethering, the boson is free to be raised. This operation is implemented algebraically by interchanging the nth row of each boson up matrix u_A with the nth row of the respective boson down matrix d_A. Thus, we implement the matrix reorganizations $(u_A)_n{}^{\hat{\imath}} \leftrightarrow (d_A)_n{}^{\hat{\imath}}$. At the same time, we must interchange the nth column of each fermion up matrix $\tilde{u}_A$ with the nth column of the respective fermionic down matrix $\tilde{d}_A$, via $(\tilde{u}_A)_{\hat{\imath}}{}^n \leftrightarrow (\tilde{d}_A)_{\hat{\imath}}{}^n$. The latter operation preserves the standard relationships $\tilde{u}_A = d_A^T$ and $\tilde{d}_A = u_A^T$. It is easy to check that the linkage matrices in Table 2 are obtained from the linkage matrices in Table 1 by appropriately interchanging the fourth rows of the boson up and down matrices according to the above discussion.

We now use the enhancement equation to analyze the eight standard $N = 4$ bosonic 3-4-1 Adinkras to ascertain if any of these can enhance to a 4D $N = 1$ nongauge matter multiplet. To begin, we start with the left Adinkra in Figure 2, by using the boson linkage matrices in Table 2 and the fermion linkage matrices determined by $\tilde{u}_A = d_A^T$ and $\tilde{d}_A = u_A^T$. Using the G^a_{AB} given in (E.10), the first condition in (3.3) reduces for the choice $(a \mid AB) = (1 \mid 11)$ to the matrix equation $u_1 u_3^T + d_3 d_1^T = 0$. Using the explicit matrices in Table 2, it is easy to

Table 2: Linkage matrices for the left 3-4-1 Adinkra shown in Figure 2. The linkage matrices for the right Adinkra in that figure are obtained from these by changing the sign of u_3 and d_3.

$$u_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & 0 \end{array}\right) \qquad d_1 = \left(\begin{array}{cc|cc} 0 & & & \\ & 0 & & \\ \hline & & 0 & \\ & & & 1 \end{array}\right)$$

$$u_2 = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & -1 \\ & & 0 & \end{array}\right) \qquad d_2 = \left(\begin{array}{cc|cc} & 0 & & \\ 0 & & & \\ \hline & & & 0 \\ & & 1 & \end{array}\right)$$

$$u_3 = \left(\begin{array}{cc|cc} & & -1 & \\ & & & -1 \\ \hline 1 & & & \\ & 0 & & \end{array}\right) \qquad d_3 = \left(\begin{array}{cc|cc} & & 0 & \\ & & & 0 \\ \hline 0 & & & \\ & 1 & & \end{array}\right)$$

$$u_4 = \left(\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & -1 & & \\ 0 & & & \end{array}\right) \qquad d_4 = \left(\begin{array}{cc|cc} & & & 0 \\ & & 0 & \\ \hline & 0 & & \\ 1 & & & \end{array}\right)$$

see that this requirement is not satisfied. This tells us that the left Adinkra in Figure 2 cannot enhance to a 4D nongauge matter multiplet.

Since the right Adinkra in Figure 2 is obtained from the left Adinkra in that figure by twisting the green edges, corresponding to replacing $Q_3 \to -Q_3$, the linkage matrices for that second 3-4-1 Adinkra are obtained from those in Table 2 by scaling the matrices u_3, d_3, $\tilde{u}_3$, and $\tilde{d}_3$ each by a multiplicative minus sign. The $(a \mid AB) = (1 \mid 1,1)$ enhancement equation, $u_1 u_3^T + d_3 d_1^T = 0$, is unchanged by this operation. So we conclude that neither Adinkra in Figure 2 can enhance to a nongauge 4D matter multiplet. It is straightforward to repeat this analysis for all cases associated with raising any possible single boson vertex starting with either of the Valise Adinkras in Figure 1. It follows, after careful analysis of each case, that the nongauge enhancement equation (3.3) is not satisfied for any of the eight bosonic 3-4-1 Adinkras.

4.3. *The* $N = 4$ *Bosonic 2-4-2 Adinkras*

Things become more interesting when we raise one of the lower bosons in 3-4-1 Adinkras to obtain 2-4-2 Adinkras. In the end there are twelve minimal $N = 4$ bosonic 2-4-2 Adinkras—six obtained by two vertex raises starting from the left Adinkra in Figure 1 and six obtainable by two vertex raises starting from the right Adinkra in Figure 1. The six possibilities in each class correspond to the six different ways to select pairs from four choices. For example, if we raise ϕ_3 and ϕ_4 in either case then what results are the two 2-4-2 Adinkras shown in Figure 3. For the left Adinkra in Figure 3, the boson linkage matrices are shown in Table 3. (It is straightforward to read these matrices off of the Adinkra. It is also straightforward to

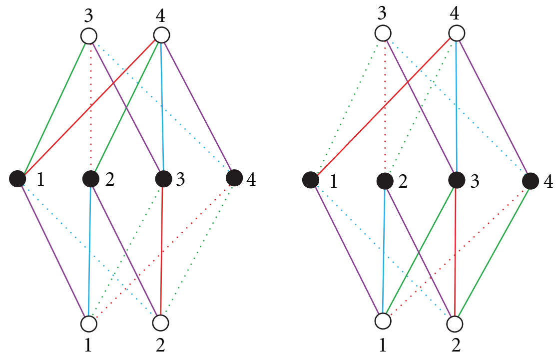


Figure 3: The $N = 4$ 2-4-2 Adinkras may be obtained from 3-4-1 Adinkras by raising one vertex. Here we have raised the third boson vertex starting with the two Adinkras shown in Figure 2.

obtain these matrices algebraically, as explained above, by interchanging the third rows of the 3-4-1 up and down matrices shown in Table 2.)

We now use the enhancement equation (3.3) to analyze the twelve standard $N = 4$ 2-4-2 Adinkras to ascertain if any of these can enhance to a 4D $N = 1$ nongauge matter multiplet. To begin, we start with the left Adinkra in Figure 3, equivalently described by the boson linkage matrices specified in Table 3 and by the fermion linkage matrices determined from these by $\widetilde{u}_A = d_A^T$ and $\widetilde{d}_A = u_A^T$.

We found above that for each of the two $N = 4$ Valise Adinkras and for each of the eight 3-4-1 Adinkras the enhancement equation corresponding to $(a \mid A, B) = (1 \mid 1, 1)$ is not satisfied. This is equivalent to the statement that the 4×4 matrix determined by $u_1 u_3^T + d_3 d_1^T$ does not vanish in these cases. The reader should compute this combination in those cases, and then also compute this combination using the linkage matrices in Table 3. It is noteworthy that in this latter case, that is, using the matrices in Table 3, the computation of $u_1 u_3^T + d_3 d_1^T$ does indeed produce the 4×4 null matrix. So the particular obstruction which we identified in the 4-4 and 3-4-1 Adinkras is notably absent for the specific 2-4-2 Adinkra shown on the left in Figure 1.

Having satisfied the $(a \mid A, B) = (1 \mid 1, 1)$ equation, it remains to analyze all of the other possible choices for $(a \mid A, B)$ and check the enhancement equations (3.3) in each case. It is interesting to comment on the case $(a \mid A, B) = (1 \mid 1, 4)$, for example. In this case the enhancement equation reads

$$u_1 u_2^T + d_2 d_1^T + u_4 u_3^T + d_3 d_4^T = 0. \tag{4.1}$$

Note that this is satisfied using the matrices in Table 3. So the left 2-4-2 Adinkra in Figure 3 passes this second enhancement test. (Thus, this Adinkra passes two out of 60 different tests, counting the both the bosonic and the fermionic enhancement conditions for each of the 30 index choices $(a \mid A, B)$.)

It is interesting that, unlike the left Adinkra in Figure 3, the right Adinkra in Figure 3 fails the test (4.1). This can be seen by noting that (4.1) is sensitive to the parity on any one of the four edge colors since the overall signs on the first two terms flip upon toggling the sign on Q_1 or Q_2 while the sign on the third and fourth terms flip upon toggling the sign on Q_3 or Q_4. More specifically, the linkage matrices for the second Adinkra in Figure 3 are obtained

Table 3: Linkage matrices for the left 2-4-2 Adinkra shown in Figure 3. The linkage matrices for the right Adinkra in that figure are obtained from these by changing the sign of u_3 and d_3.

$$u_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 0 & \\ & & & 0 \end{array}\right) \qquad d_1 = \left(\begin{array}{cc|cc} 0 & & & \\ & 0 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

$$u_2 = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 0 \\ & & 0 & \end{array}\right) \qquad d_2 = \left(\begin{array}{cc|cc} & 0 & & \\ 0 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right)$$

$$u_3 = \left(\begin{array}{cc|cc} & & -1 & \\ & & & -1 \\ \hline 0 & & & \\ & 0 & & \end{array}\right) \qquad d_3 = \left(\begin{array}{cc|cc} & & 0 & \\ & & & 0 \\ \hline 1 & & & \\ & 1 & & \end{array}\right)$$

$$u_4 = \left(\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & 0 & & \\ 0 & & & \end{array}\right) \qquad d_4 = \left(\begin{array}{cc|cc} & & & 0 \\ & & 0 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right)$$

from Table 3 by toggling the sign on Q_3, which toggles the overall sign on the matrices u_3 and d_3. If we substitute the linkage matrices for the right Adinkra in Figure 3, obtained in this way, into (4.1) we find that this equation is no longer satisfied. Thus, we conclude that the right Adinkra in Figure 3 cannot enhance to a 4D nongauge matter multiplet. Again, the reader should check these assertions by doing a few simple matrix calculations.

Further analysis of all of the remaining 58 enhancement conditions shows that the matrices in Table 3 pass every one of these tests. This is a nontrivial accomplishment, which indicates that the left Adinkra in Figure 3 does represent the shadow of a 4D $N = 1$ nongauge matter multiplet. As a representative example, consider the bosonic enhancement condition for the choice $(a \mid A, B) = (2 \mid 3, 2)$. This equation reads

$$u_2 u_2^T + u_3 u_3^T + d_2 d_2^T + d_3 d_3^T = 2, \tag{4.2}$$

where the factor of 2 on the right-hand side means twice the 4×4 unit matrix. The reader should verify that (4.2) is the bosonic enhancement equation described by (3.3) in this case. The reader should also verify that (4.2) is satisfied by the linkage matrices in Table 3.

The fact that the first Adinkra in Figure 3 describes the shadow of the 4D $N = 1$ chiral multiplet is easy to check by performing a direct dimensional reduction of the chiral multiplet. This is done explicitly in Appendix C. In that appendix we derive the shadow Adinkra, shown in Figure 4, by direct translation of the 4D chiral multiplet transformation rules. The left Adinkra in Figure 3 is obtained from the Adinkra in Figure (4) by reorganizing fields according to the following four permutation operations: $\phi_1 \leftrightarrow \phi_2$, $\phi_3 \leftrightarrow \phi_4$, $\psi_1 \leftrightarrow \psi_4$, and $\psi_2 \leftrightarrow \psi_3$. This describes a cosmetic inner automorphism, indicating that the two Adinkras are equivalent.

The fact that the left Adinkra in Figure 3 passes the enhancement criteria while the right Adinkra does not identifies the left Adinkra as the chiral multiplet shadow and identifies the right Adinkra as the so-called twisted chiral multiplet. We also find that the 2-4-2 Adinkra obtained by raising the vertices ϕ_1 and ϕ_2 starting from the left 4-4 Valise in Figure 1 also passes all of the enhancement criteria whereas the twisted analog of this does not. Scrutiny of all twelve bosonic 2-4-2 Valises confirms that only those two cases in the nontwisted family, associated with the left Valise in Figure 1, obtained by raising either the pair (ϕ_1, ϕ_2) or the pair (ϕ_3, ϕ_4) can enhance to nongauge matter multiplets in 4D.

4.4. The Two 30-Member $N = 4$ Adinkra Families

It is straighforward to systematically check the enhancement conditions for all 30 Adinkras in each of the two families—one family associated with each of the two Valises in Figure 1. In each case, the 30-member family consists of the bosonic 4-4 Adinkra (the Valise), four bosonic 3-4-1 Adinkras, six bosonic 2-4-2 Adinkras, four bosonic 1-4-3 Adinkras, and the Klein flip of each one of these 15 representatives. (The Klein flipped Adinkras are the fermionic 4-4 Valise and the 14 fermionic Adinkras obtainable from this by various vertex raises.)

In total there are exactly four out of the 60 minimal Adinkras which pass our nongauge enhancement criteria (3.3). The first two are the bosonic 2-4-2 chiral multiplet shadows obtained by raising either ϕ_1 and ϕ_2 or by raising ϕ_3 and ϕ_4 starting from the left Adinkra in Figure 1. The other two Adinkras reside in the other (relatively twisted) family, and are obtained from the right Adinkra in Figure 1 by first raising all four bosonic vertices, then raising either the fermionic vertices ψ_1 and ψ_2 or by raising the fermionic vertices ψ_3 and ψ_4. These operations produce fermionic 2-4-2 Adinkras corresponding to twisted Klein flips of the two enhanceable bosonic 2-4-2 Adinkras. Since in these cases there are at most two fermions at any given height assignment, it is clear that these cannot assemble as 4D spinors. As we explained above, the Adinkras which pass the enhancement criteria come in pairs, one element of which passes the spin-statistics test and one which does not. In this way we conclude that of the 60 minimal Adinkras specified above only the two bosonic 2-4-2 cases can describe shadows of nongauge 4D matter multiplets.

It might appear curious that the four bosonic 2-4-2 Adinkras obtained from the right Adinkra in Figure 1 by raising (ϕ_1, ϕ_3) or (ϕ_1, ϕ_4) or (ϕ_2, ϕ_3) or (ϕ_2, ϕ_4) do not pass the enhancement criteria whereas the two Adinkras obtained by raising (ϕ_1, ϕ_2) or (ϕ_3, ϕ_4) do pass this test. The reason why certain combinations of component fields appear favored relates to the fact that we have made a choice of spin structure when we selected the particular gamma matrices in (E.9). It is interesting that we lose no generality in making such a choice, however, since the freedom to choose a 4D spin basis is replaced by a corresponding freedom to perform inner automorphisms on the vector space spanned by the 1D component fields.

Upon selecting a higher-dimensional spin basis, the enhancement equations (3.3) place restrictions on the component fields which are legitimately meaningful; the result that exactly two out of the 60 minimal $N = 4$ Adinkras enhance to nongauge 4D supersymmetric matter, along with the observation that those 1D multiplets which do enhance have 2-4-2 component multiplicities says something salient about 4D supersymmetry representation theory. Specifically it says that any 4D $N = 1$ nongauge matter multiplet must have two physical bosons, four fermions, and two auxiliary bosons. This corroborates what has long been known about the minimal representations of 4D $N = 1$ supersymmetry. What is remarkable is that we have hereby shown that this information is fully extractable using merely 1D supersymmetry and a choice of 4D spin structure—that this information lies fully encoded in the 1D supersymmetry representation theory codified by the families of Adinkras, and that the key to unlocking this information is contained in our enhancement equation (3.3). (We believe that the extra structures, namely that the bosons complexify and that the fermions assemble into chiral spinors, is also encoded in our formalism, using the equations (A.5). We also believe that deeper scrutiny of those equations should provide an algebraic context for broadly resolving natural organizations of supermultiplets, including complex structures, and quaternionic structures in diverse dimensions. But this lies beyond the scope of this introductory paper on this topic.)

5. Algorithmic Complexity

Much of the analysis needed to fully delineate 4D $N = 1$ supersymmetry representation theory using our techniques succumbs to by-hand matrix manipulations, as explained above. The entirety of the computation of all Adinkras which are size-appropriate to enhance to 4D $N = 1$ supersymmetry may be analyzed, using our nonoptimized algorithm, in a matter of seconds using a simple Mathematica routine. But the reader is likely more interested in the richer (unsolved) problem implied by the unknown elements of supersymmetry representation theory, or how realistically our equations might be used to address this larger problem, with an aim to discover yet-unknown representations which might drive novel supersymmetric field theories for use in physical model building.

Using our criteria, testing enhancement to D dimensions would involve at most $(1/2)(D-1)d(d+1)$ independent $d \times d$ matrix equations per Adinkra, where d is the number of fermions or bosons in the Adinkra. The minimal-size Adinkra in the case $D = 1$, $N = 16$ is 128+128, whereby $d = 128$. To ascertain whether one of these enhances to $D = 10$, $N = 1$ supersymmetry would involve $(1/2)(9)(128)(129) = 74,304$ equations, each involving products of 128×128 matrices. This number is not prohibitively large given contemporary computer resources. But we should quantify this assertion.

To test enhancement of a generic 128+128 vertex Adinkra to a ten-dimensional space-time would require 74,304 equations, each involving a 128×128 matrix. This translates to a computation on the order of a trillion floating point operations (1 teraflop). This result derives from the fact that solving a general matrix equation of size N is an $\mathcal{O}(N^3)$ computation. However, with good choices of bases or a locality property, solving a matrix equation of size N can often be improved substantially into a mere $O(N)$ computation. On the basis of this, we envision that some interesting algorithmic work not addressed in this paper should enable the ready analysis of more general supersymmetry representations using the analytic base described in this paper.

To be more precise, an eight-processor Mac Pro is capable of about 50 Gigaflops, if all processors are used with highly optimized code. A teraflop would take only 20 seconds on such a machine. Even without highly optimized code (but using compiled code), using just one processor, this would take about five minutes. The full computation checking the enhance-ability of a size-appropriate Adinkra to ten space-time dimensions is on the order of a teraflop, which means it would take several minutes, on an eight-processor Mac Pro computer.

Thus, the rote enhancement of a given Adinkra is never difficult computationally. However, the sheer number of possibilities provides a separate concern. Certainly the number of ostensibly distinct 128+128 Adinkras is astronomical. A more precise accounting of the combinatorics is described in [42]. However, there are interesting considerations which strongly constrain the search for the subset of these with physical relevance. One of these is the feature that enhancement proceeds as a filtratation as the dimensionality of the ambient space is increased incrementally. So an algorithm based on the feature of enhancement from one-dimension to two-dimensions, and from there to three-dimensions, and so-forth would add great refinement to the brute-force approach we have used above.

Indeed, this is the approach advocated in a recent paper [42], where the enhancement problem (called the extension problem in that paper) is engaged first from one to two dimensions. The authors of that paper have already identified theorems (their Theorems 2.2 and 2.3 in that reference) which remove the great majority of Adinkras from consideration from the outset. This shows that there are considerations which can strongly influence the accounting for the difficulties inherent in implementing analyses based on our enhancement equations.

Thus, until efficient algorithms beyond the scope of this paper are developed, the problem of making systematic work of resolving interesting unknown representations of supersymmetry using our methods remains daunting in its complexity. But there are indications, partially motivated by the ruminations in this section, which give a reason to optimistically hope that this combinatoric problem is not an insurmountable impasse. The development of the relevant algorithmic approach is work-in-progress.

6. Gauge Multiplets

The nongauge enhancement condition (3.3) relies on the result (3.1), which is derived in Appendix A. An important part of that derivation uses the assumptions $\Delta_A^0 = \widetilde{u}_A^T$ and $\widetilde{\Delta}_A^0 = u_A^T$. These translate into the statement that every Adinkra edge codifies both an upward-directed term and a downward-directed term in the multiplet transformation rules. (In other words, this result applies to "standard" Adinkras.) But the presence of gauge degrees of freedom or Bianchi identities obviates this assumption. This is demonstrated explicitly by dimensionally reducing the 4D $N = 1$ Maxwell field-strength multiplet, as described in detail in Appendix D.

6.1. Introducing Phantoms

In field-strength multiplets, the vector space spanned by the boson components ϕ_i is larger than the vector space spanned by the fermion components $\psi_{\hat{\imath}}$. The physical degrees of freedom balance, however, owing to redundancies in the space of bosons, related to the constraints. This feature manifests in nonsquare linkage matrices, including sectors which decouple on the shadow. We call these "phantom sectors".

The Maxwell multiplet is characteristic of generic multiplets involving closed p-form field-strengths, when $p \geq 2$. In these cases, the field-strength divides into an "electric" sector, including components with a time-like index, and a "magnetic" sector involving components which have only space-like indices. The electric sector and the magnetic sector are correlated by the differential constraints implied by the Bianchi identity. Upon reduction to one-dimension, the magnetic sector decouples. The reason for this is that locally the magnetic fields are pure space derivatives, which vanish upon restriction to a zero-brane.

Thus, in order to enhance a one-dimensional gauge multiplet to a higher-dimensional analog, not only do we have to resurrect the spatial derivatives, ∂_a, but we also have to resurrect the gauge sector. In the case of field-strength multiplets, this means reinstating the magnetic fields. Since these are physically decoupled on the shadow, they are reintroduced in an interesting way, at the top of one-way upward-directed Adinkra edges. These nascent magnetic vertices play no role in the one-dimensional supersymmetry algebra respected by the other fields on the shadow. (But they play an important role in closing the algebra in ambient higher dimensions.) We therefore call these vertices "phantoms".

Phantoms may be introduced into one-dimensional supermultiplets to enable possible enhancement involving closed p-form multiplet components. Since closed 1-forms do not involve gauge invariance, it follows that the simplest case involves closed 2-forms, such as $F_{\mu\nu}$ subject to $\partial_{[\lambda}F_{\mu\nu]} = 0$. This allows access to the important cases involving vector multiplets. The higher-p cases may be treated similarly, but these involve additional subtlety. In order to keep our presentation relatively concise, we will not address cases $p \geq 3$ nor will we address cases involving gauge fermions. Our discussion will remain focussed on the ability to include 4D Abelian field-strengths. We also avoid other subtle technicalities by allowing 4D fermions to assemble only as spin 1/2 fields; that is we will not address the case of spin 3/2, or Rarita-Schwinger fields, in this paper. It is straightforward to generalize our technology to allow for these possibilities. But we defer discussions of these cases to future work, for reasons of bounding complexity.

The structure of a phantom sector is usefully codified by phantom link matrices, defined as

$$(P_A)_i{}^{\hat{\imath}} := \left(\widetilde{u}_A^T - \Delta_A^0\right)_i{}^{\hat{\imath}}, \tag{6.1}$$

where $\widetilde{u}_A^T$ is the transpose of the Ath fermion "up" matrix. A nonvanishing phantom matrix indicates the presence of one-way upward-directed Adinkra edges. If P_A is nonvanishing then this modifies the analysis in Appendix A precisely at the point where (A.9) is introduced as the transpose of (A.8). If the phantom matrices are included and the analysis is repeated, it is easy to show that (3.1) generalizes to

$$\Delta_A^a = -\left(\Gamma^0\Gamma^a\Delta^0\right)_A - \frac{1}{2}\left(\Gamma^0\Gamma^a P\right)_A + T^{0a}P_A - P_A\widetilde{T}^{0a}. \tag{6.2}$$

Note that the final three terms will contribute nontrivially to this equation only in the gauge sector.

It is helpful to briefly review the particular phantom sector associated with the shadow of the Maxwell field-strength multiplet. This provides the archetype for generalizations, and motivates what follows.

6.2. Maxwell's Shadow

The 4D $N = 1$ super Maxwell multiplet involves four boson degrees of freedom off-shell. These organize as the auxiliary scalar D plus the three off-shell "electromagnetic" degrees of freedom described by $F_{\mu\nu}$. It is natural to write $E_a = F_{0a}$ and $B^a = (1/2)\varepsilon^{abc}F_{bc}$. The Bianchi identity $\partial_{[\lambda}F_{\mu\nu]} = 0$ correlates E_a and B^a. Locally, we can solve the Bianchi identity in terms of a vector potential A_μ, so that $E_a = \partial_0 A_a - \partial_a A_0$ and $B^a = \varepsilon^{abc}\partial_b A_c$. Upon restriction to the zero-brane we take $\partial_a \to 0$, so that $E_a \to \partial_0 A_a$ and $B_a \to 0$. Since the magnetic fields vanish on the zero-brane, it is natural to think of the E_a as more fundamental for our purposes. The shadow is described by a fermionic 4-4 Adinkra where the bosons are (E_1, E_2, E_3, D) and the fermions are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. To enhance this multiplet we must reintroduce ∂_a and also reintroduce the fields B^a, along with constraints. To do this, we allow for "phantom" bosons on the worldline, which correspond to the B^a off of the worldline.

To accommodate the phantom bosons, we consider an enlarged bosonic vector space, $\phi_i = (E_1, E_2, E_3, D \mid B_1, B_2, B_3)$ in conjunction with the fermionic vector space $\psi_{\hat{\imath}} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. As a useful index notation, we write these as $\phi_i = (E_a, D \mid B^{\overline{a}})$, where $B^{\overline{a}}$ is the magnetic phantom associated with E_a. Thus, a and $\overline{a}$ each assume the values 1,2,3, and we have $\phi_{1,2,3} = E_{1,2,3}$ and $\phi_{5,6,7} = B^{\overline{1},\overline{2},\overline{3}}$, respectively. In this way, phantom fields are designated by an over-bar on the relevant index. Boson fields not in the phantom sector are indicated by underlined indices, so that $\phi_{\underline{1},\underline{2},\underline{3},\underline{4}} = (E_1, E_2, E_3, D)$. Matrices with two boson indices then divide into four sectors, $X_{\underline{i}}{}^{\underline{j}}$, $X_{\underline{i}}{}^{\overline{a}}$, $X_{\overline{a}}{}^{\underline{j}}$, and $X_{\overline{a}}{}^{\overline{b}}$.

The shadow transformation rules associated with the Maxwell multiplet can be written as (2.3), but the linkage matrices are not square! Instead, $(\widetilde{u}_A)_{\hat{\imath}}{}^{j}$ is 7×4 and $(\Delta_A^{\mu})_i{}^{\hat{\imath}}$ is 4×7. We exhibit the precise linkage matrices associated with the Maxwell multiplet in Appendix D. For the Super Maxwell case, the first enhancement equation in (3.3) is a 7×7 matrix equation, whereas the second is a 4×4 matrix equation. The first equation has phantom sectors which can be shuffled away canonically via use of the Bianchi identity, as explained below.

6.3. Canonical Reshuffling

Owing to the derivatives in the enhancement condition (3.3), we may use the Bianchi identity, $\partial_{[\lambda} F_{\mu\nu]} = 0$, usefully rewritten as

$$\begin{aligned} &\partial_0 B^{\overline{a}} = \varepsilon^{\overline{a}bc} \partial_b E_c, \\ &\partial_{\overline{a}} B^{\overline{a}} = 0, \end{aligned} \tag{6.3}$$

to define "canonical reorganizations" of the matrices in (3.2) under which (3.3) remains unchanged. Specifically, the first equation in (6.3) allows us to redefine

$$\begin{aligned} &\left(\Omega^0_{AB}\right)_i{}^{\overline{a}} \longrightarrow 0, \\ &\left(\Omega^a_{AB}\right)_i{}^{b} \longrightarrow \left(\Omega^a_{AB}\right)_i{}^{b} + \varepsilon^{ab}_{\overline{c}} \left(\Omega^0_{AB}\right)_i{}^{\overline{c}}, \end{aligned} \tag{6.4}$$

whereby we exchange each appearance of $\partial_0 B^{\overline{a}}$ in a supersymmetry commutator with an equivalent expression involving spatial derivatives on the electric field components. Similarly, the second equation in (6.3) allows us to redefine

$$\begin{aligned} &\left(\Omega^1_{AB}\right)_i{}^{\overline{1}} \longrightarrow 0, \\ &\left(\Omega^2_{AB}\right)_i{}^{\overline{2}} \longrightarrow \left(\Omega^2_{AB}\right)_i{}^{\overline{2}} - \left(\Omega^1_{AB}\right)_i{}^{\overline{1}}, \\ &\left(\Omega^3_{AB}\right)_i{}^{\overline{3}} \longrightarrow \left(\Omega^3_{AB}\right)_i{}^{\overline{3}} - \left(\Omega^1_{AB}\right)_i{}^{\overline{1}}. \end{aligned} \tag{6.5}$$

In this way, we define a canonical structure of the matrices $(\Omega^{\mu}_{AB})_i{}^j$, ensured by the transformations (6.4) and (6.5), enabled by the Bianchi identity (6.3), whereby $(\Omega^0_{AB})_i{}^{\overline{a}} = 0$ and $(\Omega^1_{AB})_i{}^1 = 0$. The first equation in (3.3) may now be interpreted as saying that $(\Omega^{\mu}_{AB})_i{}^j \to 0$ under these transformations.

We remark that that each of the 40 7×7 matrices $(\Omega^{\mu}_{AB})_i{}^j$ defined by (3.2), using the linkage matrices exhibited in Appendix D, do satisfy $(\Omega^{\mu}_{AB})_i{}^j \to 0$ using the transformations (6.4) and (6.5). The diligent reader is encouraged to check this assertion.

6.4. The $p = 1$ Gauge Enhancement Conditions

Based on the above, a means becomes apparent under which we can ascertain which one-dimensional multiplets may enhance to 4D gauge field-strength multiplets, based only on a knowledge of the one-dimensional transformation rules, or equivalently given an Adinkra.

For physical gauge fields, the bosonic field-strength tensor has greater engineering dimension than the corresponding gaugino fermions. Therefore, the ambient fermions transform into the magnetic fields via terms in the fermion transformation rule $\delta_Q \lambda$ given by $(1/2)\varepsilon_{abc} B^a \Gamma^{bc} \epsilon$ or by $(1/2)\varepsilon_{abc} B^a \Gamma^{bc} \Gamma_5 \epsilon$. These are the only Lorentz covariant possibilities. The former case involves a vector potential and the latter case involves an axial vector potential. We focus first on the former case, and comment on axial vectors afterwards. It is straightforward to determine the phantom "up" links and the time-like fermion "down" links using $\delta_Q \lambda = \cdots + (1/2)\varepsilon_{abc} B^a \Gamma^{bc} \epsilon$. By rearranging this term into the form $\delta_Q \lambda_i = \cdots + \epsilon^A (\tilde{u}_A)_i^{\overline{a}} B_{\overline{a}}$, we derive (Consistency of (6.6) with (6.1) implies usefully extractable information about the spin representation assignments. We will not explore this arena in this paper.)

$$
\begin{gathered}
(\tilde{u}_A)_{\hat{\imath}}{}^{\overline{a}} = \frac{1}{2} \varepsilon^{\overline{a}bc} (\Gamma_{bc})_{\hat{\imath}A}, \\
(\Delta^0_A)_{\overline{a}}{}^{\hat{\imath}} = 0,
\end{gathered}
\tag{6.6}
$$

whereby using (6.1) we determine the nonvanishing part of the phantom matrix as

$$
(P_A)_{\overline{a}}{}^{\hat{\imath}} = \frac{1}{2} \varepsilon_{\overline{a}bc} \left(\Gamma^{bc}\right)_A{}^{\hat{\imath}}.
\tag{6.7}
$$

The entire phantom matrix has nonvanishing entries only in its final three rows.

We can resolve the Δ^a_A matrices in two parts, using different methods. First we resolve the phantom part $(\Delta^a_A)_{\overline{a}}{}^{\hat{\imath}}$. Then we resolve the nonphantom part $(\Delta^a_A)_{\underline{a}}{}^{\hat{\imath}}$.

We determine the phantom part of the space-like boson down matrices using the fact that $(\Delta^0_A)_{\overline{a}}{}^{\hat{\imath}} = 0$, which says that there are no connections linking downward from the phantoms. Thus, equation (A.4) tells us $(\Delta^a_A)_{\overline{a}}{}^{\hat{\imath}} = -(T^{0a})_{\overline{a}}{}^i (\Delta^0_A)_i{}^{\hat{\imath}}$. Next, we use the fact that a boost shuffles magnetic fields into electric fields, via $(T_{0a})_b{}^{\overline{c}} B_{\overline{c}} = \varepsilon^c{}_{ab} E_c$, to determine

$$
(\Delta^a_A)_{\overline{b}}{}^{\hat{\imath}} = \varepsilon_{\overline{b}}{}^{a\underline{c}} \left(\Delta^0_A\right)_{\underline{c}}{}^{\hat{\imath}}.
\tag{6.8}
$$

This determines the $\overline{b}$th row in the phantom sector of the Ath space-like down matrices in terms of "electric" rows in the time-like down matrices.

We determine the nonphantom part of the space-like down matrices by considering the nonphantom sector in (6.2). Thus, we allow only nonphantom values for the suppressed boson index. In this case, the second and the fourth terms on the right-hand side vanish because $(P_A)_{\underline{a}}{}^{\hat{\imath}} = 0$. The third term on the right-hand side can be resolved by noting that a boost shuffles electric fields into magnetic fields, via $(T^{0a})_b{}^c E_c = \varepsilon_b{}^{a\overline{c}} B_{\overline{c}}$, whereby $(T^{0a})_{\underline{b}}{}^c = \varepsilon_{\underline{b}}{}^{a\overline{c}}$. Substituting this result, along with (6.7), we derive

$$\left(\Delta^a_A\right)_{\underline{b}}{}^{\hat{\imath}} = -\left(\Gamma^0\Gamma^a\right)_A{}^B \left(\Delta^0_A\right)_{\underline{b}}{}^{\hat{\imath}} + \left(\Gamma^a_{\underline{b}}\right)_A{}^{\hat{\imath}}. \tag{6.9}$$

This is the same as our nongauge result (3.1), modified by the second term. (Note that the second term in (6.9) can also be written as $-2\varepsilon_{\underline{b}}{}^{ac}(\mathcal{R}_c)_A{}^{\hat{\imath}}$, where $\mathcal{R}_c$ generates a rotation in the cth spatial dimension.) Taken together, (6.8) and (6.9) generate for us the entire space-like phantom-modified down matrices, generalizing our earlier nongauge result (3.1) to the case in which fields can assemble into closed 1-forms.

Note that the Maxwell field-strength shadow, including its phantom sectors, may be reproduced from the minimal $N = 4$ Adinkras using the methods described in this section. Presently, we explain a means to produce a representative in one of the two minimal $N = 4$ Adinkra families which demonstrably enhances to a 4D Maxwell multiplet. Importantly, we can verify that this Adinkra enhances to such a 4D multiplet using only one-dimensional reasoning.

This is done by starting with right Adinkra in Figure 1. (Note that this Adinkra is in the family twisted relative to the family which includes the chiral multiplet shadow.) From this starting Adinkra, we raise all four boson vertices, to obtain a fermionic 4-4 Adinkra with the four bosons at the higher level. We then perform a permutation of the first and the fourth boson vertices, and a permutation of the the second and the fourth boson vertices. We then flip both the first and the fourth boson vertices. (To flip a vertex means to scale this by an overall minus sign.) Finally we flip the third fermion vertex. We compute the time-like up matrices $\tilde{u}_A$ and the time-like down matrices Δ^0_A using the resultant fermionic 4-4 Adinkra. We designate the first three boson vertices in this final orientation as our designated "electric" components. We then apply (6.6), (6.8), and (6.9) to append phantom sectors to these time-like linkage matrices, and to generate provisional space-like down links Δ^a_A, including phantom sectors. What results from these operations are precisely the matrices shown in Appendix D. We next compute the Ω matrices using (3.2). Finally, we apply a canonical reshuffling of these Ω-matrices, using (6.4) and (6.5). Happily, we find that under this operation all of the matrices $(\Omega^\mu_{AB})_i{}^j$ and $(\tilde{\Omega}^\mu_{AB})_{\hat{\imath}}{}^{\hat{\jmath}}$ vanish. This illustrates that this representative set of operations produces an Adinkra which passes our nontrivial gauge enhancement test.

If we repeat the above search allowing for axial vector potential, we would modify all equations in this subsection with an additional factor of Γ_5. What we find is that every multiplet which enhances to provide a vector potential also enhances to provide an axial vector potential. This is loosely similar to the situation involving chiral versus antichiral multiplets, which have identical shadows. It follows that the both the vector multiplet and the axial vector multiplet shadow lie in the family of Adinkras relatively twisted relative to the chiral multiplet. We will not exhibit separate equations for the axial vector case, leaving that as an exercise for the interested reader.

6.5. Algorithm

We have already explained that it is possible to systematically generate the linkage matrices for each member of the family associated with a given Valise. For each representative, we can sequentially select triplets of vertices as potential electric field components, and use (6.6), (6.8), and (6.9) to generate postulate phantom sectors. For each of these, we compute the relevant Ω-matrices using (3.2), then shuffle these into canonical form using (6.4) and (6.5). By selecting those Adinkras for which all the canonical Ω-matrices obtained in this way vanish, we thereby obtain an algorithmic search for all multiplets in which vertices can assemble into closed 1-forms. This search is guaranteed to locate those multiplets which do properly enhance. (N.B. we have explained in the previous paragraph an example which we know works.)

In the case of closed 1-form multiplets, an enhanceable Adinkra exhibits a synergy between the postulated electric field components and the designated magnetic phantom sector, vis-a-vis the assignment of the component basis ϕ_i. This is because the enhancement criteria are sensitive to the basis choice on the component boson vector via the structure of our imposed phantom sector. (Note that designating the phantom sector using (6.8) and (6.9) does not remove generality from the search, much as choosing 4D gamma matrices does not remove generality, for reasons described above. Instead, this removes redundancies from the answer set.) Practically, this requires, for a comprehensive algorithmic search for enhanceable Adinkras, that in addition to sifting through all possible vertex raises and all possible selections of vertex triplets, we also have to sift through vertex permutations and vertex flips. Thus, inner automorphisms would seem to enlarge the effective search family. Regardless, our discussion does show that the portion of 4D supersymmetry representation theory involving closed 1-form multiplets is accessible and understandable using only 1D supersymmetry. We find this interesting.

In summary, following is a method to test an Adinkra to see if it enhances to give a 4D multiplet with a closed 1-form gauge field-strength.

(1) Select three boson vertices with a common engineering dimension as the presumed electric components, and arrange the boson vector space so that $\phi_{1,2,3}$ correspond to these.

(2) Compute time-like linkage matrices u_A, $\tilde{u}_A$, Δ^0_A, and $\tilde{\Delta}^0_A$ from the Adinkra.

(3) Augment the boson vector space by adding on a phantom sector consisting of three new fields $\phi_{\bar{1},\bar{2},\bar{3}}$ with the same engineering dimension as $\phi_{1,2,3}$.

(4) Add phantom sectors to the up matrices $\tilde{u}_A$, by adding three extra columns, and add phantom sectors to the time-like down matrices Δ^0_A by adding three extra rows.

(5) Populate the phantom sector of the up matrices using (6.6).

(6) Generate space-like down matrices, including phantom sectors using (6.8) and (6.9).

(7) Use the complete set of phantom-augmented linkage matrices to determine the matrices $(\Omega^a_{AB})_i{}^j$ and $(\tilde{\Omega}^a_{AB})_{\hat{\imath}}{}^{\hat{\jmath}}$ using (3.2).

(8) Reshuffle the boson matrix $(\Omega^{a}_{AB})_i{}^j$ using the prescription (6.4) and (6.5), to obtain a new matrix, in canonical form,

$$(\Omega^{a}_{AB})_i{}^j \longrightarrow \left(\widehat{\Omega}^{a}_{AB}\right)_i{}^j. \tag{6.10}$$

The presence of the hat indicates canonical form.

(9) The $p = 1$ gauged enhancement conditions now correspond to the original enhancement conditions (3.3) augmented by the addition of phantom sectors and a canonical reshuffle. We conclude that a necessary requirement for an Adinkra to enhance to a $p = 1$ field-strength multiplet is

$$\begin{aligned} \left(\widehat{\Omega}^{\mu}_{AB}\right)_i{}^j &= 0, \\ \left(\widetilde{\Omega}^{\mu}_{AB}\right)_{\hat{\imath}}{}^{\hat{\jmath}} &= 0. \end{aligned} \tag{6.11}$$

This is our $p = 1$ gauge enhancement condition. The key difference as compared to the nongauge case is that the linkage matrices are not square in the gauge case, owing to the presence of the phantom boson sectors. Furthermore, we must implement the canonical reshuffling maneuver to generate the hatted $\widehat{\Omega}$ matrices which describe the nongauge enhancement condition.

The way we have designed our formalism is tailored toward implementation via computer-searches. This may require supercomputers for cases with higher N, which will involve large matrix computations. We hope to enlarge our algorithms so that sifting through one-dimensional multiplets is controlled by the relevant lists of doubly-even error-correcting codes which correspond to these, as explained in the introduction. But this lies beyond the scope of the presentation in this paper. We find it sufficiently noteworthy that such algorithms exist, even in principle. Our hope is that this will shed light on unknown aspects of supersymmetry which have defied attack using previous conventions.

7. Conclusions

We have presented algebraic conditions which allow one to systematically locate those representations of one-dimensional supersymmetry which may enhance to higher dimensions. Equivalently, we have explained how to discern whether a given one-dimensional supermultiplet is a shadow of a higher-dimensional analog. This allows the representation theory of supersymmetry in diverse dimensions to be divided into the simpler representation theory of one-dimensional supersymmetry augmented with separate questions pertaining to the possibility of enhancement into higher dimensions.

We have shown through explicit examples how information pertaining to four-dimensional $N = 1$ supersymmetry may be extracted using only information from one-dimensional supersymmetry. We did this comprehensively for the case of 4D $N = 1$ nongauge matter multiplets. We have also explained how this systematics generalizes to cases involving higher-dimensional gauge invariances, specializing our discussion to the case of 4D $N = 1$ Super-Maxwell theory.

We intend to use the formalism and the algorithms developed above to seek inroads towards off-shell aspects of interesting supersymmetric contexts where the off-shell physics remains mysterious but potentially relevant.

Appendices

A. A Proof

In this appendix we prove that demanding Lorentz invariance of the linkage matrices defined in (2.3) completely determines all of the space-like linkage matrices Δ^a_A in terms of the time-like linkage matrices Δ^0_A, and does so in precisely the manner specified above as equation (3.1). We also show how this same requirements provides constraints on the spin representation content of supermultiplet component fields.

The linkage matrices $(\Delta^\mu_A)_i{}^{\hat{\imath}}$ transform under $\mathfrak{spin}(1,D-1)$, manifestly, as

$$\begin{aligned}\delta_L\left(\Delta^\mu_A\right)_i{}^{\hat{\imath}} &= \theta^\mu_\nu(\Delta^\nu_A)_i{}^{\hat{\imath}} + \frac{1}{4}\theta^{\lambda\sigma}(\Gamma_{\lambda\sigma})_A{}^B\left(\Delta^\mu_B\right)_i{}^{\hat{\imath}}\\ &\quad + \frac{1}{2}\theta^{\lambda\sigma}(T_{\lambda\sigma})_i{}^j\left(\Delta^\mu_A\right)_j{}^{\hat{\imath}} - \frac{1}{2}\theta^{\lambda\sigma}\left(\Delta^\mu_A\right)_i{}^{\hat{\jmath}}\left(\widetilde{T}_{\lambda\sigma}\right)_{\hat{\jmath}}{}^{\hat{\imath}}.\end{aligned} \tag{A.1}$$

In (A.1), the first term indicates that on $(\Delta^\mu_A)_i^{\hat{\jmath}}$ the μ index is a vector index, the second term indicates that the A index is a spinor index, and the last line accommodates the representation content of the supermultiplet component fields.

Using (A.1), we obtain the following boost and rotation transformations for the "time-like" linkage matrices $(\Delta^0_A)_i{}^{\hat{\imath}}$:

$$\begin{aligned}\delta_{\text{boost}}\left(\Delta^0_A\right)_i{}^{\hat{\imath}} &= \theta^0_a(\Delta^a_A)_i{}^{\hat{\imath}} + \frac{1}{2}\theta^{0a}(\Gamma_{0a})_A{}^B\left(\Delta^0_B\right)_i{}^{\hat{\imath}}\\ &\quad + \theta^{0a}\left(T_{0a}\Delta^0_A\right)_i{}^{\hat{\imath}} - \theta^{0a}\left(\Delta^0_A\widetilde{T}_{0a}\right)_i{}^{\hat{\imath}},\\ \delta_{\text{rotation}}\left(\Delta^0_A\right)_i{}^{\hat{\imath}} &= +\frac{1}{2}\theta^{ab}(\Gamma_{ab})_A{}^B\left(\Delta^0_B\right)_i{}^{\hat{\imath}}\\ &\quad + \theta^{ab}\left(T_{ab}\Delta^0_A\right)_i{}^{\hat{\imath}} - \theta^{ab}\left(\Delta^0_A\widetilde{T}_{ab}\right)_i{}^{\hat{\imath}},\end{aligned} \tag{A.2}$$

and the following boost and rotation transformations for the "space-like" linkage matrices:

$$\begin{aligned}\delta_{\text{boost}}(\Delta^a_A)_i{}^{\hat{\imath}} &= \theta^a_0(\Delta^0_A)_i{}^{\hat{\imath}} + \frac{1}{2}\theta^{0b}(\Gamma_0\Gamma_b)_A{}^B(\Delta^a_B)_i{}^{\hat{\imath}}\\ &\quad + \theta^{0b}(T_{0b}\Delta^a_A)_i{}^{\hat{\imath}} - \theta^{0b}(\Delta^a_A\widetilde{T}_{0b})_i{}^{\hat{\imath}},\\ \delta_{\text{rotation}}(\Delta^a_A)_i{}^{\hat{\imath}} &= \theta^a_b(\Delta^b_A)_i{}^{\hat{\imath}} + \frac{1}{4}\theta^{bc}(\Gamma_{bc})_A{}^B(\Delta^a_B)_i{}^{\hat{\imath}}\\ &\quad + \frac{1}{2}\theta^{bc}(T_{bc}\Delta^a_A)_i{}^{\hat{\imath}} - \frac{1}{2}\theta^{bc}(\Delta^a_A\widetilde{T}_{bc})_i{}^{\hat{\imath}}.\end{aligned} \tag{A.3}$$

We demand that the linkage matrices are Lorentz invariant. This imposes that each of the transformations in (A.2) and (A.3) must vanish. (This is similar to "demanding" that the gamma matrices appearing in a Salam-Strathdee superfield be Lorentz invariant—in that case they are, automatically, as a consequence of the Clifford algebra.) Requiring $\delta_{\text{boost}}\Delta^0 = 0$ imposes

$$(\Delta_A^a)_i{}^{\hat{\imath}} = -\frac{1}{2}\left(\Gamma^0\Gamma^a\right)_A{}^B\left(\Delta_B^0\right)_i{}^{\hat{\imath}} - \left(T^{0a}\Delta_A^0\right)_i{}^{\hat{\imath}} + \left(\Delta_A^0\widetilde{T}^{0a}\right)_i{}^{\hat{\imath}}. \tag{A.4}$$

This determines $(\Delta_A^a)_i{}^{\hat{\imath}}$ in terms of $(\Delta_A^0)_i{}^{\hat{\imath}}$ and in terms of the representation assignments of the supermultiplet component fields.

The remaining consequences of imposing Lorentz invariance on the linkage matrices $(\Delta_A^\mu)_i{}^{\hat{\imath}}$ are the following:

$$\begin{aligned}\frac{1}{2}(\Gamma_{ab})_A{}^B\Delta_B^0 &= \Delta_A^0\widetilde{T}_{ab} - T_{ab}\Delta_A^0,\\ \delta_b{}^a\Delta_A^0 &= \frac{1}{2}(\Gamma_0\Gamma_b)_A{}^B\Delta_B^a + T_{0b}\Delta_A^a - \Delta_A^a\widetilde{T}_{0b},\\ \eta^{a[b}\Delta_A^{c]} + \frac{1}{4}\left(\Gamma^{bc}\right)_A{}^B\Delta_B^a &= \frac{1}{2}\Delta_A^a\widetilde{T}^{bc} - \frac{1}{2}T^{bc}\Delta_A^a,\end{aligned} \tag{A.5}$$

where the $(\cdot)_i^{\hat{\imath}}$ index structure has been suppressed on each term. These correspond, respectively, to $\delta_{\text{rotation}}\Delta^0 = 0$, $\delta_{\text{boost}}\Delta^a = 0$, and $\delta_{\text{rotation}}\Delta^a = 0$, for arbitrary transformation parameters θ^{0a} and θ^{ab}. Equations (A.5) place significant restrictions on the spin representation content of the component fields. As explained above, we suspect that these equations encode useful and extractable information regarding allowable complements of spin structures in supermultiplets in diverse dimensions.

The linkage matrices $(u_A)_i{}^{\hat{\imath}}$ transform under $\mathfrak{spin}(1, D-1)$, manifestly, as

$$\delta(u_A)_i{}^{\hat{\imath}} = \frac{1}{4}\theta^{\mu\nu}(\Gamma_{\mu\nu})_A{}^B(u_B)_i{}^{\hat{\imath}} + \frac{1}{2}\theta^{\mu\nu}(T_{\mu\nu}u_A)_i{}^{\hat{\imath}} - \frac{1}{2}\theta^{\mu\nu}\left(u_A\widetilde{T}_{\mu\nu}\right)_i{}^{\hat{\imath}}. \tag{A.6}$$

Requiring that these transformations vanish imposes

$$\frac{1}{2}(\Gamma_{\mu\nu})_A{}^B(u_B)_i{}^{\hat{\imath}} = \left(u_A\widetilde{T}_{\mu\nu}\right)_i{}^{\hat{\imath}} - (T_{\mu\nu}u_A)_i{}^{\hat{\imath}}. \tag{A.7}$$

This indicates correlations between the up linkage matrices and the representation content of the component fields.

Similar conditions result from demanding invariance of $(\widetilde{u}_A)_{\hat{\imath}}{}^i$ and $(\widetilde{\Delta}_A^\mu)_{\hat{\imath}}{}^i$. These are obtained from the above constraints by placing tildes on all matrices which do not have tildes and removing tildes from those that do. For example, invariance of $(\widetilde{u}_A)_{\hat{\imath}}{}^i$ imposes

$$\frac{1}{2}(\Gamma_{\mu\nu})_A{}^B(\widetilde{u}_B)_{\hat{\imath}}{}^i = (\widetilde{u}_AT_{\mu\nu})_{\hat{\imath}}{}^i - \left(\widetilde{T}_{\mu\nu}\widetilde{u}_A\right)_{\hat{\imath}}{}^i. \tag{A.8}$$

Note that for standard Adinkras we have $\Delta^0_A = \tilde{u}^T_A$. (For nonstandard Adinkras, such as those which accommodate gauge invariances, this relationship does not hold. Instead, we define $\tilde{u}_A = \Delta^0_A + P_A$, where P_A is a so-called phantom matrix, which encodes the nexus of one-way upward-directed Adinkra edges. This generalization is addressed in Section 6.) Thus, using (A.8) we determine

$$\frac{1}{2}(\Gamma_{\mu\nu})_A{}^B\left(\Delta^0_B\right)_i{}^{\hat{\imath}} = \left(T^T_{\mu\nu}\Delta^0_A\right)_i{}^{\hat{\imath}} - \left(\Delta^0_A\tilde{T}^T_{\mu\nu}\right)_i{}^{\hat{\imath}}. \tag{A.9}$$

This equation can be used in conjunction with (A.4) to replace that equation with an analog in which the representation matrices are not included.

The boost matrices $(T_{0a})_i{}^j$ and $(\tilde{T}_{0a})_{\hat{\imath}}{}^{\hat{\jmath}}$ are generically symmetric (For example, if the fermions assemble as spinors then $\tilde{T}_{0a} = (1/2)\Gamma_0\Gamma_a$. In the Majorana basis Γ_0 is antisymmetric and real while Γ_a are symmetric and real, and since Γ_0 and Γ_a anticommute, it follows that $\tilde{T}_{0a}$ is symmetric in that case. For vectors V_a we have $(T_{0a})_0{}^b = \delta_a{}^b$ and $(T_{0a})_b{}^0 = -\eta_{ab}$; for our metric choice $\eta_{ab} = -1$, so that these T_{0a} are symmetric. This reasoning generalizes to higher-rank tensors and to all products of tensor and spinor representations. Note too, that if the boost matrices were antisymmetric, then (A.9) and (A.4) could be used together to prove the inconsistent result that $\Delta^a = 0$.) Therefore, (A.9) can be rewritten as

$$\left(T_{0a}\Delta^0_A\right)_i{}^{\hat{\imath}} - \left(\Delta^0_A\tilde{T}_{0a}\right)_i{}^{\hat{\imath}} = \frac{1}{2}(\Gamma_0\Gamma_a)_A{}^B\left(\Delta^0_B\right)_i{}^{\hat{\imath}}. \tag{A.10}$$

Substituting this result for the last two terms in (A.4), we determine

$$\left(\Delta^a_A\right)_i{}^{\hat{\imath}} = -\left(\Gamma^0\Gamma^a\right)_A{}^B\left(\Delta^0_B\right)_i{}^{\hat{\imath}}. \tag{A.11}$$

Remarkably, this relationship is completely independent of the representation content of the component fields. This is an interesting result, which says that the space-like linkage matrices Δ^a are determined from the time-like linkage matrices Δ^0.

B. Adinkra Conventions

In this appendix we give a very concise overview of the graphical technology of Adinkra diagrams. These were introduced in [17], and have formed the basis of a multidisciplinary research endeavor, aimed at resolving a mathematically rigorous basis for supersymmetry [18, 19, 21, 23–25]. Some of the conventions, notably as regards sign choices, have varied in these references, in part because some of these papers aim at a physics audience and some at a mathematics audience. Thus, one reason for this appendix is to clarify our conventions, as used above, so that this paper can be appreciated without undue confusion. Another is to allow this paper to be functionally self-contained.

A representation of N-extended supersymmetry in one time-like dimension consists of d boson fields ϕ_i and d fermion fields $\psi_{\hat{\imath}}$ endowed with a set of transformation rules, generated by $\delta_Q(\epsilon)$, where ϵ^A are a set of N anticommuting parameters, which respect the N-extended supersymmetry algebra specified by the commutator $[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] =$

$2i\delta_{AB}\epsilon_1^A\epsilon_2^B\partial_\tau$. The transformation rules can be written for boson fields as $\delta_Q\phi_i = -i\epsilon^A(Q_A)_i{}^{\hat{\imath}}\psi_{\hat{\imath}}$ and for fermion fields as $\delta_Q\psi_{\hat{\imath}} = -i\epsilon^A(\widetilde{Q}_A)_{\hat{\imath}}{}^{i}\phi_i$, where $(Q_A)_i^{\hat{\imath}}$ and $(\widetilde{Q}_A)_{\hat{\imath}}^{i}$ are two sets of N abstract $d\times d$ matrix generators of supersymmetry. By definition these represent

$$\begin{aligned}\left(Q_{(A}\widetilde{Q}_{B)}\right)_i{}^j &= i\delta_i{}^j\partial_\tau,\\ \left(\widetilde{Q}_{(A}Q_{B)}\right)_{\hat{\imath}}{}^{\hat{\jmath}} &= i\delta_{\hat{\imath}}{}^{\hat{\jmath}}\partial_\tau,\end{aligned}\tag{B.1}$$

where the symmetrization brackets are defined with “weight-one”, whereby $X_{(A}Y_{B)} = (1/2)(X_AY_B + X_BY_B)$.

It is possible to use cosmetic field redefinitions to redefine the component fields ϕ_i and $\psi_{\hat{\imath}}$ into a “frame” where the generators $(Q_A)_i{}^{\hat{\jmath}}$ and $(\widetilde{Q}_A)_{\hat{\imath}}{}^{\hat{\jmath}}$ are first-order differential operators with a specialized matrix structure. Specifically, it is possible to write

$$\begin{aligned}(Q_A)_i{}^{\hat{\imath}} &= (u_A)_i{}^{\hat{\imath}} + (d_A)_i{}^{\hat{\imath}}\partial_\tau,\\ (\widetilde{Q}_A)_{\hat{\imath}}{}^{i} &= i(\widetilde{u}_A)_{\hat{\imath}}{}^{i} + i(\widetilde{d}_A)_{\hat{\imath}}{}^{i}\partial_\tau,\end{aligned}\tag{B.2}$$

where $(u_A)_i{}^{\hat{\imath}}$, $(d_A)_i{}^{\hat{\imath}}$, $(\widetilde{u}_A)_{\hat{\imath}}{}^{j}$, and $(\widetilde{d}_A)_{\hat{\imath}}{}^{j}$ are four sets of N real $d\times d$ “linkage matrices” with the features that every entry of each of these matrices takes only one of three values, 0, 1, or -1, and such that there is at most one nonvanishing entry in every row and at-most one nonvanishing entry in every column of each of these matrices. Remarkably, we lose no generality by specializing to generators of the sort (B.2) with these particular properties. A mathematical proof that any one-dimensional supermultiplet can be written in this manner is provided in [18].

A simple example in the context of $N = 2$ supersymmetry is given by the following transformation rules,

$$\begin{aligned}\delta_Q\phi_1 &= -i\epsilon^1\psi_1 - i\epsilon^2\psi_2,\\ \delta_Q\phi_2 &= -i\epsilon^1\partial_\tau\psi_2 + i\epsilon^2\partial_\tau\psi_1,\\ \delta_Q\psi_1 &= \epsilon^1\partial_\tau\phi_1 - \epsilon^2\partial_\tau\phi_2,\\ \delta_Q\psi_2 &= \epsilon^1\phi_1 + \epsilon^2\partial_\tau\phi_2.\end{aligned}\tag{B.3}$$

It is straightforward to verify that these satisfy the commutator relationship specified above.

The operator ∂_τ carries unit engineering dimension, while supersymmetry parameters ϵ^A carry engineering dimension one-half. (In a system where $\hbar = c = 1$, a field with engineering dimension q carries units of $(\text{Mass})^q$.) Thus, in order to balance units in the transformation rules (B.3) it follows that the two fermions $\psi_{1,2}$ have a common engineering dimension one-half greater than ϕ_1, and that ϕ_2 has an engineering dimension one-half greater than the fermions, and one unit greater than ϕ_1.

The transformation rules (B.3) can be expressed equivalently, in terms of linkage matrices, as

$$
\begin{aligned}
(u_1)_i{}^{\hat{\jmath}} &= \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, & (u_2)_i{}^{\hat{\imath}} &= \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, \\
(d_1)_i{}^{\hat{\imath}} &= \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}, & (d_2)_i{}^{\hat{\imath}} &= \begin{pmatrix} & 0 \\ -1 & \end{pmatrix}, \\
(\tilde{u}_1)_{\hat{\imath}}{}^{j} &= \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}, & (\tilde{u}_2)_{\hat{\imath}}{}^{j} &= \begin{pmatrix} & -1 \\ 0 & \end{pmatrix}, \\
(\tilde{d}_1)_{\hat{\imath}}{}^{j} &= \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, & (\tilde{d}_2)_{\hat{\imath}}{}^{j} &= \begin{pmatrix} & 0 \\ 1 & \end{pmatrix},
\end{aligned}
\tag{B.4}
$$

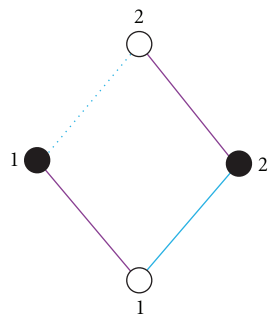
where blank matrix entries represent zeros. This set of eight matrices is completely equivalent to the transformation rules (B.3). It is straightforward to verify, using (B.2), that the algebra (B.1) is properly represented using these matrices.

As an example, to illustrate what these matrices mean, consider the matrix $\tilde{u}_2$ defined in (B.4). This is the "second fermion up matrix", where the qualifier "second" refers to the subscript on $\tilde{u}_2$ and indicates that this matrix encodes a mapping under the second supersymmetry, while the qualifier "fermionic" refers to the tilde, and indicates that this matrix encodes transformations of the fermions. The single nonvanishing term in this matrix is in the first row, second column, which indicates that, of the two fermions, only the first fermion ψ_1 transforms under the second supersymmetry, into the second boson ϕ_2. The fact that this matrix entry is -1 indicates a minus sign in the transformation rule $\delta_Q\psi_1$ on the term proportional to ϕ_2, that is, $\delta_Q\psi_1 = \cdots - \phi_2\epsilon^2$, as seen in (B.3). The reason why this is called an "up" matrix is that it encodes a mapping "upward" from from a field with lower engineering dimension—ψ_1 in this case—to a field with higher engineering dimension—ϕ_2 in this case.

The matrices in (B.4) exhibit the properties $\tilde{u}_A = d_A^T$ and $\tilde{d}_A = u_A^T$. It is easy to see that this indicates a symmetric feature in the transformation rules (B.3), whereby a fermion appearing in a boson transformation rule is correlated with that boson appearing in the transformation rule for that fermion. In other words, terms in these transformation rules come paired. This feature is satisfied by a wide and important class of supermultiplets, which we call "standard". (These are also called "Adinkraic" in the literture.)

There is a third equivalent way to represent the supersymmetry transformations given by (B.3) and by (B.4). This method uses the observation that the generic properties of linkage matrices facilitate a concise system under which the entire collection of linkage matrices for a given multiplet can be faithfully represented by a graph. Such a graph, called an Adinkra, consists of d white vertices (one for each boson) and d black vertices (one for each fermion). Two vertices are connected by an Ath colored edge if the two fields corresponding to those vertices are interrelated by the Ath supersymmetry. The edge is rendered solid if the corresponding Q_A matrix entries are +1 and are rendered dashed if the corresponding Q_A matrix entries are -1. Finally, the vertices are arranged so that their heights on the graph correlate faithfully with the respective engineering dimension.

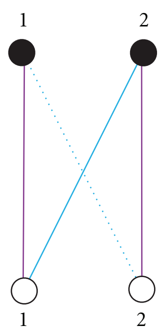
Thus, if we designate Q_1 using purple edges and Q_2 using blue edges, then the example multiplet described by (B.4), equivalent to (B.3), would have the following Adinkra,



(B.5)

where the numerals on the vertices specify the fields, for example, the black vertex labeled 2 represents the fermion field ψ_2. As an easy exercise, the reader should confirm that (B.4) can be recovered from (B.5) using the rules described above. There is a striking economy exhibited by this graphical method, empowered by the fact that these graphs completely encode every aspect of the transformation rules, in a way which allows for ready translation from any Adinkra into linkage matrices or into parameter-dependent transformation rules.

As another example, consider the following Adinkra,



(B.6)

This describes a supermultiplet distinct from the previous example, as evidenced by the fact that (B.6) spans only two different engineering dimensions, whereas (B.5) spans three.

We can readily extract the linkage matrices equivalent to (B.6). For example, the boson down matrices d_1 and d_2 obviously vanish because the two bosons do not connect "downward" to any lower fermion vertices. Similarly, the two fermion up matrices $\tilde{u}_1$ and $\tilde{u}_2$ also obviously vanish, since there are no links "upward" from the black vertices. We can determine the nonvanishing linkage matrices by "reading" the diagram. For example, the boson up matrix u_2 encodes blue edges connecting upward from boson vertices. Thus, since the boson ϕ_1 links upward via blue edge only to the fermion ψ_2, and does so with a solid edge, this tells us that the matrix entry $(u_2)_1^{\,2} = +1$. In this way, we can translate the Adinkra (B.6) into the linkage matrices described by $d_A = 0$, $\tilde{u}_A = 0$,

$$(u_1)_i^{\,\hat{\jmath}} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad (u_2)_i^{\,\hat{\imath}} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \tag{B.7}$$

and $\tilde{d}_A = u_A^T$.

Note that the Adinkra (B.5) can be obtained from (B.6) by an interesting operation: by moving the vertex ϕ_2 to a new position located one level above the fermions, while continuously maintaining all intervertex edge connections, so that the edges swivel upward during this process. This macrame-like move encodes a transformation which maps one supermultiplet into another, and is called a vertex raising operation.

One of our results concerning Adinkras is a mathematical proof that any standard supermultiplet can be obtained by a sequence of vertex raising operations starting from an Adinkras with vertices which span only two different heights, for example, (B.6). Accordingly, the representation theory of 1D standard multiplets breaks naturally into two parts; first to classify all of the possible two-height Adinkras for a given value of N, and then to systematize the possible sequences of vertex raises using each of these as a starting point.

Owing to the special role played by the two-height Adinkras, we have given these a special name. Standard Adinkras which span only two height assignments are called Valise Adinkras, or Valises for short. The reason for this nomenclature is based on the observation that a large number of multiplets can be “unpacked”, as from a suitcase (or a valise), by judicious choices of vertex raises. (We credit Tristan Hübsch for inventing this catchy and useful term.)

Using the information above, the reader ought to be able to verify the relationships between the Adinkras shown in Figures 1, 2, and 3, with the corresponding linkage matrices exhibited in the respective Tables 1, 2, and 3, and should appreciate our use of the terms Adinkra, Valise, and the concept of vertex raising.

C. The Shadow of the Chiral Multiplet

In this appendix we explain how to dimensionally-reduce the 4D chiral multiplet to extract its shadow.

The 4D $N = 1$ chiral multiplet has the following transformation rules,

$$\begin{aligned} \delta_Q \phi &= 2i\bar{\epsilon}_L \chi_R, \\ \delta_Q \chi_R &= \not{\partial}\phi \epsilon_L + F\epsilon_R, \\ \delta_Q F &= 2i\bar{\epsilon}_R \not{\partial} \chi_R, \end{aligned} \tag{C.1}$$

where ϕ is a complex scalar, F is a complex auxiliary scalar, and χ_R is a right-chiral Weyl spinor field. The parameter ϵ_R is also a right-chiral spinor, while ϵ_L describes the corresponding Majorana conjugate, that is, $\epsilon_L = C^{-1}\bar{\epsilon}_R^T$. The transformation rules (C.1) satisfy $[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = 4i\bar{\epsilon}_{[2L}\not{\partial}\epsilon_{1]L}$ on all component fields ϕ, F, and χ_R. Note that we can define a Majorana spinor parameter via $\epsilon = \epsilon_R + \epsilon_L$, so that $\epsilon_{R,L} = (1/2)(1 \pm \Gamma_5)\epsilon$ are the corresponding right- and left-chiral projections. In terms of the Majorana spinor, the algebra is $[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = 2i\bar{\epsilon}_2\not{\partial}\epsilon_1$.

We express spinors in the Majorana basis described in Appendix E. (The choice of basis is immaterial for the computing the dimensional reduction; we obtain identical results using any other basis. We use the Majorana basis here in order to maintain consistency with other derivations in this paper.) Accordingly, we write the spinor field and the spinor supersymmetry parameter as

$$\chi_R = \frac{1}{2}\begin{pmatrix} \chi_1 + i\chi_2 \\ \chi_2 - i\chi_1 \\ \hline \chi_3 + i\chi_4 \\ \chi_4 - i\chi_3 \end{pmatrix}, \qquad \epsilon_R = \frac{1}{2}\begin{pmatrix} \epsilon_1 + i\epsilon_2 \\ \epsilon_2 - i\epsilon_1 \\ \hline \epsilon_3 + i\epsilon_4 \\ \epsilon_4 - i\epsilon_3 \end{pmatrix}, \tag{C.2}$$

where $\chi_{1,2,3,4}$ are each real anti-commuting fields and $\epsilon_{1,2,3,4}$ are each real anti-commuting constant parameters. We also write the complex boson fields as $\phi = \phi_1 + i\phi_2$, where $\phi_{1,2}$ are real bosons, and $F = F_1 + iF_2$, where $F_{1,2}$ are real auxiliary bosons.

Using these definitions, setting $\partial_a = 0$, and using the spinor identities in Appendix E, the transformation rules (C.1) become

$$\begin{aligned}
\delta_Q\phi_1 &= -i\epsilon_1\chi_3 + i\epsilon_2\chi_4 + i\epsilon_3\chi_1 - i\epsilon_4\chi_2, \\
\delta_Q\phi_2 &= -i\epsilon_1\chi_4 - i\epsilon_2\chi_3 + i\epsilon_3\chi_2 + i\epsilon_4\chi_1, \\
\delta_Q\chi_1 &= F_1\epsilon_1 - F_2\epsilon_2 - \dot{\phi}_1\epsilon_3 - \dot{\phi}_2\epsilon_4, \\
\delta_Q\chi_2 &= F_2\epsilon_1 + F_1\epsilon_2 - \dot{\phi}_2\epsilon_3 + \dot{\phi}_1\epsilon_4, \\
\delta_Q\chi_3 &= \phi_1\epsilon_1 + \phi_2\epsilon_2 + F_1\epsilon_3 - F_2\epsilon_4, \\
\delta_Q\chi_4 &= \phi_2\epsilon_1 - \phi_1\epsilon_2 + F_2\epsilon_3 + F_1\epsilon_4, \\
\delta_Q F_1 &= -i\epsilon_1\dot{\chi}_1 - i\epsilon_2\dot{\chi}_2 - i\epsilon_3\dot{\chi}_3 - i\epsilon_4\dot{\chi}_4, \\
\delta_Q F_2 &= -i\epsilon_1\dot{\chi}_2 + i\epsilon_2\dot{\chi}_1 - i\epsilon_3\dot{\chi}_4 + i\epsilon_4\dot{\chi}_3.
\end{aligned} \tag{C.3}$$

These rules describe the shadow of (C.1). We organize the boson fields so that $\phi_i = (\phi_1, \phi_2, F_1, F_3)$ and the fermion fields so that $\psi_{\hat{\imath}} = (\chi_1, \chi_2, \chi_3, \chi_4)$. Using the Adinkra conventions described in Appendix B, along with our edge coloration scheme whereby $Q_{1,2,3,4}$ are respectively described by purple, blue, green, and red colored edges, we can unambiguously represent (C.3) as the Adinkra shown in Figure 4.

It is easy to translate the Adinkra in Figure 4 into equivalent up and down matrices. For example, we determine the boson up matrix u_1 by looking at the purple colored edges extending upward from boson vertices. There are two such edges: a solid edge connecting ϕ_1 with ψ_3 and solid edge connecting ϕ_2 with ψ_4. Thus, there are two nonvanishing entries in u_1: one in the first row, third column, and the other in the second row, fourth column. These both take the value +1 because both edges are solid. In this way we determine the up matrices shown in Table 4 and the time-like down matrices shown in Table 5.

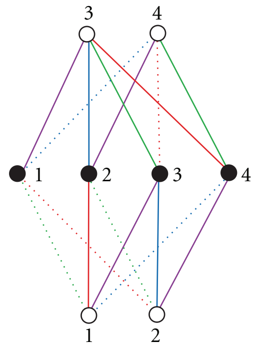


Figure 4: The shadow of the chiral multiplet, expressed as an Adinkra equivalent to the transformation rules (C.3).

Table 4: The boson up linkage matrices for the chiral multiplet shadow. The fermion time-like down matrices are determined from these via $\tilde{d}_A = u_A^T$.

$$u_1 = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 0 \\ & & 0 & \end{array}\right) \qquad u_2 = \left(\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & 0 & & \\ 0 & & & \end{array}\right)$$

$$u_3 = \left(\begin{array}{cc|cc} -1 & & & \\ & -1 & & \\ \hline & & 0 & \\ & & & 0 \end{array}\right) \qquad u_4 = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 0 \\ & & 0 & \end{array}\right)$$

We can also determine the time-like down linkage matrices directly from (C.1). For example, to determine the femion 1-sector down matrices, $\tilde{\Delta}_A^1$ we isolate those terms in the fermion transformation rule $\delta_Q \psi_R$ involving the derivative ∂_1. These are given by $\delta_Q^{(1)} \chi_R = \Gamma^1 \epsilon_L \partial_1 \phi$. We then use the explicit matrix Γ^1 specified in (E.9), the spinor components specified in (C.2), and we write $\phi = \phi_1 + i\phi_2$. This allows us, after a small amount of algebra, to rewrite the 1-sector fermion transformation rule as $\delta_Q \psi_{\hat{\imath}} = \epsilon^A (\tilde{\Delta}_A^1)_{\hat{\imath}}^{\,j} \partial_1 \phi_j$, from which we can read off the four matrices $(\tilde{\Delta}_A^1)_{\hat{\imath}}^{\,j}$. It is straightforward to perform this calculation, and then to verify that the matrices thereby obtained satisfy $\tilde{\Delta}_A^1 = -(\Gamma^0 \Gamma^1)_A^{\ B} \tilde{\Delta}_B^0$, where $\tilde{\Delta}_A^0 = u_A^T$. Similar calculations can be done for all of the time-like down matrices, providing a nice consistency check on our powerful assertion (3.1).

D. The Shadow of the Maxwell Field Strength Multiplet

In this appendix we determine the linkage matrices for the 4D $N = 4$ Maxwell field-strength multiplet. This provides a means to exhibit precisely how 1D phantom sectors arise upon restriction of a $p = 1$ gauge multiplet to a zero-brane. This appendix is complementary to Section 6 in the main text, above.

Table 5: The time-like dowm linkage matrices for the chiral multiplet. The fermion up matrices are determined from these via $\tilde{u}_A = d_A^T$.

$\Delta_1^0 = \left(\begin{array}{cc\|cc} & & 0 & \\ & & & 0 \\ \hline 1 & & & \\ & 1 & & \end{array}\right)$	$\Delta_2^0 = \left(\begin{array}{cc\|cc} & & & 0 \\ & & 0 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right)$
$\Delta_3^0 = \left(\begin{array}{cc\|cc} 0 & & & \\ & 0 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$	$\Delta_4^0 = \left(\begin{array}{cc\|cc} & 0 & & \\ 0 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right)$

The 4D $N = 1$ super-Maxwell field-strength multiplet has the following transformation rules,

$$\begin{aligned} \delta_Q \lambda &= \frac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon - i D \Gamma_5 \epsilon, \\ \delta_Q F_{\mu\nu} &= -2i \overline{\epsilon} \Gamma_{[\mu} \partial_{\nu]} \lambda, \\ \delta_Q D &= \overline{\epsilon} \Gamma_5 \not{\partial} \lambda, \end{aligned} \tag{D.1}$$

where λ is a Majorana spinor gaugino field, D is a real auxiliary (pseudo)scalar, and the field-strength tensor $F_{\mu\nu}$ is subject to the Bianchi identity $\partial_{[\lambda} F_{\mu\nu]} = 0$. We obtain the linkage matrices equivalent to (D.1) by deconstructing these rules using a specific spinor basis, and rewriting them in terms of individual degrees of freedom as specified in (2.3). It follows simply that that $\widetilde{\Delta}_A^a = 0$ and $u_A = 0$, since the fermions λ_A share a common engineering dimension of $3/2$ while the bosons $F_{\mu\nu}$ and D share a common engineering dimension of 2.

We use the specific Majorana basis defined in Appendix E by the gamma matrices given in (E.9). To determine the "up" linkage matrices, it is helpful to rewrite the fermion transformation rule in (D.1) as

$$\delta_Q \lambda = 2 E_a \mathcal{B}^a \epsilon + 2 B^a \mathcal{R}_a \epsilon - i D \Gamma_5 \epsilon, \tag{D.2}$$

where we have used the definitions $E_a = F_{0a}$ and $B^a = (1/2)\varepsilon^{abc} F_{bc}$, for the electric and magnetic fields, respectively. We have also used the definitions $\mathcal{B}_a = (1/2)\Gamma^0 \Gamma^a$, and $\mathcal{R}_a = (1/4)\varepsilon_{abc} \Gamma^{bc}$ for the boost and rotation generators also given in Appendix E. We now use the explicit matrices $\mathcal{B}_a$, $\mathcal{R}^a$, and Γ_5 specified in (E.11) and (E.9), to recast (D.2) in matrix form: the left side as a four-component column matrix $\lambda_A = (\lambda_1, \lambda_2 \mid \lambda_3, \lambda_4)^T$ and the right-hand side as a 4×7 matrix multiplying another four-component column matrix given by $\epsilon_A = (\epsilon_1, \epsilon_2 \mid \epsilon_3, \epsilon_4)^T$. A small amount of algebra then allows us to rewrite the result in the form $\delta_Q \lambda_{\hat{\imath}} = \epsilon^A (\tilde{u}_A)_{\hat{\imath}}{}^j \phi_j$, where $\phi_i := (E_1, E_2 \mid E_3, D \| B_1, B_2, B_3)^T$, whereupon we can read off each of the four matrices $(\tilde{u}_A)_{\hat{\imath}}{}^j$. The result is shown in Table 6. The fact that λ transforms nontrivially into B^a manifests in the nontriviality of the rightmost three columns in these results.

Table 6: The four "up" linkage matrices associated with the Maxwell field-strength multiplet.

$$\tilde{u}_1 = \left(\begin{array}{cc|cc||ccc} & & 1 & & 0 & 0 & 0 \\ & & & -1 & 0 & 0 & 1 \\ \hline 1 & & & & 0 & -1 & 0 \\ & 1 & & & 1 & 0 & 0 \end{array}\right) \qquad \tilde{u}_2 = \left(\begin{array}{cc|cc||ccc} & & & 1 & 0 & 0 & -1 \\ & & 1 & & 0 & 0 & 0 \\ \hline & -1 & & & -1 & 0 & 0 \\ 1 & & & & 0 & -1 & 0 \end{array}\right)$$

$$\tilde{u}_3 = \left(\begin{array}{cc|cc||ccc} 1 & & & & 0 & 1 & 0 \\ & -1 & & & 1 & 0 & 0 \\ \hline & & -1 & & 0 & 0 & 0 \\ & & & -1 & 0 & 0 & -1 \end{array}\right) \qquad \tilde{u}_4 = \left(\begin{array}{cc|cc||ccc} & 1 & & & -1 & 0 & 0 \\ 1 & & & & 0 & 1 & 0 \\ \hline & & & 1 & 0 & 0 & 1 \\ & & -1 & & 0 & 0 & 0 \end{array}\right)$$

We then do a similar thing to the boson fields to determine the down matrices Δ^{μ}_{A}. We do this separately for each of the four choices for μ, referring to these as the μ-sector down matrices. For example, to extract the 0-sector down matrices, we isolate those terms in the boson transformation rules in (D.1) proportional to the derivative $\partial_0\lambda$. These are given by

$$\begin{aligned} \delta^{(0)}_{Q} E_a &= i\overline{\epsilon}\Gamma_a\partial_0\lambda, \\ \delta^{(0)}_{Q} D &= \overline{\epsilon}\Gamma_5\Gamma^0\partial_0\lambda, \\ \delta^{(0)}_{Q} B^a &= 0. \end{aligned} \tag{D.3}$$

Note that the magnetic fields $B^a = (1/2)\varepsilon^{abc}F_{bc}$ do not transform into time derivatives of the gaugino field. This is not surprising since the magnetic field is expressible locally as $B^a = \varepsilon^{abc}\partial_b A_c$. But it is worth noting that (D.3) follows simply from (D.1). This tells us that upon restriction to a zero-brane, there are no downward Adinkra links connecting the three magnetic field components to any other fields; in the shadow these degrees of freedom sit at the top of one-way upward edges. In this way the magnetic fields decouple from the multiplet upon reduction to one-dimension. By utilizing the specific gamma matrices given in Appendix E we can use the same techniques described above to rewrite the 0-sector transformation rules (D.3) as $\delta_Q\phi_i = -i\epsilon^A(\Delta^0_A)_i{}^{\hat{\imath}}\partial_0\lambda_{\hat{\imath}}$, and then read-off the the matrices $(\Delta^0_A)_i{}^{\hat{\imath}}$. The result of this straightforward process is exhibited in Table 7. (It is easy to see that $\widetilde{u}^T_A \neq \Delta^0_A$, so that in this case the phantom matrix defined in (6.1) is nonvanishing. Although the phantom sector is irrelevant to any one-dimensional physics, it is necessary to resurrect this sector should we wish to enhance the shadow theory to its full ambient analog.)

By isolating the terms in (D.1) respectively proportional to $\partial_{1,2,3}\lambda$, writing these explicitly using the Majorana basis gamma matrices shown in (E.9), and then reconfiguring the rules as $\delta_Q\phi_i = -i\epsilon^A(\Delta^a_A)_i{}^{\hat{\imath}}\partial_a\lambda_{\hat{\imath}}$, allows us to read off the remaining space-like linkage matrices. The results of this straightforward process are exhibited in Tables 8, 9, and 10.

E. 4D Spinor Bases

gamma matrices satisfy the Clifford relationship $\{\Gamma_\mu,\Gamma_\nu\}_A{}^B = -2\eta_{\mu\nu}\delta_A^B$, where $\eta_{\mu\nu} = \mathrm{diag}(+---)$. These act from the left on spinors ψ_A and from the right on barred spinors $\overline{\psi}^A := (\psi^\dagger\Gamma_0)^A$.

Table 7: The time-like down matrices associated with the Maxwell field-strength multiplet.

$$
\Delta_1^0 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & -1 & & \\ \hline\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad
\Delta_2^0 = \left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & 1 & & \\ 1 & & & \\ \hline\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

$$
\Delta_3^0 = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & -1 \\ \hline\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad
\Delta_4^0 = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \\ \hline\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

Table 8: The 1-sector space-like down matrices for the Maxwell field-strength multiplet.

$$
\Delta_1^1 = \left(\begin{array}{cc|cc} -1 & & & \\ & 0 & & \\ \hline & & 0 & \\ & & & 1 \\ \hline\hline 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad
\Delta_2^1 = \left(\begin{array}{cc|cc} & -1 & & \\ 0 & & & \\ \hline & & & 0 \\ & & -1 & \\ \hline\hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)
$$

$$
\Delta_3^1 = \left(\begin{array}{cc|cc} & & -1 & \\ & & & 0 \\ \hline 0 & & & \\ & 1 & & \\ \hline\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \qquad
\Delta_4^1 = \left(\begin{array}{cc|cc} & & & -1 \\ & & 0 & \\ \hline & 0 & & \\ -1 & & & \\ \hline\hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right)
$$

In four-dimensions, the minimal solution involves 4×4 matrices, so the spinor index A takes on four values. The 4D charge conjugation matrix C is defined by $C\Gamma^a C^{-1} = -(\Gamma^a)^T$. In addition, the matrix C is real, antisymmetric, and has unit determinant. A chirality operator is defined by $\Gamma_5 := i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$.

Table 9: The 2-sector space-like down matrices for the Maxwell field-strength multiplet.

$$\Delta_1^2 = \left(\begin{array}{cc|cc} & 0 & & \\ -1 & & & \\ \hline & & & 0 \\ & & -1 & \\ \hline\hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \quad \Delta_2^2 = \left(\begin{array}{cc|cc} 0 & & & \\ & -1 & & \\ \hline & & 0 & \\ & & & -1 \\ \hline\hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$$\Delta_3^2 = \left(\begin{array}{cc|cc} & & & 0 \\ & & -1 & \\ \hline & 0 & & \\ 1 & & & \\ \hline\hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right) \quad \Delta_4^2 = \left(\begin{array}{cc|cc} & & 0 & \\ & & & -1 \\ \hline 0 & & & \\ & 1 & & \\ \hline\hline 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

Table 10: The 3-sector space-like down matrices for the Maxwell field-strength multiplet.

$$\Delta_1^3 = \left(\begin{array}{cc|cc} & & 0 & \\ & & & 0 \\ \hline -1 & & & \\ & 1 & & \\ \hline\hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Delta_2^3 = \left(\begin{array}{cc|cc} & & & 0 \\ & & 0 & \\ \hline & -1 & & \\ -1 & & & \\ \hline\hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Delta_3^3 = \left(\begin{array}{cc|cc} 0 & & & \\ & 0 & & \\ \hline & & -1 & \\ & & & -1 \\ \hline\hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Delta_4^3 = \left(\begin{array}{cc|cc} & 0 & & \\ 0 & & & \\ \hline & & & -1 \\ & & 1 & \\ \hline\hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We can change bases by replacing

$$\begin{aligned} \Psi_A &\longrightarrow {M_A}^B \psi_B, \\ {(\Gamma^\mu)_A}^B &\longrightarrow {\left(M\Gamma^\mu M^{-1}\right)_A}^B, \\ C^{-1}_{AB} &\longrightarrow \frac{1}{\sqrt{\det(M)}}\left(MC^{-1}M^T\right)_{AB}, \end{aligned} \tag{E.1}$$

where M is any nonsingular 4×4 matrix. Two especially useful bases are the Weyl basis and the Majorana basis. These are explained below. Note that $G^\mu = -\Gamma^\mu C^{-1}$ transforms as

$$G^\mu \longrightarrow \frac{1}{\sqrt{\det(M)}} M G^\mu M^T. \tag{E.2}$$

Given any basis, this allows us to find a similarity transformation to render all spinor components real (a Majorana basis), and moreover one for which $G^0_{AB} = \delta_{AB}$. The resultant basis is then specially tailored for dimensional reduction to 1D, for the simple reason that the four real components of the Majorana supercharge operator Q_A supply natural real 1D shadow supercharges, which satisfy the algebra $\{Q_A, Q_B\} = i\delta_{AB}\partial_\tau$.

E.1. The Weyl Basis

In the Weyl basis we choose 4×4 matrices using the following convention:

$$\begin{gathered}
\Gamma_0 = \left(\begin{array}{c|c} & -\mathbb{1} \\ \hline \mathbb{1} & \end{array}\right), \qquad \Gamma_a = \left(\begin{array}{c|c} & \sigma_a \\ \hline \sigma_a & \end{array}\right), \\
C = \left(\begin{array}{c|c} \varepsilon & \\ \hline & \varepsilon \end{array}\right), \qquad \Gamma_5 = \left(\begin{array}{c|c} \mathbb{1} & \\ \hline & -\mathbb{1} \end{array}\right),
\end{gathered} \tag{E.3}$$

where $\mathbb{1}$ is the 2×2 unit matrix, $a = 1, 2, 3$, and σ_a are the Pauli matrices and $\varepsilon = i\sigma_2$.

Right- and left-handed Weyl spinors satisfy the respective constraints $\Gamma_5 \psi_{R,L} = \pm\psi_{R,L}$. In terms of the Weyl basis (E.3), this means that right- and left-handed spinors are respectively configured as

$$\chi_R = \left(\begin{array}{c} \chi_1 \\ \chi_2 \\ \hline 0 \\ 0 \end{array}\right), \qquad \varphi_L = \left(\begin{array}{c} 0 \\ 0 \\ \hline \varphi_1 \\ \varphi_2 \end{array}\right), \tag{E.4}$$

where χ_1, χ_2, φ_1, and φ_2 are complex anticommuting fields. Note that Weyl spinors take an especially tidy form in the Weyl basis, since half of the four complex spinor components vanish.

A Majorana spinor satisfies $\psi = C^{-1}\overline{\psi}^T$. In terms of the Weyl basis (E.3), this means

$$\psi = \left(\begin{array}{c} \psi_1 \\ \psi_2 \\ \hline \psi_2^* \\ -\psi_1^* \end{array}\right), \tag{E.5}$$

where ψ_1 and ψ_2 are complex anticommuting fields. Note that Majorana spinors are relatively awkward in the Weyl basis.

E.2. The Majorana Basis

Change bases from the Weyl basis to the Majorana basis, using (E.1), by choosing

$$M = \frac{1}{2}\left(\begin{array}{cccc} 1 & & & -1 \\ -i & & & -i \\ \hline & 1 & 1 & \\ & -i & i & \end{array}\right). \tag{E.6}$$

Using the transformation (E.1), right- and left-handed Weyl spinors in the Weyl basis transform into right- and left-handed Weyl spinors in the Majorana basis, as specified respectively by

$$\chi_{R,L} = \left(\begin{array}{c} \chi_1 \\ \mp i\chi_1 \\ \hline \chi_2 \\ \mp i\chi_2 \end{array}\right), \tag{E.7}$$

where χ_1 and χ_2 are complex fields. Note that the difference between left- and right-handedness in this basis manifests in the relative phases appearing in (E.7). Note that Weyl spinors are relatively awkward in the Majorana basis.

Using the transformation (E.1), a Majorana spinor in the Weyl basis, (E.5), transforms into a Majorana spinor in the Majorana basis, as given by

$$\psi_A = \left(\begin{array}{c} \operatorname{Re}\psi_1 \\ \operatorname{Im}\psi_1 \\ \hline \operatorname{Re}\psi_2 \\ \operatorname{Im}\psi_2 \end{array}\right). \tag{E.8}$$

Note that Majorana spinors take an especially tidy form in the Majorana basis, since all four components are independent and real. In this basis, the gamma matrices and the charge conjugation matrices are

$$
\Gamma_0 = \left(\begin{array}{cccc} & & -1 & \\ & & & 1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right), \qquad \Gamma_1 = \left(\begin{array}{cccc} -1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right),
$$

$$
\Gamma_2 = \left(\begin{array}{cccc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right), \qquad \Gamma_3 = \left(\begin{array}{cccc} & & 1 & \\ & & & -1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right), \tag{E.9}
$$

$$
C = \left(\begin{array}{cccc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right), \qquad \Gamma_5 = \left(\begin{array}{cccc} & i & & \\ -i & & & \\ \hline & & & i \\ & & -i & \end{array}\right),
$$

as obtained by transforming (E.3) using (E.1). The corresponding G-Matrices $G^a_{AB} = -\eta^{ab}(\Gamma_b C^{-1})_{AB}$ are

$$
G^0 = \left(\begin{array}{cccc} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right), \qquad G^1 = \left(\begin{array}{cccc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right),
$$

$$
G^2 = \left(\begin{array}{cccc} & & & 1 \\ & & -1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right), \qquad G^3 = \left(\begin{array}{cccc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right). \tag{E.10}
$$

Note that G-matrices are symmetric, real, and traceless.

Also useful are the "boost" operators $\mathcal{B}^a = (1/2)\Gamma^0\Gamma^a$ and the "rotation" operators $\mathcal{R}_a = (1/4)\varepsilon_{abc}\Gamma^{bc}$. In the Majorana basis (E.9) these are

$$
\begin{gathered}
\mathcal{B}^1 = \frac{1}{2}\left(\begin{array}{cccc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right), \qquad \mathcal{R}_1 = \frac{1}{2}\left(\begin{array}{cccc} & & & -1 \\ & & 1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right), \\
\mathcal{B}^2 = \frac{1}{2}\left(\begin{array}{cccc} & & & 1 \\ & & -1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right), \qquad \mathcal{R}_2 = \frac{1}{2}\left(\begin{array}{cccc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right), \\
\mathcal{B}^3 = \frac{1}{2}\left(\begin{array}{cccc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right), \qquad \mathcal{R}_3 = \frac{1}{2}\left(\begin{array}{cccc} & -1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right).
\end{gathered}
\tag{E.11}
$$

Note that the boost operators are symmetric while the rotation operators are antisymmetric. Note too that $G^a = 2\mathcal{B}^a$. The operators in (E.11) satisfy the Lorentz algebra

$$
\begin{aligned}
[\mathcal{R}_a, \mathcal{R}_b] &= -\varepsilon^{c}_{ab}\mathcal{R}_c, \\
\left[\mathcal{B}^a, \mathcal{B}^b\right] &= \varepsilon^{abc}\mathcal{R}_c, \\
[\mathcal{B}^a, \mathcal{R}_b] &= -\varepsilon^{a}_{bc}\mathcal{B}^c.
\end{aligned}
\tag{E.12}
$$

The Lorentz algebra (E.12) can be written concisely, and in a manner which is manifestly covariant, as

$$
\left[M_{\mu\nu}, M^{\lambda\sigma}\right] = \delta^{\lambda}_{\mu}M^{\sigma}_{\nu} - \delta^{\sigma}_{\mu}M^{\lambda}_{\nu} + \delta^{\sigma}_{\nu}M^{\lambda}_{\mu} - \delta^{\lambda}_{\nu}M^{\sigma}_{\mu},
\tag{E.13}
$$

where $M^{0a} = \mathcal{B}^a$ and $M_{ab} = \varepsilon_{abc}\mathcal{R}^c$. (Note that using our conventions, $M_{0a} = \eta_{00}\eta_{ab}M^{0b} = -\delta_{ab}M^{0b}$, whereas $\mathcal{B}_a = \delta_{ab}\mathcal{B}^b$.)

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