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Research Article

Constancy of $\overline{\phi}$-Holomorphic Sectional Curvature for an Indefinite Generalized $g \cdot f \cdot f$-Space Form

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Bonome et al., 1997, provided an algebraic characterization for an indefinite Sasakian manifold to reduce to a space of constant ϕ-holomorphic sectional curvature. In this present paper, we generalize the same characterization for indefinite $g \cdot f \cdot f$-space forms.

1. Introduction

For an almost Hermitian manifold (M^{2n}, g, J) with $\dim(M) = 2n > 4$, Tanno [1] has proved the following.

Theorem 1.1. *Let* $\dim(M) = 2n > 4$, *and assume that almost Hermitian manifold* (M^{2n}, g, J) *satisfies*

$$R(JX, JY, JZ, JX) = R(X, Y, Z, X) \tag{1.1}$$

for every tangent vector X, Y, and Z. Then (M^{2n}, g, J) has a constant holomorphic sectional curvature at x if and only if

$$R(X, JX)X \text{ is proportional to } JX \tag{1.2}$$

for every tangent vector X at $x \in M$.

Tanno [1] has also proved an analogous theorem for Sasakian manifolds as follows.

Theorem 1.2. *A Sasakian manifold ≥5 has a constant ϕ-sectional curvature if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X \tag{1.3}$$

for every tangent vector X such that $g(X, \xi) = 0$.

Nagaich [2] has proved the generalized version of Theorem 1.1, for indefinite almost Hermitian manifolds as follows.

Theorem 1.3. *Let (M^{2n}, g, J) $(n > 2)$ be an indefinite almost Hermitian manifold that satisfies* (1.1)*, then (M^{2n}, g, J) has a constant holomorphic sectional curvature at x if and only if*

$$R(X, JX)X \text{ is proportional to } JX \tag{1.4}$$

for every tangent vector X at $x \in M$.

Bonome et al. [3] generalized Theorem 1.2 for an indefinite Sasakian manifold as follows.

Theorem 1.4. *Let $(M^{2n+1}, \phi, \eta, \xi, g)$ $(n \geq 2)$ be an indefinite Sasakian manifold. Then M^{2n+1} has a constant ϕ-sectional curvature if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X \tag{1.5}$$

for every vector field X such that $g(X, \xi) = 0$.

In this paper, we generalize Theorem 1.4 for an indefinite generalized $g \cdot f \cdot f$-space form by proving the following.

Theorem 1.5. *Let $(\overline{M}^{2n+r}, F_1, F_2, \mathcal{F})$ be an indefinite generalized $g \cdot f \cdot f$-space form. Then $\overline{M}^{2n+r}$ is of constant ϕ-sectional curvature if and only if*

$$\overline{R}\left(X, \overline{\phi} X\right)X \text{ is proportional to } \overline{\phi} X \tag{1.6}$$

for every vector field X such that $\overline{g}(X, \overline{\xi_\alpha}) = 0$, for any $\alpha \in \{1, \ldots, r\}$.

2. Preliminaries

A manifold $\overline{M}$ is called a *globally framed f-manifold (or $g \cdot f \cdot f$-manifold)* if it is endowed with a nonnull $(1,1)$-tensor field $\overline{\phi}$ of constant rank, such that $\ker \overline{\phi}$ is parallelizable; that is, there exist global vector fields $\overline{\xi_\alpha}$, $\alpha \in \{1, \ldots, r\}$, with their dual 1-forms $\overline{\eta}^\alpha$, satisfying $\overline{\phi}^2 = -I + \sum_{\alpha=1}^{r} \overline{\eta}^\alpha \otimes \overline{\xi_\alpha}$ and $\overline{\eta}^\alpha(\overline{\xi_\beta}) = \delta^\alpha_\beta$.

The $g \cdot f \cdot f$-manifold $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha)$, $\alpha \in \{1, \dots, r\}$, is said to be an indefinite metric $g \cdot f \cdot f$-manifold if $\overline{g}$ is a semi-Riemannian metric with index ν $(0 < \nu < 2n + r)$ satisfying the following compatibility condition:

$$\overline{g}\left(\overline{\phi}X, \overline{\phi}Y\right) = \overline{g}(X, Y) - \sum_{\alpha=1}^{r} \epsilon_\alpha \overline{\eta}^\alpha(X) \overline{\eta}^\alpha(Y), \tag{2.1}$$

for any $X, Y \in \Gamma(T\overline{M})$, being $\epsilon_\alpha = \pm 1$ according to whether $\overline{\xi}_\alpha$ is spacelike or timelike. Then, for any $\alpha \in \{1, \dots, r\}$, one has $\overline{\eta}^\alpha(X) = \epsilon_\alpha \overline{g}(X, \overline{\xi}_\alpha)$. Following the notations in [4, 5], we adopt the curvature tensor R, and thus we have $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ and $\overline{R}(X, Y, Z, W) = \overline{g}(\overline{R}(Z, W, Y), X)$, for any $X, Y, Z, W \in \Gamma(TM)$.

We recall that, as proved in [6], the Levi-Civita connection $\overline{\nabla}$ of an indefinite $g \cdot f \cdot f$-manifold satisfies the following formula:

$$\begin{aligned} 2\overline{g}\left(\left(\overline{\nabla}_X \overline{\phi}\right)Y, Z\right) &= 3d\Phi\left(X, \overline{\phi}Y, \overline{\phi}Z\right) - 3d\Phi(X, Y, Z) \\ &\quad + \overline{g}\left(N(Y, Z), \overline{\phi}X\right) + \epsilon_\alpha N_\alpha^{\overline{\phi}}(Y, Z)\overline{\eta}^\alpha(X) \\ &\quad + 2\epsilon_\alpha d\overline{\eta}^\alpha\left(\overline{\phi}Y, X\right)\overline{\eta}^\alpha(Z) - 2\epsilon_\alpha d\overline{\eta}^\alpha\left(\overline{\phi}Z, X\right)\overline{\eta}^\alpha(Y), \end{aligned} \tag{2.2}$$

where $N_\alpha^{\overline{\phi}}$ is given by $N_\alpha^{\overline{\phi}}(X, Y) = 2d\overline{\eta}^\alpha(\overline{\phi}X, Y) - 2d\overline{\eta}^\alpha(\overline{\phi}Y, X)$.

An indefinite metric $g \cdot f \cdot f$-manifold is called an *indefinite $\mathcal{S}$-manifold* if it is normal and $d\overline{\eta}^\alpha = \Phi$, for any $\alpha \in \{1, \dots, r\}$, where $\Phi(X, Y) = \overline{g}(X, \overline{\phi}Y)$ for any $X, Y \in \Gamma(T\overline{M})$. The normality condition is expressed by the vanishing of the tensor field $N = N_{\overline{\phi}} + \sum_{\alpha=1}^{r} 2d\overline{\eta}^\alpha \otimes \overline{\xi}_\alpha$, $N_{\overline{\phi}}$ being the Nijenhuis torsion of $\overline{\phi}$.

Furthermore, the Levi-Civita connection of an indefinite $\mathcal{S}$-manifold satisfies

$$\left(\overline{\nabla}_X \overline{\phi}\right)Y = \overline{g}\left(\overline{\phi}X, \overline{\phi}Y\right)\overline{\xi} + \overline{\eta}(Y)\overline{\phi}^2(X), \tag{2.3}$$

where $\overline{\xi} = \sum_{\alpha=1}^{r} \overline{\xi}_\alpha$ and $\overline{\eta} = \sum_{\alpha=1}^{r} \epsilon_\alpha \overline{\eta}^\alpha$. We recall that $\overline{\nabla}_X \overline{\xi}_\alpha = -\epsilon_\alpha \overline{\phi}X$ and $\ker \overline{\phi}$ is an integrable flat distribution since $\overline{\nabla}_{\overline{\xi}_\alpha} \overline{\xi}_\beta = 0$ (see more details in [6]).

A plane section in $T_p\overline{M}$ is a $\overline{\phi}$-holomorphic section if there exists a vector $X \in T_p\overline{M}$ orthogonal to $\overline{\xi}_1, \dots, \overline{\xi}_r$ such that $\{X, \overline{\phi}X\}$ span the section. The sectional curvature of a $\overline{\phi}$-holomorphic section, denoted by $c(X) = R(X, \overline{\phi}X, \overline{\phi}X, X)$, is called a $\overline{\phi}$-holomorphic sectional curvature.

Proposition 2.1 (see [7]). *An indefinite Sasakian manifold* $(\overline{M}^{2n+1}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *has* $\overline{\phi}$*-sectional curvature* c *if and only if its curvature tensor verifies*

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{(c+3)}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\ & + \frac{(c-1)}{4}\Big\{\Phi(X,Z)\overline{\phi}Y - \Phi(Y,Z)\overline{\phi}X + 2\Phi(X,Y)\ \overline{\phi}Z \\ & - \overline{g}(Z,Y)\overline{\eta}(X)\overline{\xi} + \overline{g}(Z,X)\overline{\eta}(Y)\overline{\xi} - \overline{\eta}(Y)\overline{\eta}(Z)X + \overline{\eta}(Z)\overline{\eta}(X)Y\Big\}\end{aligned} \tag{2.4}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$.

A Sasakian manifold $\overline{M}^{2n+1}$ *with constant* $\overline{\phi}$*-sectional curvature* $c \in \mathbb{R}$ *is called a Sasakian space form, denoted by* $\overline{M}^{2n+1}(c)$.

Definition 2.2. An almost contact metric manifold $(\overline{M}^{2n+1}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is an *indefinite generalized Sasakian space form*, denoted by $\overline{M}^{2n+1}(f_1, f_2,\ \text{and}\ f_3)$, if it admits three smooth functions f_1, f_2, f_3 such that its curvature tensor field verifies

$$\begin{aligned}\overline{R}(X,Y)Z = {} & f_1\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\ & + f_2\Big\{\Phi(X,Z)\overline{\phi}Y - \Phi(Y,Z)\overline{\phi}X + 2\Phi(X,Y)\overline{\phi}Z\Big\} \\ & + f_3\Big\{-\overline{g}(Z,Y)\overline{\eta}(X)\overline{\xi} + \overline{g}(Z,X)\overline{\eta}(Y)\overline{\xi} \\ & - \overline{\eta}(Y)\overline{\eta}(Z)X + \overline{\eta}(Z)\overline{\eta}(X)Y\Big\}\end{aligned} \tag{2.5}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$.

Remark 2.3. Any indefinite generalized Sasakian space form has $\overline{\phi}$-sectional curvature $c = f_1 + 3f_2$. Indeed, $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4$.

Proposition 2.4 (see [6]). *An indefinite* $\mathcal{S}$*-manifold* $\overline{M}^{2n+r}$ *has* $\overline{\phi}$*-sectional curvature* c *if and only if its curvature tensor verifies*

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{(c+3\epsilon)}{4}\Big\{\overline{g}\Big(\overline{\phi}X,\overline{\phi}Z\Big)\overline{\phi}^2Y - \overline{g}\Big(\overline{\phi}Y,\overline{\phi}Z\Big)\overline{\phi}^2X\Big\} \\ & + \frac{(c-\epsilon)}{4}\Big\{\Phi(Z,Y)\overline{\phi}X - \Phi(Z,X)\overline{\phi}Y + 2\Phi(X,Y)\overline{\phi}Z\Big\} \\ & + \Big\{\overline{\eta}(Z)\overline{\eta}(X)\overline{\phi}^2Y - \overline{\eta}(Y)\overline{\eta}(Z)\overline{\phi}^2X + \overline{g}\Big(\overline{\phi}Z,\overline{\phi}Y\Big)\overline{\eta}(X)\overline{\xi} - \overline{g}\Big(\overline{\phi}Z,\overline{\phi}X\Big)\overline{\eta}(Y)\overline{\xi}\Big\}\end{aligned} \tag{2.6}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$ *and* $\epsilon = \sum \epsilon_\alpha$.

An indefinite $\mathcal{S}$-manifold $\overline{M}^{2n+r}$ with constant $\overline{\phi}$-sectional curvature $c \in \mathbb{R}$ is called a $\mathcal{S}$-space form, denoted by $\overline{M}^{2n+r}(c)$. One remarks that for $r = 1$ (2.6) reduces to (2.4).

3. An Indefinite Generalized $g \cdot f \cdot f$-Manifold

Let $\mathcal{F}$ denote any set of smooth functions F_{ij} on $\overline{M}^{2n+r}$ such that $F_{ij} = F_{ji}$ for any $i, j \in \{1, \ldots, r\}$.

Definition 3.1. An *indefinite generalized $g \cdot f \cdot f$-space-form*, denoted by $(\overline{M}^{2n+r}, F_1, F_2, \mathcal{F})$, is an indefinite $g \cdot f \cdot f$-manifold $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$ which admits smooth function $F_1, F_2, \mathcal{F}$ such that its curvature tensor field verifies

$$\begin{aligned}\overline{R}(X,Y)Z = {} & F_1\left\{\overline{g}\left(\overline{\phi}X,\overline{\phi}Z\right)\overline{\phi}^2 Y - \overline{g}\left(\overline{\phi}Y,\overline{\phi}Z\right)\overline{\phi}^2 X\right\} \\ & + F_2\left\{\Phi(Z,Y)\overline{\phi}X - \Phi(Z,X)\overline{\phi}Y + 2\Phi(X,Y)\overline{\phi}Z\right\} \\ & + \sum_{\alpha,\beta=1}^{r} F_{\alpha\beta}\left\{\overline{\eta}^\alpha(X)\overline{\eta}^\beta(Z)\overline{\phi}^2 Y - \overline{\eta}^\alpha(Y)\overline{\eta}^\beta(Z)\overline{\phi}^2 X\right. \\ & \left. + \overline{g}\left(\overline{\phi}Z,\overline{\phi}Y\right)\overline{\eta}^\alpha(X)\overline{\xi_\beta} - \overline{g}\left(\overline{\phi}Z,\overline{\phi}X\right)\overline{\eta}^\alpha(Y)\overline{\xi_\beta}\right\}\end{aligned} \tag{3.1}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$.

For $r = 1$, we obtain an indefinite Sasakian space form $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with $f_1 = F_1$, $f_2 = F_2$, and $f_3 = F_1 - F_{11}$. In particular, if the given structure is Sasakian, (3.1) holds with $F_{11} = 1$, $F_1 = (c+3)/4$, $F_3 = (c-1)/4$, and $f_3 = F_1 - F_{11} = (c-1)/4 = f_2$.

Theorem 3.2. *Let $(\overline{M}^{2n+r}, F_1, F_2, \mathcal{F})$ be an indefinite generalized $g \cdot f \cdot f$-space form. Then $\overline{M}^{2n+r}$ is of constant ϕ-sectional curvature if and only if*

$$\overline{R}\left(X,\overline{\phi}X\right)X \text{ is proportional to } \overline{\phi}X \tag{3.2}$$

for every vector field X such that $\overline{g}(X, \overline{\xi_\alpha}) = 0$, for any $\alpha \in \{1, \ldots, r\}$.

Proof. Let $(\overline{M}^{2n+r}, F_1, F_2, \mathcal{F})$ be an indefinite generalized $g \cdot f \cdot f$-space form. To prove the theorem for $n \geq 2$, we will consider cases when $n = 2$ and when $n > 2$, that is, when $n \geq 3$.

Case 1 ($\overline{g}(X,X) = \overline{g}(Y,Y)$). The proof is similar as given by Lee and Jin [8], so we drop the proof.

Case 2 ($\overline{g}(X,X) = -\overline{g}(Y,Y)$). Here, if X is spacelike, then Y is timelike or vice versa. First of all, assume that $\overline{M}$ is of constant $\overline{\phi}$-holomorphic sectional curvature. Then (3.1) gives

$$\overline{R}\left(X,\overline{\phi}X\right)X = \{F_1 + 3F_2\}\overline{\phi}X = c\overline{\phi}X. \tag{3.3}$$

Conversely, let $\{X, Y\}$ be an orthonormal pair of tangent vectors such that $\overline{g}(\overline{\phi}X, Y) = \overline{g}(X, Y) = \overline{g}(Y, \overline{\xi_\alpha}) = 0$, $\alpha \in \{1, \ldots, r\}$, and $n \geq 3$. Then $\ddot{X} = (X + iY)/\sqrt{2}$ and $\ddot{Y} = (i\overline{\phi}X + \overline{\phi}Y)/\sqrt{2}$ also form an orthonormal pair of tangent vectors such that $\overline{g}(\overline{\phi}\ddot{X}, \ddot{Y}) = 0$. Then (3.1) and curvature properties give

$$\begin{aligned} 0 &= \overline{R}\left(\ddot{X}, \overline{\phi}\ddot{X}, \ddot{Y}, \ddot{X}\right) \\ &= \overline{g}\left(\overline{R}\left(X, \overline{\phi}X\right)X, \overline{\phi}X\right) - \overline{g}\left(\overline{R}\left(Y, \overline{\phi}Y\right)Y, \overline{\phi}Y\right) \\ &\quad - 2\overline{g}\left(\overline{R}\left(X, \overline{\phi}Y\right)Y, \overline{\phi}Y\right) + 2\overline{g}\left(\overline{R}\left(X, \overline{\phi}X\right)Y, \overline{\phi}X\right). \end{aligned} \tag{3.4}$$

From the assumption, we see that the last two terms of the right-hand side vanish. Therefore, we get $c(X) = c(Y)$.

Now, if $\operatorname{span}\{U, V\}$ is $\overline{\phi}$-holomorphic, then for $\overline{\phi}U = aU + bV$, where a and b are constant, we have

$$\operatorname{span}\left\{U, \overline{\phi}U\right\} = \operatorname{span}\{U, aU + bV\} = \operatorname{span}\{U, V\}. \tag{3.5}$$

Similarly,

$$\operatorname{span}\left\{V, \overline{\phi}V\right\} = \operatorname{span}\{U, V\}, \qquad \operatorname{span}\left\{U, \overline{\phi}U\right\} = \operatorname{span}\left\{V, \overline{\phi}V\right\}. \tag{3.6}$$

These imply

$$\overline{R}\left(U, \overline{\phi}U, U, \overline{\phi}U\right) = \overline{R}\left(V, \overline{\phi}V, V, \overline{\phi}V\right), \quad \text{or} \quad c(U) = c(V). \tag{3.7}$$

If $\operatorname{span}\{U, V\}$ is not $\overline{\phi}$-holomorphic section, then we can choose unit vectors $X \in \operatorname{span}\{U, \overline{\phi}U\}^\perp$ and $Y \in \operatorname{span}\{V, \overline{\phi}V\}^\perp$ such that $\operatorname{span}\{X, Y\}$ is $\overline{\phi}$-holomorphic. Thus we get

$$c(U) = c(X) = c(Y) = c(V), \tag{3.8}$$

which shows that any $\overline{\phi}$-holomorphic section has the same $\overline{\phi}$-holomorphic sectional curvature.

Now, let $n = 2$, and let $\{X, Y\}$ be a set of orthonormal vectors such that $\overline{g}(X, X) = -\overline{g}(Y, Y)$ and $\overline{g}(X, \overline{\phi}X) = 0$, and we have $c(X) = c(Y)$ as before. Using the property (3.2), we get

$$\begin{gathered}
\overline{R}\left(X, \overline{\phi}X\right)X = -\{F_1 + 3F_2\}\overline{\phi}X = -c(X)\overline{\phi}X, \\
\overline{R}\left(X, \overline{\phi}X\right)Y = -2F_2\overline{\phi}Y, \\
\overline{R}\left(X, \overline{\phi}Y\right)X = -F_1\overline{\phi}Y, \\
\overline{R}\left(X, \overline{\phi}Y\right)Y = F_2\overline{\phi}X, \\
\overline{R}\left(Y, \overline{\phi}X\right)Y = F_1\overline{\phi}X, \\
\overline{R}\left(Y, \overline{\phi}X\right)X = -F_2\overline{\phi}Y, \\
\overline{R}\left(Y, \overline{\phi}Y\right)X = 2F_2\overline{\phi}X, \\
\overline{R}\left(Y, \overline{\phi}Y\right)Y = \{F_1 + 3F_2\}\overline{\phi} = c(Y)\overline{\phi}Y = c(X)\overline{\phi}Y.
\end{gathered} \tag{3.9}$$

Now, define $\widehat{X} = aX + bY$ such that $a^2 - b^2 = 1$ and $a^2 \neq b^2$. Using the above relations, we get

$$R\left(\widehat{X}, \overline{\phi}\widehat{X}\right)\widehat{X} = C_1\overline{\phi}X + C_2\overline{\phi}Y. \tag{3.10}$$

Therefore, we have

$$\begin{gathered}
C_1 = -a^3c(X) + ab^2c(X), \\
C_2 = b^3c(X) - a^2bc(X).
\end{gathered} \tag{3.11}$$

On the other hand,

$$\overline{R}\left(\widehat{X}, \overline{\phi}\widehat{X}\right)\widehat{X} = c\left(\widehat{X}\right)\overline{\phi}\widehat{X} = c\left(\widehat{X}\right)\left\{a\overline{\phi}X + b\overline{\phi}Y\right\}. \tag{3.12}$$

Comparing (3.11) and (3.12), we get

$$\begin{gathered}
-a^2c(X) + b^2c(X) = c\left(\widehat{X}\right), \\
b^2c(X) - a^2c(X) = c\left(\widehat{X}\right).
\end{gathered} \tag{3.13}$$

On solving (3.13), we have

$$c(X) = c\left(\widehat{X}\right). \tag{3.14}$$

Similary, we can prove

$$c(Y) = c\left(\widehat{Y}\right). \tag{3.15}$$

Therefore, $\overline{M}$ has constant $\overline{\phi}$-holomorphic sectional curvature.

Case 3 ($\overline{g}(U,U) = 0$). It is enough to show a sufficient condition. Let Y_α be a unit vector tangent to $\overline{\xi}_\alpha$, for any $\alpha \in \{1,\ldots,r\}$, such that $\overline{g}(Y_\alpha, Y_\alpha) = -\overline{g}(\xi_\alpha, \xi_\alpha) = -\epsilon_\alpha$, and consider the null vector $U_\alpha = \xi_\alpha + Y$. From (3.2),

$$\begin{aligned} c(U_\alpha)\overline{\phi}U_\alpha &= c(U_\alpha)\overline{\phi}(\xi_\alpha + Y_\alpha) \\ &= \overline{R}\left(\xi_\alpha + Y_\alpha, \overline{\phi}(\xi_\alpha + Y_\alpha)\right)(\xi_\alpha + Y_\alpha). \end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned} c(U_\alpha) &= \overline{g}\left(c(U_\alpha)\overline{\phi}(\xi_\alpha + Y_\alpha), \epsilon_\alpha\overline{\phi}Y_\alpha\right) \\ &= \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha + Y_\alpha, \overline{\phi}(\xi_\alpha + Y_\alpha)\right)(\xi_\alpha + Y_\alpha), \overline{\phi}Y_\alpha\right) \\ &= \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}\xi_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}\xi_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}\xi_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}\xi_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) + 2\epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) \\ &= \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right). \end{aligned} \tag{3.17}$$

From Cases 1 and 2, depending on the sign of ϵ_α, $\overline{g}(\overline{R}(Y_\alpha, \overline{\phi}Y_\alpha)Y_\alpha,\ \overline{\phi}Y_\alpha) = \epsilon_\alpha c(Y_\alpha)$ is constant, and hence $c(U_\alpha) = c(Y_\alpha)$ is constant. □

Theorem 3.3 (see [9]). *Let* $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\eta}^{\alpha}, \overline{\xi_{\alpha}}, \overline{g})(n \geq 2)$ *be an indefinite* $\mathcal{S}$*-manifold. Then* M^{2n+r} *is of constant* ϕ*-sectional curvature if and only if*

$$R\Big(X, \overline{\phi} X\Big) X \text{ is proportional to } \overline{\phi} X \tag{3.18}$$

for every vector field X *such that* $g(X, \overline{\xi_{\alpha}}) = 0$*, for any* $\alpha \in \{1, \ldots, r\}$.

Proof. An $\mathcal{S}$-space form is a special case of $g \cdot f \cdot f$-space form, and hence the proof follows from Theorem 3.2 and (2.6). □

Theorem 3.4 (cf. Bonome et al. [3]). *Let* $(M^{2n+1}, \phi, \eta, \xi, g)(n \geq 2)$ *be an indefinite Sasakian manifold. Then* M^{2n+1} *is of constant* ϕ*-sectional curvature if and only if*

$$R(X, \phi X) X \text{ is proportional to } \phi X \tag{3.19}$$

for every vector field X *such that* $g(X, \xi) = 0$.

Proof. When $r = 1$, an indefinite $\mathcal{S}$-space form $M^{2n+1}(c)$ reduces to a Sasakian space form. The proof follows from (2.4) and Theorem 3.3. □

References

[1] S. Tanno, "Constancy of holomorphic sectional curvature in almost Hermitian manifolds," *Kodai Mathematical Seminar Reports*, vol. 25, pp. 190–201, 1973.

[2] R. K. Nagaich, "Constancy of holomorphic sectional curvature in indefinite almost Hermitian manifolds," *Kodai Mathematical Journal*, vol. 16, no. 2, pp. 327–331, 1993.

[3] A. Bonome, R. Castro, E. Garcia-Rio, and L. Hervella, "Curvature of indefinite almost contact manifolds," *Journal of Geometry*, vol. 58, no. 1-2, pp. 66–86, 1997.

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Interscience Publishers, New York, NY, USA, 1963.

[5] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2, Interscience Publishers, New York, NY, USA, 1969.

[6] L. Brunetti and A. M. Pastore, "Curvature of a class of indefinite globally framed f-manifolds," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 51(99), no. 3, pp. 183–204, 2008.

[7] T. Ikawa and J. B. Jun, "On sectional curvatures of normal contact Lorentzian manifold," *Journal of the Korean Mathematical Society*, vol. 4, pp. 27–33, 1997.

[8] J. W. Lee and D. H. Jin, "Constancy of ϕ—holomorphic scetional curvature in generalized $g \cdot f \cdot f$—manifolds ," http://arxiv.org/abs/1103.5266v1 .

[9] J. W. Lee, "Constancy of ϕ-sectional curvature in indefinite S-manifolds," *British Journal of Mathematics and Computer Science*, vol. 1, no. 3, pp. 121–128, 2011.