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Research Article

Quantum Groupoids Acting on Semiprime Algebras

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Following Linchenko and Montgomery's arguments we show that the smash product of an involutive weak Hopf algebra and a semiprime module algebra, satisfying a polynomial identity, is semiprime.

1. Introduction

Group actions, Lie algebras acting as derivations and finite group gradings are typical examples of Hopf algebra actions which have been studied for many years. Several generalizations of Hopf algebras have emerged in recent years, like weak Hopf algebras (or quantum groupoids) introduced by Böhm et al. [1]. The action of such objects on algebras, as given by quantum groupoids acting on C^*-algebras, [2] or weak Hopf algebras arising from Jones towers [3] are particularly interesting. New examples of weak Hopf algebras arose from double groupoids [4], which were also used to find new weak Hopf actions (see [2]).

A long-standing open problem in the theory of Hopf action is to show that the smash product $A\#H$ of a semiprime module algebra A and a semisimple Hopf algebra H is again semiprime (see [5]) (an algebra A is semiprime if it does not contain nonzero nilpotent ideals.). The case of A being commutative had been settled in [6]. The most recent partial answer to this problem has been given by Linchenko and Montgomery in [7] where they prove the semiprimness of $A\#H$ under the condition of A satisfying a polynomial identity. The purpose of this note is that their result carries over to actions of weak Hopf algebras. We reach more generality by considering actions of linear operators that satisfy certain intertwining relations with the regular multiplications on the algebra.

Let k be a commutative ring and let A be an associative unital k-algebra. For any $a \in A$ define two linear operators $L(a)$ and $R(a)$ in $\mathrm{End}_k(A)$ given by $\langle L(a), x\rangle = ax$ and $\langle R(a), x\rangle = xa$ for all $x \in A$. We identify A with the subalgebra $L(A)$ of $\mathrm{End}_k(A)$ generated by all left multiplications $L(a)$ and denote the subalgebra generated by all operators $L(a)$ and $R(a)$ by $M(A)$, which is also sometimes referred to as the *multiplication algebra* of A. As a left $L(A)$-module, A is isomorphic to $L(A)$ since we assume A to be unital. We will be interested in certain actions on an algebra A that may stem from a bialgebra or more generally a bialgebroid. The situation we will encounter is the one where we have an extension $A \subseteq B$ where B acts on A through a ring homomorphism $\phi : B \to \mathrm{End}_k(A)$ such that $(a)\phi = L(a)$ for all $a \in A$. For the intrinsic properties of A under this action it is enough to look at the subalgebra $\phi(B)$ in $\mathrm{End}_k(A)$ generated by this action and we might consider intermediate algebras $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ instead. Hence let B be a subalgebra of $\mathrm{End}_k(A)$ that contains $L(A)$. Then A becomes a cyclic faithful left B-module by evaluating endomorphisms, that is, for all $b \in B, a \in A : b \cdot a := \langle b, a\rangle$. Note that for any $a' \in A$ we have $L(a') \cdot a = \langle L(a'), a\rangle = a'a$.

Since we assume A to be unital, the map $\Psi : \mathrm{End}_B(A) \to A$ with $\Psi(f) = (1)f$, for all $f \in \mathrm{End}_B(A)$—evaluating endomorphisms at 1—is an injective ring homomorphism, since for all $f, g \in \mathrm{End}_B(A) : \Psi(f \circ g) = ((1)f)g = ((1)f \cdot 1)g = (1)f \cdot (1)g = \Psi(f)\Psi(g)$. Moreover if $\Psi(f) = (1)f = 0$, then $(a)f = a(1)f = 0$ and $f = 0$. The subalgebra $\Psi(\mathrm{End}_B(A))$ can be described as the set of elements $a \in A$ such that for any $b \in B : b \cdot a = (b \cdot 1)a$, which we will denote by A^B. On one hand if $a = (1)f = \Psi(f)$ for some $f \in \mathrm{End}_B(A)$, then for any $b \in B : b \cdot a = (b \cdot 1)f = L((b \cdot 1)) \cdot (1)f = (b \cdot 1)a$ and on the other hand if $a \in A^B$, then $f = R(a)$ is left B-linear since for any $b \in B$ and $x \in A$:

$$\begin{aligned} b \cdot (x)f &= b \cdot (xa) = b \cdot (L(x) \cdot a) = (bL(x)) \cdot a = (bL(x) \cdot 1)a \\ &= (b \cdot (L(x) \cdot 1))a = (b \cdot x)a = (b \cdot x)f. \end{aligned} \tag{1.1}$$

Thus $\mathrm{End}_B(A) \simeq A^B$.

Let M be any left B-module and define $M^B = \{m \in M : \forall b \in B : b \cdot m = (b \cdot 1)m\}$. With the same argument as above one sees that $\Psi_M : \mathrm{Hom}_B(A, M) \to M^B$ with $\Psi(f) = (1)f$ is an isomorphism of abelian groups, hence yielding a left A^B-module structure on M^B. Moreover it is possible to show that $\mathrm{Hom}_B(A, -)$ is isomorphic to $(-)^B$ as functors from B-Mod to A^B-Mod.

A subset I of A is called B-stable if $B \cdot I \subseteq I$. The B-stable left ideals are precisely the (left) B-submodules of A. In particular $\mathrm{Hom}_B(A, I) \simeq I^B = I \cap A^B$, for any B-stable left ideal of A.

Examples 1.1. The following list illustrates that our aproach reflects many interesting cases of algebras with actions.

(1) Let $B = M(A)$ be the multiplication algebra of A, then A is a faithful cyclic left B-module. The left B-modules are precisely the A-bimodules, in particular the left ideals of A are the two-sided ideals of A, and $\mathrm{Hom}_B(A, M) \simeq M^B = Z(M) = \{m \in M \mid \forall a \in A : am = ma\}$ holds for any A-bimodule M. The operator algebra B is a quotient of the enveloping algebra $A^e = A \otimes A^{\mathrm{op}}$ through the map $a \otimes b \mapsto L(a) \circ R(b)$, for all $a \otimes b \in A^e$.

(2) Let G be a group acting as (k-linear) automorphisms on A, that is, there exists a group homomorphism $\eta : G \to \mathrm{Aut}_k(A)$. Set ${}^g a = \langle \eta(g), a\rangle$ for any $a \in A, g \in G$.

Define $B = \langle L(A) \cup \{\eta(g) \mid g \in G\}\rangle \subseteq \mathrm{End}_k(A)$. Then the left B-submodules I of A are precisely the G-stable left ideals of A and $\mathrm{Hom}_B(A, I) \simeq I^B = \{x \in I \mid {}^g x = x\}$. B is a quotient of the skew group ring $A\#G$ whose underlying A-submodule is the free left A-module with basis $\{\overline{g} \mid g \in G\}$ and whose multiplication is given by $(a\#\overline{g})(b\#\overline{h}) = a^g b\#\overline{gh}$. Note that for any left $A\#G$-module M we have, $\mathrm{Hom}_{A\#G}(A, M) = \mathrm{Hom}_B(A, M) \simeq M^G$ is the set of fixed elements of M.

(3) Let A be an k-algebra with involution $*$ and let B be the subalgebra of $\mathrm{End}_A(\ )$ generated by A and $*$. Since for any $a \in A : R(a) = * \circ L(a^*) \circ *$ we got $M(A) \subseteq B$. This means (as it is well-known) that any left ideal of A which is stable under $*$ is a two-sided ideal. Note that B can be seen as the factor ring of the skew-group ring $A^e\#G$ where $G = \{\mathrm{id}, \overline{*}\}$ is the cyclic group of order two and $\overline{*} \in \mathrm{Aut}(A^e)$ is given by $(a \otimes b)^{\overline{*}} := b^* \otimes a^*$.

(4) Let $\delta \in \mathrm{Der}_k(A)$ be an k-linear derivation of A and consider $B = \langle L(A) \cup \{\delta\}\rangle \subseteq \mathrm{End}_k(A)$. The left B-submodules of A are the left ideals I that satisfy $\delta(I) \subseteq I$. The operator algebra B is a factor of the ring of differential operator $A[z;\delta]$, which as a left A-module is equal to $A[z]$ and its multiplication is given by $za = az + \delta(a)$. The map $A[z;\delta] \to B$ with $\sum_{i=0}^n a_i z^i \mapsto \sum_{i=0}^n L(a_i) \circ \delta^i \in B$ is a surjective k-algebra homomorphism and for any left $A[z;\delta]$-module M we have $\mathrm{Hom}_{A[z;\delta]}(A, M) = \mathrm{Hom}_B(A, M) = M^\delta = \{m \in M \mid zm = 0\}$. In particular $\mathrm{End}_{A[z;\delta]}(A) \simeq A^\delta = \mathrm{Ker}(\delta)$. The subring A^δ of A is called the ring of constants of δ.

(5) Let H be an k-Hopf algebra action on A. Let us denote the action of an element $h \in H$ on A by $\lambda_h \in \mathrm{End}_k(A)$ and define $B = \langle L(A) \cup \{\lambda_h \mid h \in H\}\rangle \subseteq \mathrm{End}_k(A)$. The smash product $A\#H$ is an extension with additional module structure. Define $\varphi : A\#H \to \mathrm{End}_k(A)$ by $\varphi(a\#h) := L(a) \circ \lambda_h$.

2. Linear Operators Acting on Algebras Satisfying a Polynomial Identity

Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ be any intermediate algebra as above.

The first technical lemma generalizes a corresponding result of Linchenko [8, Theorem 3.1] for Hopf actions and Nikshych [9, Theorem 6.1.3] for weak Hopf actions. Recall that an ideal whose elements are nilpotent is called a nil ideal.

Lemma 2.1. *Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ and suppose that for all $\psi \in B$ there exist $m \geq 1$ and elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ such that*

$$L(\langle \psi, a\rangle) = \sum_{i=1}^m \psi_i^1 \circ L(a) \circ \psi_i^2, \sum_{i=1}^n \psi_i^2 \circ \psi_i^1 \in L(A) \tag{2.1}$$

for any $a \in A$. If A is finite dimensional over a field of characteristic 0 and if I is a nil ideal, then $B \cdot I$ is nil. In particular the Jacobson radical of A is B-stable.

Proof. Denote the trace of a k-linear endomorphism f of A by $\mathrm{Tr}(f)$. Let $\psi \in B$, $a \in A$. Using $\mathrm{Tr}(fgh) = \mathrm{Tr}(hfg)$ and the hypotheses we get

$$\mathrm{Tr}\big(L(\langle \psi, a\rangle)\big) = \mathrm{Tr}\left(\left(\sum_{i=1}^{n}\psi_i^2 \circ \psi_i^1\right) \circ L(a)\right) = \mathrm{Tr}\big(L(y) \circ L(a)\big) = \mathrm{Tr}\big(L(ya)\big) \tag{2.2}$$

for some $y \in A$. Suppose that $a \in I$ with I a nil ideal, then $ya \in I$ is nilpotent, hence $\mathrm{Tr}(L(\langle \psi, a\rangle)) = \mathrm{Tr}(L(ya)) = 0$. For any $k > 0$ set $z^k := \langle \psi, a\rangle^k$. Then

$$z^k = \left\langle L(\langle \psi, a\rangle), z^{k-1}\right\rangle = \sum_{i=1}^{n}\left\langle \psi_i^1, a_i\right\rangle \tag{2.3}$$

for $a_i = a\langle \psi_i^2, z^{k-1}\rangle$. Since I is an ideal, $a_i \in I$. Hence

$$\mathrm{Tr}\Big(L(z)^k\Big) = \mathrm{Tr}\Big(L\Big(z^k\Big)\Big) = \sum_{i=1}^{n}\mathrm{Tr}\Big(L\Big(\Big\langle \psi_i^1, a_i\Big\rangle\Big)\Big) = 0. \tag{2.4}$$

Since A is finite dimensional, $\mathrm{char}(k) = 0$ and the trace of all powers of $L(z)$ is zero, $L(z)$ is a nilpotent operator, that is, $z = \langle \psi, a\rangle$ is nilpotent. Thus $B \cdot I$ is a nil ideal. Since the Jacobson radical of an Artinian ring is the largest nilpotent ideal, we have $B \cdot \mathrm{Jac}(A) = \mathrm{Jac}(A)$. □

The last lemma, which had been proven first by Linchenko for Hopf actions and then by Nikshych for weak Hopf actions allows us to show the stability of the Jacobson radical of an algebra A which satisfies a polynomial identity and on which act some operator algebra B which is finitely generated over A. The hypotheses of the following theorem allow the reduction to finite-dimensional factors.

Theorem 2.2. *Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ over some field k of characteristic 0 with B being finitely generated as right A-module. Suppose that for all $\psi \in B$ there exist $n \geq 1$ and elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ satisfying*

$$L(\langle \psi, a\rangle) = \sum_{i=1}^{m}\psi_i^1 \circ L(a) \circ \psi_i^2, \sum_{i=1}^{m}\psi_i^2 \circ \psi_i^1 \in L(A) \tag{2.5}$$

for any $a \in A$. If A satisfies a polynomial identity or if k is an uncountable algebraically closed field, A countably generated and all left primitive factor rings of A are Artinian, then $B \cdot I \subseteq \mathrm{Jac}(A)$ for all nil ideals I of A.

Proof. Let I be a nil ideal. It is enough to show that $(B \cdot I)V = 0$ for all simple left A-modules V, then $B \cdot I \subseteq \mathrm{Jac}(A)$. Let V be a simple left A-module and $P = \mathrm{Ann}_A(V)$ be its annihilator. If k is an uncountable algebraically closed field and A is countably generated, then it satisfies the Nullstellensatz, hence $\mathrm{End}_k(V) = k$ (see [10, 9.1.8]). If primitive factors of A are Artinian, then by the Weddeburn-Artin Theorem $A/P \simeq M_n(k)$ for some n, hence V is a finite-dimensional simple left A-module. On the other hand, if A satisfies a polynomial identity, then $A/P \simeq M_n(D)$ where D is a finite-dimensional division algebra over $F = Z(A/P)$ by Kaplansky's theorem [10, 13.3.8]. Tensoring A by F yields an F-algebra $\widetilde{A} := A \otimes F$ with F-action on the right. Then

$$L\left(\widetilde{A}\right) = L(A) \otimes F \subseteq B \otimes F \subseteq \operatorname{End}_k(A) \otimes F \subseteq \operatorname{End}_F(A \otimes F) = \operatorname{End}_F\left(\widetilde{A}\right). \tag{2.6}$$

Moreover V is a finite-dimensional simple left $\widetilde{A}$-module since $\operatorname{Ann}_{\widetilde{A}}(V) = P \otimes F =: \widetilde{P}$ and $\widetilde{A}/\widetilde{P} \simeq M_n(D)$ is finite dimensional over F. Note also that the nil ideal I extends to a nil ideal $\widetilde{I} := I \otimes F$ since by [11, Theorem 5] I is a locally nilpotent algebra and hence any element $\sum_{i=1}^{m} a_i \otimes f_i$ lies in a nilpotent finitely generated subalgebra generated by the a_i's and F.

To summarize, our hypothesis on A allows us to consider V to be a finite dimensional simple left A-module, where A and B are algebras over some field k of characteristic 0. Denote by $W = B \otimes_A V$ the induced left B-module. Since B_A is finitely generated and V is finite dimensional, W is finite dimensional. Note that the left B-action on W is given by $\psi \cdot (\phi \otimes v) := (\psi \circ \phi) \otimes v$. Let $Q = \operatorname{Ann}_A(W)$. Then Q is B-stable, because if $a \in Q$ and $\psi \in B$, then by hypothesis there exist elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ satisfying (2.5). Thus for any $w = \phi \otimes v \in W$ we have

$$\begin{aligned}\langle \psi, a \rangle \cdot (\phi \otimes v) &= \left(L(\langle \psi, a \rangle) \circ \phi\right) \otimes v \\ &= \sum_{i=1}^{m} \psi_i^1 \circ L(a) \circ \psi_i^2 \circ \phi \otimes v \\ &= \sum_{i=1}^{m} \psi_i^1 \cdot \left(a \cdot \left(\psi_i^2 \cdot w\right)\right) = 0,\end{aligned} \tag{2.7}$$

since $\psi_i^2 \cdot w \in W$ and $a \cdot W = 0$. Hence $B \cdot Q \subseteq Q$. Let $Q_B = \operatorname{Ann}_B(A/Q)$. Then

$$\frac{A}{Q} \simeq L\left(\frac{A}{Q}\right) \subseteq \frac{B}{Q_B} \subseteq \operatorname{End}_k\left(\frac{A}{Q}\right). \tag{2.8}$$

Since W is finite dimensional, A/Q is finite dimensional. Note that V is a simple left A/Q-module. Any nil ideal I of A yields a nil ideal $(I+Q)/Q$ of A/Q. Moreover every element $\psi + Q_B \in B/Q_B$ satisfies (2.5). By Lemma 2.1, $((B \cdot I) + Q)/Q = B/Q_B \cdot (I+Q)/Q$ is included in $\operatorname{Jac}(A/Q)$. Thus

$$(B \cdot I)V = \left(\frac{B}{Q_B} \cdot \frac{(I+Q)}{Q}\right) \cdot V \subseteq \operatorname{Jac}\left(\frac{A}{Q}\right) \cdot V = 0. \tag{2.9}$$

Hence $B \cdot I \subseteq \operatorname{Jac}(A)$ for any nil ideal I of A. □

3. Weak Hopf Actions on Algebras Satisfying a Polynomial Identity

Before we apply the results from the previous section, we recall the definition of weak Hopf algebras (or quantum groupoids) as introduced by Böhm et al. in [1].

Definition 3.1. An associative k-algebra H with multiplication m and unit 1 which is also a coassociative coalgebra with comultiplication Δ and counit ϵ is called a weak Hopf algebra if it satisfies the following properties:

(1) the comultiplication is multiplicative, that is, for all $g, h \in H$:

$$\Delta(gh) = \Delta(g)\Delta(h), \tag{3.1}$$

(2) the unit and counit satisfy:

$$\begin{gathered}(\Delta \otimes \mathrm{id})\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1), \\ \epsilon(fgh) = \epsilon(fg_1)\epsilon(g_2h) = \epsilon(fg_2)\epsilon(g_1h),\end{gathered} \tag{3.2}$$

(3) there exists a linear map $S : A \to A$, called antipode, such that

$$\begin{gathered}h_1S(h_2) = (\epsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)) =: \epsilon_t(h), \\ S(h_1)h_2 = (\mathrm{id} \otimes \epsilon)((1 \otimes h)\Delta(1)) =: \epsilon_s(h), \\ S(h) = S(h_1)h_2S(h_3).\end{gathered} \tag{3.3}$$

Note that we will use Sweedler's notation for the comultiplication with suppressed summation symbol.

The image of ϵ_t and ϵ_s are subalgebras H_t and H_s of H which are separable over k [15, 2.3.4] and their images commute with each other. Those subalgebras are also characterized by $H_t = \{h \in H : \Delta(h) = 1_1h \otimes 1_2\}$, respectively, $H_s = \{h \in H : \Delta(h) = 1_1 \otimes 1_2h\}$.

A left H-module algebra A over a weak Hopf algebra H is an associative unital algebra A such that A is a left H-module and for all $a, b \in A, h \in H$:

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \qquad h \cdot 1_A = \epsilon_t(h) \cdot 1_A. \tag{3.4}$$

Let A be a left H-module algebra over a weak Hopf algebra H and let λ be the ring homomorphism from H to $\mathrm{End}_k(A)$ that defines the left module structure on A, that is, $\langle \lambda(h), a \rangle := h \cdot a$ for all $h \in H, a \in A$. Property (3.4) of the definition above can be interpreted as an intertwining relation $\lambda(h) \circ L(a) = L(h_1 \cdot a) \circ \lambda(h_2)$ of left multiplications $L(a)$ and left H-actions $\lambda(h)$.

The following properties are now deduced from the axioms.

Lemma 3.2. *Let A be a left H-module algebra over a weak Hopf algebra H. Then*

(1) *for all $z \in H_t : \lambda(z) = L(z \cdot 1_A)$ and for all $z \in H_s : \lambda(z) = R(z \cdot 1_A)$,*

(2) *for all $h \in H, a \in A : L(h \cdot a) = \lambda(h_1) \circ L(a) \circ \lambda(S(h_2))$,*

(3) *if $S^2 = \mathrm{id}$, then $\lambda(S(h_2)) \circ \lambda(h_1) \in L(A)$ for all $h \in H$.*

Proof. (1) Let $z \in H_t$. Since $\Delta(z) = 1_1 z \otimes 1_2$, we have for all $a \in A$:

$$z \cdot a = z \cdot (1_A a) = (1_1 z \cdot 1_A)(1_2 \cdot a) = 1_H \cdot ((z \cdot 1_A)a) = (z \cdot 1_A)a. \tag{3.5}$$

The proof of the second statement is analogous.

(2) For $h \in H, a, x \in A$ we have

$$\begin{aligned}(h \cdot a)x &= (h_1 \cdot a)(\epsilon_t(h_2) \cdot 1_A)x \\ &= (h_1 \cdot a)(\epsilon_t(h_2) \cdot x) = (h_1 \cdot a)(h_2 S(h_3) \cdot x) = h_1 \cdot (a(S(h_3) \cdot x)).\end{aligned} \tag{3.6}$$

(3) Suppose $S^2 = \mathrm{id}$, then $S(\epsilon_s(h)) = S(h_2)h_1$ and as $S(H_s) \subseteq H_t$, we have using (1):

$$\lambda(S(h_2)) \circ \lambda(h_1) = \lambda(S(\epsilon_s(h))) = L(S(\epsilon_s(h)) \cdot 1_A) \in L(A). \tag{3.7}$$

□

We say that a weak Hopf algebra H is involutive if its antipode is an involution. Any groupoid algebra is an involutive weak Hopf algebra. Moreover any semisimple Hopf algebra over a field of characteristic zero is involutive. Say that H acts finitely on a left H-module algebra A if the image of $\lambda : H \to \mathrm{End}_k(A)$ is finite dimensional. The following statement follows from the last lemma and Theorem 2.2.

Theorem 3.3. *Let H be an involutive weak Hopf algebra over a field k of characteristic zero acting finitely on a left H-module algebra A. If A satisfies a polynomial identity or if k is an uncountable algebraically closed field, A is countably generated and all left primitive factor rings of A are Artinian, then the Jacobson radical of A is H-stable.*

Proof. Let $\lambda : H \to \mathrm{End}_k(A)$ be the ring homomorphism inducing the left H-module structure on A. Denote by B the subalgebra of $\mathrm{End}_k(A)$ generated by $L(A)$ and $\lambda(H)$. Let $h_1, \dots, h_m$ be elements of H such that $\{\lambda(h_1), \dots, \lambda(h_m)\}$ forms a basis of $\lambda(H)$. We claim that any element of B is of the form $\sum_{i=1}^m \lambda(h_i) \circ L(a_i)$ for some $a_i \in A$. It is enough to show $L(A)\lambda(H) \subseteq \lambda(H)L(A)$. So take elements $h \in H$ and $a, b \in A$. Then using Lemma 3.2(2), $S^{-1} = S$ and $S(H_t) = H_s$ we have

$$\begin{aligned}h_2 \cdot ((S(h_1) \cdot a)b) &= (h_2 S(h_1) \cdot a)(h_3 \cdot b) \\ &= (S(\epsilon_t(h_1)) \cdot a)(h_2 \cdot b) = a(\epsilon_t(h_1) \cdot 1_A)(h_2 \cdot b) = a(h \cdot b).\end{aligned} \tag{3.8}$$

This shows the intertwining relation $L(a) \circ \lambda(h) = \lambda(h_2) \circ L(S(h_1) \cdot a)$ in B which yields that B is finitely generated as a right A-module. By the definition of module algebras, we also have that $\lambda(h) \circ L(a) = L(h_1 \cdot a) \circ \lambda(h_2)$. Hence $\lambda(H)L(A) = L(A)\lambda(H)$. For any $a \in A$ and $\psi = \sum_{i=1}^m L(a_i) \circ \lambda(h_i) \in B$ we have by Lemma 3.2(3) and by (3.8):

$$\begin{aligned}L(\langle \psi, a \rangle) &= \sum L(a_i) \circ L(\langle h_i, a \rangle) \\ &= \sum L(a_i) \circ \lambda(h_{i1}) \circ L(a) \circ \lambda(S(h_{i2})) \\ &= \sum \lambda(h_{i2}) \circ L((S(h_{i1}) \cdot a_i)) \circ L(a) \circ \lambda(S(h_{i3})) = \sum_j \psi_j^1 \circ L(a) \circ \psi_j^2\end{aligned} \tag{3.9}$$

for $\psi_j^1 = \lambda(h_{i2}) \circ L(S(h_{i1}) \cdot a_i)$, $\psi_j^2 = \lambda(S(h_{i3})$, and some appropriate choice of indices j. Moreover

$$\sum \psi_j^2 \circ \psi_j^1 = \sum \lambda(S(h_{i3})) \circ \lambda(h_{i2}) \circ L(S(h_{i1}) \cdot a_i) = \sum L\left(y_{h_{i2}}\right) \circ L(S(h_{i1}) \cdot a_i) \in L(A) \quad (3.10)$$

for some elements $y_{h_{i2}} \in A$ that exist by Lemma 3.2(4). Therefore the hypotheses of Theorem 2.2 are fulfilled and the statement follows. □

3.1. Smash Products of Weak Hopf Actions

Recall that the smash product $A\#H$ of a left H-module algebra A and a weak Hopf algebra H is defined on the tensor product $A \otimes_{H_t} H$ where A is considered a right H_t-module by $a \cdot z = a(z \cdot 1_A)$ for $a \in A, z \in H_t$. The (k-linear) dual H^* of H becomes also a weak Hopf algebra and acts on $A\#H$ by $\phi \cdot (a\#h) := a\#\phi(\rightharpoonup h)$, where $\phi \rightharpoonup h = h_1\langle \phi, h_2\rangle$. Using the Montgomery-Blattner duality theorem for weak Hopf algebras proven by Nikshych we have the following.

Lemma 3.4. *Let H be a finite-dimensional weak Hopf algebra and A a left H-module algebra. Then $A\#H$ is a finitely generated projective right A-module and $A\#H\#H^* \simeq eM_n(A)e$ for some idempotent $e \in M_n(A)$ where $M_n(A)$ denotes the ring of $n \times n$-matrixes for some number $n > 0$.*

Proof. By [14, Theorem 3.3] $A\#H\#H^* \simeq \mathrm{End}(A\#H_A)$. Since H_t is a separable k-algebra, it is semisimple Artinian. Hence H is a (finitely generated) projective right H_t-module and H is a direct summand of H_t^n for some $n > 0$. Moreover it follows from the proof of Lemma 3.2 that $A\#H = (1\#H)(A\#1)$. Thus $H \otimes_{H_t} A \simeq A\#H$ as right A-modules by $h \otimes a \mapsto (1\#h)(a\#1)$. On the other hand $H \otimes_{H_t} A$ is a direct summand of $H_t^n \otimes_{H_t} A \simeq A^n$ as right A-module. Hence $A\#H$ is a projective right A-module of rank n and $\mathrm{End}(A\#H)_A \simeq eM_n(A)e$ for some idempotent $e \in M_n(A)$. □

3.2. Semiprime Smash Products for Weak Hopf Actions

We can now transfer Linchenko and Montgomery's result [7, Theorem 3.4] on the semiprimness of smash products to weak Hopf actions.

Theorem 3.5. *Let A be a left H-module algebra over a finite dimensional involutive weak Hopf algebra H over a field of characteristic zero. If A is semiprime and satisfies a polynomial identity, then $A\#H$ is semiprime.*

Proof. Set $B = A\#H\#H^*$. Note that H^* is also involutive since its antipode is defined by $\langle S^*(\phi), h\rangle := \langle \phi, S(h)\rangle$ for all $\phi \in H^*, h \in H$. By [12, Corollary 6.5] H^* is semisimple and by [1, 3.13] there exists a normalized left integral $\Lambda \in H^*$. This implies that $A\#H$ is a projective left B-module as the left B-linear map $A\#H \to B$ with $a\#h \mapsto a\#h\#\Lambda$ splits the projection $B \to A\#H$ given by $a\#h\#\phi \mapsto a\#h(\phi \rightharpoonup 1)$.

First suppose that $\mathrm{Jac}(A) = 0$. By Lemma 3.4, $\mathrm{Jac}(B) \simeq eM_n(\mathrm{Jac}(A))e = 0$ for some idempotent e. This implies also that $\mathrm{Rad}_B(A\#H) = 0$ as well, since $A\#H$ is supposed to be a projective left B-module. Recall that the radical $\mathrm{Rad}(M)$ of a module M is the intersection of all maximal submodules of M or equivalently the sum of all small submodules, that is, of those submodules N of M such that $N + L \neq M$ for all $L \neq M$.

Since $A\#H$ is a finite extension of A, also $A\#H$ satisfies a polynomial identity and since H^* is finite dimensional it acts finitely on $A\#H$. Thus Theorem 3.3 applies and for any nil ideal I of $A\#H$ we have $B \cdot I \subseteq \mathrm{Jac}(A\#H)$. On the other hand any B-submodule N of $\mathrm{Jac}(A\#H)$ is contained in $\mathrm{Rad}_B(A\#H)$, which is zero. Hence $I = 0$ and $A\#H$ is semiprime.

In general, if A is semiprime, we can extend the H-action of A to the polynomial ring $A[x]$ by identifying $A[x]$ with $A \otimes_{H_t} H_t[x]$, which is a left H-module algebra, where H acts on x by $h \cdot x = (\epsilon_t(h) \cdot 1_A)x$. Since A is semiprime, satisfying a polynomial identity, $\mathrm{Jac}(A[x]) = 0$ by [13]. Moreover $A[x]$ also satisfies a polynomial identity and by the argument above $A[x]\#H$ is semiprime. As any ideal I of $A\#H$ can be extended to an ideal $I[x]$ of $A\#H[x] = A[x]\#H$, also $A\#H$ is semiprime. □

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