Research Article

Study of the Generalized Quantum Isotonic Nonlinear Oscillator Potential

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We study the generalized quantum isotonic oscillator Hamiltonian given by $H = -d^2/dr^2 + l(l+1)/r^2 + w^2 r^2 + 2g(r^2 - a^2)/(r^2 + a^2)^2$, $g > 0$. Two approaches are explored. A method for finding the quasipolynomial solutions is presented, and explicit expressions for these polynomials are given, along with the conditions on the potential parameters. By using the asymptotic iteration method, we show how the eigenvalues of this Hamiltonian for arbitrary values of the parameters g, w, and a may be found to high accuracy.

1. Introduction

Recently, Cariñena et al. [1] studied a quantum nonlinear oscillator potential whose Schrödinger equation reads

$$\left[-\frac{d^2}{dx^2} + x^2 + 8\frac{2x^2 - 1}{(2x^2 + 1)^2} \right] \psi_n(x) = E_n \psi(x). \tag{1.1}$$

The interest in this problem came from the fact that it is exactly solvable in a sense that the exact eigenenergies and eigenfunctions can be obtained explicitly. Indeed, Cariñena et al. [1] were able to show that

$$\psi_n(x) = \frac{P_n(x)}{(2x^2+1)}e^{-x^2/2},$$

$$E_n = -3 + 2n, \quad n = 0, 3, 4, 5, \ldots,$$

(1.2)

where the polynomials factors $P_n(x)$ are related to the Hermite polynomials by means of

$$P_n(x) = \begin{cases} 1, & \text{if } n = 0, \\ H_n(x) + 4nH_{n-2}(x) + 4n(n-3)H_{n-4}(x), & \text{if } n = 3, 4, 5, \ldots. \end{cases}$$

(1.3)

In a more recent work, Fellows and Smith [2] showed that the potential $V(x) = x^2 + 8(2x^2 - 1)/(2x^2 + 1)^2$ as well as, for certain values of the parameters w, g, and a, the potential $V(x) = w^2x^2 + 2g(x^2 - a^2)/(x^2 + a^2)^2$ of the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + w^2x^2 + 2g\frac{x^2 - a^2}{(x^2 + a^2)^2} \right]\psi_n(x) = 2E_n\psi(x)$$

(1.4)

are indeed supersymmetric partners of the harmonic oscillator potential. Using the supersymmetric approach, the authors were able to construct an infinite set of exact soluble potentials, along with their eigenfunctions and eigenvalues. Very recently, Sesma [3], using a Möbius transformation, was able to transform (1.4) into a confluent Heun equation [4], and thereby obtain an efficient algorithm to solve the Schrödinger equation (1.4) numerically.

The purpose of the present work is to provide a detailed solution, by means of the quasipolynomial solutions and the application of the asymptotic iteration method [5–8], for the Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + w^2r^2 + 2g\frac{r^2 - a^2}{(r^2 + a^2)^2} \right]\psi(r) = 2E\psi(r),$$

(1.5)

where l is the angular momentum number $l = -1, 0, 1, \ldots$. Our results show that the quasiexact solutions of Sesma [3] as well the results of Cariñena et al. [1] follow as special cases of our general approach. The present paper is organized as follows. In Section 2, some preliminary analysis of the Schrödinger equation (1.5) is presented. A general approach for finding polynomial solutions of (1.5), for certain values of parameters w and g, is presented and is based on a recent work of Ciftci et al. [6] for solving the second-order linear differential equation

$$\left(\sum_{i=0}^{3} a_{3,i}x^i \right)y'' + \left(\sum_{i=0}^{2} a_{2,i}x^i \right)y' - \left(\sum_{i=0}^{1} \tau_{1,i}x^i \right)y = 0.$$

(1.6)

More general quasiexact solutions, including the results of Sesma [3], are discussed in Section 3. Unrestricted solutions of (1.5) based on the asymptotic iteration method are discussed in Section 4.

2. Generalized Quantum Isotonic Oscillator—Preliminary Results

A simple scaling argument, using $r = a^2 x$, allows us to write (1.5) as

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + \left(wa^2\right)^2 x^2 + 2g\frac{x^2 - 1}{(x^2+1)^2} \right] \psi(x) = 2Ea^2 \psi(x). \tag{2.1}$$

A further substitution $z = x^2 + 1$ yields a differential equation with two regular singular points at $z = 0, 1$ and one irregular singular point of rank 2 at $z = \infty$. The roots μ's of the indicial equation for the regular singular point $z = 0$ reads $\mu_\pm = 1/2(1 \pm \sqrt{1+4g})$, while the roots of the indicial equation at $z = 1$ are $\mu_+ = (l+1)/2$ and $\mu_- = -l/2$. Since the singularity for $z \to \infty$ corresponds to that for $x \to \infty$, it is necessary that the solution for $z \to \infty$ behave as $\psi(x) \sim \exp(-wa^2 x^2/2)$. Consequently, we may assume the general solution of (2.1) which vanishes at the origin and at infinity takes the form

$$\psi_n(x) = x^{l+1} \left(x^2 + 1\right)^\mu e^{-(wa^2/2)x^2} f_n(x). \tag{2.2}$$

A straightforward calculation shows that $f_n(x)$ are the solutions of the second-order homogeneous linear differential equation

$$f''(x) + \left(\frac{2(l+1)}{x} + \frac{4\mu x}{x^2+1} - 2wa^2 x \right) f'(x)$$

$$+ \left[2Ea^2 - wa^2(2l + 3 + 4\mu) + \frac{2\mu(2l+3+2wa^2) + 4\mu(\mu-1) - 2g}{x^2+1} + \frac{4(g - \mu(\mu-1))}{(x^2+1)^2} \right]$$

$$\times f(x) = 0. \tag{2.3}$$

In the next sections, we attempt to give a general solution of this equation. For now, we assume that μ takes the value of the indicial root

$$\mu \equiv \mu_- = \frac{1}{2}\left(1 - \sqrt{1+4g}\right), \tag{2.4}$$

which allows us to write (2.3) as

$$f_n''(x) + \left(\frac{2(l+1)}{x} + \frac{4\mu x}{x^2+1} - 2wa^2 x \right) f_n'(x)$$

$$+ \left[2Ea^2 - wa^2(2l + 3 + 4\mu) + \frac{2\mu(2l+3+2wa^2) + 2\mu(\mu-1)}{x^2+1} \right] f_n(x) = 0. \tag{2.5}$$

We now consider the cases where the following two equations are satisfied:

$$2\mu\left(2l + 3 + 2wa^2\right) + 2\mu(\mu - 1) = 0,$$

$$g = \mu(\mu - 1).$$

(2.6)

The solutions of this system, for g and μ, are given explicitly by

$$g = 0, \quad \text{or} \quad g = 2(1 + l + a^2w)(3 + 2l + 2a^2w),$$

$$\mu = 0, \quad \text{or} \quad \mu = -2(1 + l + a^2w).$$

(2.7)

Next, we consider each case of these two sets of solutions.

2.1. Case 1

The first set of solutions $(g, \mu) = (0, 0)$ reduces the differential equation (2.3) to

$$x f_n''(x) + \left[-2wa^2x^2 + 2(l + 1)\right] f_n'(x) + \left(2Ea^2 - wa^2(2l + 3)\right) x f_n(x) = 0,$$

(2.8)

which is a special case of the general differential equation

$$\left(a_{3,0}x^3 + a_{3,1}x^2 + a_{3,2}x + a_{3,3}\right)y'' + \left(a_{2,0}x^2 + a_{2,1}x + a_{2,2}\right)y' - (\tau_{1,0}x + \tau_{1,1})y = 0,$$

(2.9)

with $a_{3,0} = a_{3,1} = a_{3,3} = a_{2,1} = \tau_{1,1} = 0$, $a_{3,2} = 1$, $a_{2,0} = -2wa^2$, $a_{2,2} = 2(l + 1)$, and $\tau_{1,0} = -2Ea^2 + wa^2(2l + 3)$. The necessary and sufficient conditions for polynomial solutions of (2.9) are given by the following theorem [6].

Theorem 2.1. *The second-order linear differential equation* (2.9) *has a polynomial solution of degree* n *if*

$$\tau_{1,0} = n(n - 1)a_{3,0} + na_{2,0}, \quad n = 0, 1, 2, \ldots,$$

(2.10)

along with the vanishing of $(n + 1) \times (n + 1)$-*determinant* Δ_{n+1} *given by*

$$\Delta_{n+1} = \begin{vmatrix} \beta_0 & \alpha_1 & \eta_1 & & & & \\ \gamma_1 & \beta_1 & \alpha_2 & \eta_2 & & & \\ & \gamma_2 & \beta_2 & \alpha_3 & \eta_3 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \gamma_{n-2} & \beta_{n-2} & \alpha_{n-1} & \eta_{n-1} \\ & & & & \gamma_{n-1} & \beta_{n-1} & \alpha_n \\ & & & & & \gamma_n & \beta_n \end{vmatrix} = 0,$$

(2.11)

where

$$\beta_n = \tau_{1,1} - n((n-1)a_{3,1} + a_{2,1}),$$

$$\alpha_n = -n((n-1)a_{3,2} + a_{2,2}),$$

$$\gamma_n = \tau_{1,0} - (n-1)((n-2)a_{3,0} + a_{2,0}), \tag{2.12}$$

$$\eta_n = -n(n+1)a_{3,3},$$

and $\tau_{1,0}$ is fixed for a given n in the determinant $\Delta_{n+1} = 0$.

Thus, the necessary condition for the differential equation (2.8) to have polynomial solutions $f_n(x) = \sum_{i=0}^{n} c_i x^i$ is

$$2E_n a^2 = wa^2(2n' + 2l + 3), \quad n' = 0,1,2,\ldots, \tag{2.13}$$

while the sufficient condition, (2.12), is

$$\Delta_{n+1} = \begin{vmatrix} 0 & \alpha_1 & 0 & 0 & & & & \\ \gamma_1 & 0 & \alpha_2 & 0 & & & & \\ & \gamma_2 & 0 & \alpha_3 & 0 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & \gamma_{n-2} & 0 & \alpha_{n-1} & 0 \\ & & & & \gamma_{n-1} & 0 & \alpha_n \\ & & & & & \gamma_n & 0 \end{vmatrix} = \begin{cases} 0, & \text{if } n = 0,2,4,\ldots, \\ \displaystyle\prod_{j=0}^{(n-1)/2} (-1)^{2j+1}\alpha_{2j+1}\gamma_{2j+1} = 0, & \text{if } n = 1,3,5,\ldots, \end{cases} \tag{2.14}$$

where $\beta_n = 0$, $\alpha_n = -n(n + 2l + 1)$ and $\gamma_n = 2wa^2(n - n' - 1)$.

If $l = -1$, the determinant Δ_{n+1} is identically zero for all n, which is equivalent to the exact solutions of the one-dimensional harmonic oscillator problem.

For $l \neq -1$, we have for $n = 0,2,4,\ldots$, $\Delta_{n+1} \equiv 0$, and we obtain the exact solutions of the Gol'dman and Krivchenkov (or Isotonic) Hamiltonian H_0, where

$$H_0 \psi_{nl}(x) \equiv \left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + w^2 a^4 x^2 \right] \psi_{nl}(x) = 2E_{nl}^{g=0} a^2 \psi_{nl}(x), \quad 0 \leq x < \infty. \tag{2.15}$$

These exact solutions are given by [9]

$$2a^2 E_{nl}^{g=0} = wa^2(4n + 2l + 3), \quad n = 0,1,2,\ldots,$$

$$\psi_{nl}(x) = x^{l+1} e^{-wa^2 x^2 / 2} {}_1F_1\left(-n; l + \frac{3}{2}; wa^2 x^2 \right), \quad n = 0,1,2,\ldots, \tag{2.16}$$

where the confluent hypergeometric function $_1F_1(-n; a; z)$ defined in terms of the Pochhammer symbol (or Gamma function $\Gamma(a)$)

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1, & \text{if } (k = 0, \ a \in \mathbb{C} \setminus \{0\}), \\ a(a+1)(a+2)\cdots(a+k-1), & \text{if } (k = \mathbb{N}, \ a \in \mathbb{C}), \end{cases} \tag{2.17}$$

as

$$_1F_1(-n; a; z) = \sum_{k=0}^{n} \frac{(-n)_k z^k}{(a)_k k!}. \tag{2.18}$$

The polynomial solutions $f_n(x) = {}_1F_1(-n; l + (3/2); wa^2x^2)$ are easily obtained by using the asymptotic iteration method (AIM), which is summarized by means of the following theorem.

Theorem 2.2 (Ciftci et al. [7, equations (2.13)-(2.14)]). *Given $\lambda_0 \equiv \lambda_0(x)$ and $s_0 \equiv s_0(x)$ in C^∞, the differential equation*

$$f''(x) = \lambda_0(x) f'(x) + s_0(x) f(x) \tag{2.19}$$

has the general solution

$$f(x) = \exp\left(-\int^x \alpha(t) dt\right) \left[C_2 + C_1 \int^x \exp\left(\int^t (\lambda_0(\tau) + 2\alpha(\tau)) d\tau\right) dt\right], \tag{2.20}$$

if for some $n \in \mathbb{N}^+ = \{1, 2, \ldots\}$

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} = \alpha(x), \quad or \quad \delta_n(x) = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0, \tag{2.21}$$

where

$$\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_n,$$
$$s_n = s'_{n-1} + s_0 \lambda_n. \tag{2.22}$$

For the differential equation (2.8) with

$$\lambda_0(x) = -\frac{(-2wa^2x^2 + 2(l+1))}{x},$$
$$s_0(x) = -\left(2Ea^2 - wa^2(2l+3)\right), \tag{2.23}$$

the first few iterations with $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$, using (2.20), imply

$$
\begin{aligned}
f_0(x) &= 1, \\
f_1(x) &= 2wa^2x^2 - (2l+3), \\
f_2(x) &= 4w^2a^4x^4 - 4wa^2(2l+5)x^2 + (2l+3)(2l+5),
\end{aligned}
\tag{2.24}
$$

$$
\vdots
$$

which we may easily generalized using the definition of the confluent hypergeometric function, (2.18), as

$$
f_n(x) = {}_1F_1\left(-n; l + \frac{3}{2}; wa^2x^2\right),
\tag{2.25}
$$

up to a constant.

2.2. Case 2

The second set of solutions

$$
(g, \mu) = \left(2\left(1 + l + a^2w\right)\left(3 + 2l + 2a^2w\right), -2\left(1 + l + a^2w\right)\right)
\tag{2.26}
$$

allow us to write the differential equation (2.3) as

$$
\begin{aligned}
&f_n''(x) + \left(\frac{2(l+1)}{x} - \frac{8(l+1+a^2w)x}{x^2+1} - 2wa^2x\right)f_n'(x) \\
&\quad + \left(2Ea^2 + wa^2\left(6l+5+8wa^2\right)\right)f_n(x) = 0.
\end{aligned}
\tag{2.27}
$$

A further change of variable $z = x^2 + 1$ allows us to write the differential equation (2.27) as

$$
\begin{aligned}
&4z(z-1)f''(z) - \left(4a^2wz^2 + 2\left(6l+5+6wa^2\right)z - 16\left(l+1+wa^2\right)\right)f'(z) \\
&\quad + \left(2Ea^2 + wa^2\left(6l+5+8wa^2\right)\right)zf(z) = 0,
\end{aligned}
\tag{2.28}
$$

Again, (2.28) is a special case of the differential equation (2.9) with $a_{3,0} = a_{3,3} = \tau_{1,1} = 0$, $a_{3,1} = 4$, $a_{3,2} = -4$, $a_{2,0} = -4wa^2$, $a_{2,1} = -2(6l+5+6wa^2)$, $a_{2,2} = 16(l+1+wa^2)$, and $\tau_{1,0} = -2Ea^2 - wa^2(6l+5+8wa^2)$. Consequently, the polynomial solutions $f_n(x)$ of (2.28) are subject to the following two conditions: the necessary condition (2.10) reads

$$
2E_na^2 = wa^2\left(4n' - 6l - 5 - 8wa^2\right), \quad n' = 0, 1, 2, \ldots,
\tag{2.29}
$$

and the sufficient condition; namely, the vanishing of the tridiagonal determinant (2.12), reads

$$\Delta_{n+1} = \begin{vmatrix} \beta_0 & \alpha_1 & & & & & \\ \gamma_1 & \beta_1 & \alpha_2 & & & & \\ & \gamma_2 & \beta_2 & \alpha_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma_{n-2} & \beta_{n-2} & \alpha_{n-1} & \\ & & & & \gamma_{n-1} & \beta_{n-1} & \alpha_n \\ & & & & & \gamma_n & \beta_n \end{vmatrix} = 0, \tag{2.30}$$

where

$$\beta_n = -2n\left(2n - 6l - 7 - 6wa^2\right),$$

$$\alpha_n = 4n\left(n - 4l - 5 - 4a^2 w\right), \tag{2.31}$$

$$\gamma_n = 4wa^2\left(n - n' - 1\right),$$

and $n' = n$ is fixed for the given dimension of the determinant Δ_{n+1}. From the sufficient condition (2.31), we obtain the following conditions on the parameters:

$$\Delta_2 = 0 \Longrightarrow a^2 w\left(l + 1 + a^2 w\right) = 0,$$

$$\Delta_3 = 0 \Longrightarrow a^2 w\left(l + 1 + a^2 w\right)\left(1 + 2l + 2a^2 w\right) = 0,$$

$$\Delta_4 = 0 \Longrightarrow a^2 w\left(l + 1 + a^2 w\right)\left(1 + 2l + 2a^2 w\right)\left(3(1 + 6l) + 14a^2 w\right) = 0,$$

$$\Delta_5 = 0 \Longrightarrow a^2 w\left(l + 1 + a^2 w\right)\left(1 + 2l + 2a^2 w\right)\left(3(6l - 1)(6l + 1) + 4(38l + 1)a^2 w + 44a^4 w^2\right) = 0,$$

$$\Delta_6 = 0 \Longrightarrow a^2 w\left(l + 1 + a^2 w\right)\left(1 + 2l + 2a^2 w\right)$$

$$\times \left(3(2l - 1)(6l - 1)(6l + 1) + 2\left(208l^2 - 54l - 5\right)a^2 w + 200la^4 w^2\right) = 0,$$

$$\vdots = \vdots$$

$$\tag{2.32}$$

For a physically meaningful solution, we must have $a^2w > 0$. This is possible for a very restricted value of the angular momentum number l. Since $\beta_0 = 0$, we may observe that

$$
\Delta_{n+1} = \left(l + 1 + a^2w\right)\left(1 + 2l + 2a^2w\right) \times
\begin{vmatrix}
\beta_2 & \alpha_3 & & & & & \\
\gamma_3 & \beta_3 & \alpha_4 & & & & \\
& \gamma_4 & \beta_4 & \alpha_5 & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \gamma_{n-2} & \beta_{n-2} & \alpha_{n-1} & \\
& & & & \gamma_{n-1} & \beta_{n-1} & \alpha_n \\
& & & & & \gamma_n & \beta_n
\end{vmatrix}
$$

$$
= \left(l + 1 + a^2w\right)\left(1 + 2l + 2a^2w\right) \times Q_{n-1}^l\left(a^2w\right),
$$

(2.33)

where $Q_{n-1}^l(a^2w)$ are polynomials in the parameter product a^2w.

For physically acceptable solutions, we must have $l = -1$ and the factor $(l + 1 + a^2w)$ yields $a^2w = 0$, which is not physically acceptable, so we ignore it. The second factor $(1 + 2l + 2a^2w)$ implies a special value of $a^2w = 1/2$, for all n, which we will study shortly in full detail. Meanwhile, the polynomials $Q_n^l(a^2w)$

$$
Q_{n-1}^{l=-1}\left(a^2w\right) =
\begin{cases}
1, & \text{if } n = 2, \\
14a^2w - 15, & \text{if } n = 3, \\
44a^4w^2 - 148a^2w + 105, & \text{if } n = 4, \\
200a^4w^2 - 514a^2w + 315, & \text{if } n = 5, \\
\vdots
\end{cases}
$$

(2.34)

give new values, not reported before, of a^2w that yield quasiexact solutions of the Schrödinger equation (with one eigenstate)

$$
-\psi_n''(x) + \left[\left(wa^2\right)^2 x^2 + 4a^2w\left(1 + 2a^2w\right)\frac{(x^2 - 1)}{(x^2 + 1)^2}\right]\psi_n(x) = wa^2\left(4n + 1 - 8a^2w\right)\psi_n(x),
$$

(2.35)

where

$$
\psi_n(x) = \left(x^2 + 1\right)^{-2a^2w} e^{-wa^2x^2/2} f_n(x),
$$

(2.36)

Figure 1: Plot of the unnormalized wave function $\psi_3(x)$ and the potential $V_3 = (225/196)x^2 + (660/49)(x^2 - 1)/(x^2 + 1)^2$.

and $f_n(x)$ are the solutions of

$$4z(z-1)f''(z) - \left(4a^2wz^2 + 2\left(-1 + 6wa^2\right)z - 16wa^2\right)f'(z) + 4nwa^2zf(z) = 0, \quad z = x^2 + 1. \tag{2.37}$$

For example, $\Delta_4 = 0$ implies, using (2.34), that $a^2w = 15/14$, and thus, we have for

$$-\psi_3''(x) + \left[\frac{225}{196}x^2 + \frac{660}{49}\frac{(x^2-1)}{(x^2+1)^2}\right]\psi_3(x) = \frac{465}{98}\psi_3(x), \tag{2.38}$$

the exact solution

$$\psi_3(x) = \left(x^2 + 1\right)^{-15/7} e^{-(15/28)x^2}\left(45x^6 + 225x^4 + 315x^2 - 49\right), \tag{2.39}$$

with a plot of the wave function and potential given in Figure 1.

Further, $\Delta_5 = 0$, equation (2.34) implies

$$a^2w = \frac{37}{22} \pm \frac{\sqrt{214}}{22}, \tag{2.40}$$

and we have for

$$
-\psi_4''(x) + \left[\left(\frac{37}{22} \pm \frac{\sqrt{214}}{22} \right)^2 x^2 + 2\left(\frac{37}{11} \pm \frac{\sqrt{214}}{11} \right)\left(\frac{48}{11} \pm \frac{\sqrt{214}}{11} \right) \frac{(x^2-1)}{(x^2+1)^2} \right] \psi_4(x)
$$
$$
= \left(\frac{37}{22} \pm \frac{\sqrt{214}}{22} \right)\left(\frac{39}{11} \mp \frac{4\sqrt{214}}{11} \right) \psi_4(x),
$$

(2.41)

the exact solutions

$$
\psi_4^\pm(x) = \left(x^2 + 1 \right)^{-\left((37/11) \pm (\sqrt{214}/11) \right)} e^{-\left((37/44) \pm (\sqrt{214}/44) \right)x^2}
$$
$$
\times \left(1575x^8 + \left(9660 \pm 420\sqrt{214} \right)x^6 + \left(26250 \pm 2100\sqrt{214} \right)x^4 + \left(29820 \pm 2940\sqrt{214} \right)x^2 \right.
$$
$$
\left. - \left(1129 \pm 188\sqrt{214} \right) \right).
$$

(2.42)

Similar results can be obtained for $\Delta_{n+1} = 0$, for $n \geq 5$.

2.3. Exactly Solvable Quantum Isotonic Nonlinear Oscillator

As mentioned above, for $l = -1$ and $a^2 w = 1/2$, it clear that $\Delta_{n+1} = 0$ for all n and the one-dimensional Schrödinger equation

$$
\left[-\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{4(x^2-1)}{(x^2+1)^2} \right] \psi_n(x) = \left(2n - \frac{3}{2} \right) \psi_n(x), \quad n = 0, 1, 2, \ldots
$$

(2.43)

has the exact solutions

$$
\psi_n(x) = \left(x^2 + 1 \right)^{-1} e^{-x^2/4} f_n(x),
$$

(2.44)

where $f_n(x)$ are the polynomial solutions of the following second-order linear differential equation ($z = x^2 + 1$)

$$
4z(z-1)f_n''(z) - \left(2z^2 + 4z - 8 \right) f_n'(z) + 2nz f_n(z) = 0.
$$

(2.45)

By using AIM (Theorem 2.2, (2.20)), we find that the polynomial solutions $f_n(x)$ of (2.45) are given explicitly as

$$
\begin{aligned}
f_0(x) &= 1, \\
f_1(x) &= x^2 - 2, \\
f_2(x) &= x^3 - 6x^2 + 8, \\
f_3(x) &= x^4 - 16x^3 + 52x^2 - 52, \\
f_4(x) &= x^5 - 30x^4 + 250x^3 - 580x^2 + 464, \\
&\vdots
\end{aligned}
\tag{2.46}
$$

a set of polynomial solutions that can be generated using

$$
f_0(x) = 1,
$$

$$
f_n(x) = -3x(2n+1)\,_1F_1\left(-n; \frac{3}{2}; \frac{1}{2}(x-1)\right) + 6((n+1)x - 1)\,_1F_1\left(-n+1; \frac{3}{2}; \frac{1}{2}(x-1)\right),
\tag{2.47}
$$

up to a constant factor, where, again, $_1F_1$ refers to the confluent hypergeometric function defined by (2.18). Note that the polynomials $f_n(x)$ in (2.47) can be expressed in terms of the associated Laguerre polynomials [10] as

$$
f_0(x) = 1,
$$

$$
f_n(x) = \frac{3(-1)^n \sqrt{\pi}\, \Gamma(n)}{2\Gamma(n+3/2)}
$$

$$
\times \left[\left((1+n)(x-1)^2 + n\right) L_n^{1/2}\left(\frac{x-1}{2}\right) - (x-1)((1+n)x - 1) L_n^{3/2}\left(\frac{x-1}{2}\right)\right].
\tag{2.48}
$$

3. Quasipolynomial Solutions of the Generalized Quantum Isotonic Oscillator

In this section, we study the quasipolynomial solutions of the differential equation (2.3). We note first, using the change of variable $z = x^2$, equation (2.3) can be written as

$$
\begin{aligned}
f_n''(z) &+ \left(\frac{2l+3}{2z} + \frac{2\mu}{z+1} - wa^2\right) f_n'(z) \\
&+ \left[\frac{2Ea^2 - wa^2(2l+3+4\mu)}{4z} + \frac{\mu(2l+3+2wa^2)}{2z(z+1)} - \frac{g}{2}\frac{(z-1)}{z(z+1)^2} + \frac{\mu(\mu-1)}{(z+1)^2}\right] f_n(z) = 0.
\end{aligned}
\tag{3.1}
$$

By means of the Möbius transformation $z = t/(1-t)$ that maps the singular points $\{-1, 0, \infty\}$ into $\{0, 1, \infty\}$, we obtain

$$f_n''(t) + \left(\frac{2l+3}{2t(1-t)} + \frac{2(\mu-1)}{1-t} - \frac{wa^2}{(1-t)^2} \right) f_n'(t)$$

$$+ \left[\frac{\mu(2l+3+2wa^2)}{2t(1-t)^2} - \frac{g}{2}\frac{(2t-1)}{t(1-t)^2} + \frac{\mu(\mu-1)}{(1-t)^2} \right] f_n(t) = 0, \tag{3.2}$$

where we assume that

$$2Ea^2 - (2l+3+4\mu)wa^2 = 0. \tag{3.3}$$

The differential equation (3.2) can be written as

$$\left(t^3 - 2t^2 + t \right) f_n''(t) + \left[-2(\mu-1)t^2 + \left(2\mu - wa^2 - l - \frac{7}{2} \right)t + \left(l + \frac{3}{2} \right) \right] f_n'(t)$$

$$+ \left[(\mu(\mu-1) - g)t + \frac{g}{2} + \mu\left(l + \frac{3}{2} + wa^2 \right) \right] f_n(t) = 0, \tag{3.4}$$

which we may now compare with equation (2.9) in Theorem 2.1 with $a_{3,0} = 1$, $a_{3,1} = -2$, $a_{3,2} = 1$, $a_{3,3} = 0$, $a_{2,0} = -2(\mu-1)$, $a_{2,1} = (2\mu - wa^2 - l - 7/2)$, $a_{2,2} = (l+3/2)$, $\tau_{1,0} = -(\mu(\mu-1)-g)$, $\tau_{1,1} = -g/2 - \mu(l+3/2+wa^2)$. We, thus, conclude that the quasipolynomial solutions $f_n(t)$ of (3.4) are subject to the following conditions:

$$g = (\mu - k)(\mu - k - 1), \quad k = 0, 1, 2, \ldots, \tag{3.5}$$

along with the vanishing of the tridiagonal determinant $\Delta_{n+1} = 0$

$$\begin{vmatrix} \beta_0 & \alpha_1 & & & & \\ \gamma_1 & \beta_1 & \alpha_2 & & & \\ & \gamma_2 & \beta_2 & \alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{n-1} & \beta_{n-1} & \alpha_n \\ & & & & \gamma_n & \beta_n \end{vmatrix} = 0, \tag{3.6}$$

where

$$\beta_n = -\frac{1}{2}\left(g + (\mu - n)\left(3 + 2l + 4n + 2a^2w\right)\right),$$

$$\alpha_n = -n\left(n + l + \frac{1}{2}\right), \tag{3.7}$$

$$\gamma_n = g - (\mu - n + 1)(\mu - n).$$

Here, again, $g = (\mu - k)(\mu - k - 1)$ is fixed for given $k = n$, the *fixed* size of the determinant Δ_{n+1}.

3.1. Particular Case: $n = 0$

For $k(fixed) \equiv n = 0$, the differential equation (3.4) has the exact solution $f_0(t) = 1$ if g and μ satisfy, simultaneously, the following system of equations:

$$g + \mu\left(3 + 2l + 2a^2w\right) = 0, \quad g = \mu(\mu - 1). \tag{3.8}$$

Solving this system of equations for g and μ, we obtain the following values of

$$g = 2\left(1 + l + a^2w\right)\left(3 + 2l + 2a^2w\right), \quad \mu = -2\left(l + 1 + wa^2\right), \tag{3.9}$$

and the ground-state energy, in this case, is given by (3.3); namely,

$$Ea^2 = -\frac{1}{2}a^2w\left(5 + 6l + 8a^2w\right), \tag{3.10}$$

which in complete agreement with the results of Section 2.2.

3.2. Particular Case: $n = 1$

For $k(fixed) \equiv n = 1$, the determinant $\Delta_2 = 0$ of (3.7) yields

$$g^2 + g\left(-1 + 10\mu + 2l(2\mu + 1) + 2a^2w(2\mu - 1)\right)$$
$$+ \mu(\mu - 1)\left(15 + 4l^2 + 8l(2 + a^2w) + 4a^2w(5 + a^2w)\right) = 0, \tag{3.11}$$

$$g - (\mu - 1)(\mu - 2) = 0,$$

where the energy is given by use of (3.3), for the computed values of μ and g, by

$$E = \left(l + \frac{3}{2} + 2\mu\right)w. \tag{3.12}$$

Further, (3.11) yields the solutions for l as functions of μ and $a^2 w$

$$l = \frac{2 - (5 + 4a^2 w)\mu - 2\mu^2 \pm \sqrt{4 - 4(3 + 8a^2 w)\mu + 9\mu^2}}{4\mu} \geq -1, \tag{3.13}$$

where the energy states are now given by (3.12) along with l given by (3.13). We may also note that for

$$a^2 w = \frac{1}{2}(k + 1), \quad k = 0, 1, 2, \ldots, \tag{3.14}$$

$$a^2 E_\pm = -\frac{1}{8\mu}(k + 1)\left(-2 + (2k + 1)\mu - 6\mu^2 \pm \sqrt{4 - 4(4k + 7)\mu + 9\mu^2}\right). \tag{3.15}$$

Further, for $g = (\mu - 1)(\mu - 2)$, we obtain the unnormalized wave function (see (2.2))

$$\psi_{1,l}(x) = x^{l+1}\left(1 + x^2\right)^{\mu-1} e^{-wa^2 x^2/2}\left(1 + \frac{1 + 2l + \mu + 2a^2 w}{5 + 2l + \mu + 2a^2 w}x^2\right). \tag{3.16}$$

Thus, we may summarize these results as follows. The exact solutions of the Schrödinger equation (2.1) are given by (3.15) and (3.16) *only if g and μ are the solutions of the system given by (3.11)*. In Tables 1 and 2, we report few quasiexact solutions that can be obtained using this approach.

3.2.1. Particular Case $n = 2$

For $k(fixed) \equiv n = 2$, the determinant $\Delta_3 = 0$ along with the necessary condition (3.7) yields

$$g^3 + 3g^2\left(7\mu - 1 + 2l(1 + \mu) + 2a^2 w(\mu - 1)\right)$$

$$- g\left[18 + 56l + 8l^2 + 18(7 + 2l)\mu - 3(5 + 2l)(7 + 2l)\mu^2 - 12a^2 w(\mu - 1)((7 + 2l)\mu - 4)\right.$$

$$\left. - 4a^4 w^2(2 + 3(\mu - 2)\mu)\right]$$

$$+ \mu(\mu - 2)(\mu - 1)\left(105 + 142l + 60l^2 + 8l^3 + 6a^2 w(5 + 2l)(7 + 2l)\right.$$

$$\left. + 12a^4 w^2(7 + 2l) + 8a^6 w^3\right) = 0,$$

$$g - (\mu - 2)(\mu - 3) = 0, \tag{3.17}$$

where, again, the energy is given, for the computed values of μ and g using (3.3) and (3.17), by

$$E = \left(l + \frac{3}{2} + 2\mu\right)w. \tag{3.18}$$

Table 1: Conditions on the value of the parameters g and μ for the quasipolynomial solutions in the case of $\Delta_2 = 0$ with different values of wa^2 and l.

n	l	wa^2	Conditions	$E^{wa^2}_{n,l} \equiv E^{wa^2}_{n,l}(\mu, g)$
1	−1	1/2	$\mu = 1/3(-3 - 15A^{-1/3} - A^{-1/3})$, $A = 3(36 - \sqrt{961})$ $g = 1/9A^{-2/3}(15 + 6A^{1/3} + A^{2/3})(15 + 9A^{1/3} + A^{2/3})$	$E^{1/2}_{1,-1} = -w(3/2 + 2/3A^{1/3} + 10A^{-1/3})$
		1	$\mu = 1/3(-5 - 19A^{-1/3} - A^{1/3})$, $A = 161 - 3\sqrt{2118}$ $g = 1/9A^{-2/3}(19 + 8A^{1/3} + A^{2/3})(19 + 11A^{1/3} + A^{2/3})$	$E^{1}_{1,-1} = -w(17/6 + 2/3A^{1/3} + 38/3A^{-1/3})$
		3/2	$\mu = 1/3(-7 - 25A^{-1/3} - A^{1/3})$, $A = 199 - 18\sqrt{74}$ $g = 1/9A^{-2/3}(25 + 10A^{1/3} + A^{2/3})(25 + 13A^{1/3} + A^{2/3})$	$E^{3/2}_{1,-1} = -w(25/6 + 2/3A^{1/3} + 50/3A^{-1/3})$
		2	$\mu = 1/3(-9 - 33A^{-1/3} - A^{1/3})$, $A = 3(72 - \sqrt{1191})$ $g = 1/9A^{-2/3}(33 + 12A^{1/3} + A^{2/3})(33 + 15A^{1/3} + A^{2/3})$	$E^{2}_{1,-1} = -w(11/2 + 2/3A^{1/3} + 22A^{-1/3})$
	0	1/2	$\mu = 0$ $g = 2$	$E^{1/2}_{1,0} = 3/2w$
			$\mu = -1/2(7 + \sqrt{17})$ $g = 29 + 5\sqrt{17}$	$E^{1/2}_{1,0} = -1/2(11 + 2\sqrt{17})w$
			$\mu = -1/2(7 - \sqrt{17})$ $g = 29 - 5\sqrt{17}$	$E^{1/2}_{1,0} = -1/2(11 - 2\sqrt{17})w$
		1	$\mu = -3 + B$ $g = (-4 + B)(-5 + B)$, $A = -108 + 3i\sqrt{2697}$ $B = 1/3\Re(A^{1/3} + 33A^{-1/3})$	$E^{1}_{1,0} = -(9/2 - 2B)w$
			$\mu = -3 - B,$ $g = (5 + B)(4 + B)$ $B = \Re((11(1 + i\sqrt{3})A^{-1/3}/2) + ((1 - i\sqrt{3})A^{1/3}/6))$, $A = -108 + 3i\sqrt{2697}$	$E^{1}_{1,0} = -(9/2 + 2B)w$
			$\mu = -3 - B,$ $g = (5 + B)(4 + B)$ $B = \Re((11(1 - i\sqrt{3})A^{-1/3}/2 + ((1 + i\sqrt{3})A^{1/3}/6))$, $A = -108 + 3i\sqrt{2697}$	$E^{1}_{1,0} = -(9/2 + 2B)w$

Table 2: Conditions on the value of the parameters g and μ for the quasipolynomial solutions in the case of $\Delta_2 = 0$ with different values of wa^2 and l.

n	l	wa^2	Conditions	$E_{n,l}^{wa^2} \equiv E_{n,l}^{wa^2}(\mu, g)$
0	3/2		$\mu = -11/3 + B$ $g = (-14/3 + B)(-17/3 + B)$ $B = 1/3\Re(A^{1/3} + 43A^{-1/3}), A = -98 + 9i\sqrt{863}$	$E_{1,0}^{3/2} = -1/6(35 - 12B)w$
			$\mu = -11/3 - B,$ $g = 1/9(17 + 3B)(14 + 3B)$ $B = 1/6\Re(43(1 + i\sqrt{3})A^{-1/3} + (1 - i\sqrt{3})A^{1/3}), A = -98 + 9i\sqrt{863}$	$E_{1,0}^{1} = -1/6(35 + 12B)w$
			$\mu = -11/3 - B,$ $g = 1/9(17 + 3B)(14 + 3B)$ $B = 1/6\Re(43(1 - i\sqrt{3})A^{-1/3} + (1 + i\sqrt{3})A^{1/3}), A = -98 + 9i\sqrt{863}$	$E_{1,0}^{1} = -1/6(35 + 12B)w$
	2		$\mu = -13/3 + B$ $g = 1/9(-16 + 3B)(-19 + 3B)$ $B = 1/3\Re(A^{1/3} + 55A^{-1/3}), A = -55 + 165i\sqrt{6}$	$E_{1,0}^{3/2} = -1/6(43 - 12B)w$
			$\mu = -13/3 - B,$ $g = 1/9(16 + 3B)(19 + 3B)$ $B = 1/6\Re(55(1 + i\sqrt{3})A^{-1/3} + (1 - i\sqrt{3})A^{1/3}), A = -55 + 165i\sqrt{6}$	$E_{1,0}^{1} = -1/6(43 + 12B)w$
			$\mu = -13/3 - B,$ $g = 1/9(16 + 3B)(19 + 3B)$ $B = 1/6\Re(55(1 - i\sqrt{3})A^{-1/3} + (1 + i\sqrt{3})A^{1/3}), A = -55 + 165i\sqrt{6}$	$E_{1,0}^{1} = -1/6(43 + 12B)w$

In Table 3, we report the numerical results for some of the exact solutions of μ and g using (3.17) and the values of $(l, wa^2) = (-1, 1/2)$, $(l, wa^2) = (-1, 1)$, $(l, wa^2) = (-1, 3/2)$, $(l, wa^2) = (-1, 2)$, $(l, wa^2) = (0, 1/2)$, and $(l, wa^2) = (0, 2)$, respectively. We have also computed the corresponding eigenvalues $E_{2,l}^{wa^2} \equiv E_{2,l}^{wa^2}(\mu, g)$.

4. Numerical Computation by the Use of the Asymptotic Iteration Method

For the potential parameters w, a^2, and g, not necessarily obeying the conditions for quasi-polynomial solutions discussed in the previous sections, the asymptotic iteration method can

Table 3: Exact eigenvalues for different values of l and wa^2 in the case $\Delta_3 = 0$.

n	l	wa^2	Conditions	$E_{n,l} \equiv E_{n,l}^{wa^2}(\mu, g)$
2	-1	$1/2$	$\mu_1 = -6.301870878994198$ $g_1 = 77.22293097048609$	$E_{2,-1}^{1/2} = -6.051870878994198$
			$\mu_2 = -2.4855365082108594$ $g_2 = 24.605574274703333$	$E_{2,-1}^{1/2} = -2.2355365082108594$
		1	$\mu_1 = -7.398182984326876$ $g_1 = 97.7240263912181$	$E_{2,-1}^{1} = -7.148182984326876$
			$\mu_2 = -3.3550579014968194$ $g_2 = 34.03170302988033$	$E_{2,-1}^{1} = -3.1050579014968194$
			$\mu_3 = 0.9498105417574756$ $g_3 = 2.1530873564462514$	$E_{2,-1}^{2} = 1.1998105417574756$
		$3/2$	$\mu_1 = -8.469623341124414$ $g_1 = 120.08263624614156$	$E_{2,-1}^{3/2} = -8.219623341124414$
			$\mu_2 = -4.27750521216504$ $g_2 = 45.684576900924284$	$E_{2,-1}^{1} = -4.02750521216504$
			$\mu_3 = 0.9282653601757613$ $g_3 = 2.2203497780234294$	$E_{2,-1}^{1} = 1.1782653601757613$
		2	$\mu_1 = -9.525122115065386$ $g_1 = 144.35356188223463$	$E_{2,-1}^{2} = -9.275122115065383$
			$\mu_2 = -5.226942179911145$ $g_2 = 59.45563545168999$	$E_{2,-1}^{2} = -4.976942179911145$
			$\mu_3 = 0.9186508169859244$ $g_3 = 2.250665238619284$	$E_{2,-1}^{2} = 1.1686508169859244$
2	0	$1/2$	$\mu_1 = -8.032243023438463$ $g_1 = 110.67814310476818$	$E_{2,-1}^{2} = -7.282243023438463$
			$\mu_2 = -4.32825470612182$ $g_2 = 46.37506233167478$	$E_{2,-1} = -3.57825470612182$
		2	$\mu_1 = -11.307737259773461$ $g_1 = 190.4036082349363$	$E_{2,-1}^{2} = -10.557737259773461$
			$\mu_2 = -7.180564905703867$ $g_2 = 93.46333689354533$	$E_{2,-1}^{2} = -6.430564905703867$
			$\mu_3 = 0.9472009101393033$ $g_3 = 2.1611850134722084$	$E_{2,-1}^{2} = 1.6972009101393033$

be employed to compute the eigenvalues of Schrödinger equation (2.1) for arbitrary values w, a^2, and g. The functions λ_0 and s_0, using (3.2) and (3.3), are given by

$$\lambda_0(t) = -\left(\frac{2l+3}{2t(1-t)} + \frac{2((Ea^2/2wa^2) - ((2l+3)/4) - 1)}{(1-t)} - \frac{wa^2}{(1-t)^2} \right),$$

$$s_0(t) = -\left(\frac{((Ea^2/2wa^2) - ((2l+3)/4))(2l+3+2wa^2)}{2t(1-t)^2} - \frac{g}{2}\frac{(2t-1)}{t(1-t)^2} \right. \tag{4.1}$$

$$\left. + \frac{((Ea^2/2wa^2) - ((2l+3)/4))((Ea^2/(2wa^2)) - ((2l+3)/4) - 1)}{(1-t)^2} \right),$$

Table 4: Energies of the four lowest states of the generalized isotonic oscillator of parameters w and a given for $l = -1$ as $wa^2 = 2$ and for different values of the parameter g. The subscript numbers represents the number of iterations used by AIM.

wa^2	g	$E_0 a^2$	$E_1 a^2$	$E_2 a^2$	$E_3 a^2$
2	0.000 01	$0.999\,993\,709\,536_{(39)}$	$2.999\,997\,742\,768_{(25)}$	$4.999\,998\,464\,613_{(32)}$	$6.999\,998\,987\,906_{(23)}$
	0.1	$0.936\,865\,790\,085_{(43)}$	$2.977\,274\,273\,728_{(33)}$	$4.984\,713\,354\,070_{(45)}$	$6.989\,892\,949\,082_{(32)}$
	1	$0.349\,595\,330\,721_{(51)}$	$2.758\,891\,177\,876_{(36)}$	$4.851\,946\,642\,761_{(42)}$	$6.900\,301\,395\,128_{(35)}$
	2	$-0.337\,237\,264\,447_{(51)}$	$2.487\,025\,791\,777_{(38)}$	$4.709\,976\,255\,628_{(42)}$	$6.803\,992\,334\,705_{(34)}$
	5	$-2.549\,035\,191\,007_{(53)}$	$1.494\,183\,218\,341_{(39)}$	$4.268\,043\,172\,724_{(45)}$	$6.534\,685\,249\,316_{(35)}$
	10	$-6.529\,142\,779\,202_{(60)}$	$-0.660\,939\,314\,881_{(40)}$	$3.318\,493\,978\,272_{(46)}$	$6.100\,400\,048\,017_{(38)}$
	12	$-8.182\,546\,155\,166_{(65)}$	$-1.659\,292\,230\,771_{(44)}$	$2.838\,014\,627\,229_{(48)}$	$5.905\,881\,549\,211_{(39)}$
	50	$-41.876\,959\,736\,225_{(37)}$	$-26.863\,072\,307\,493_{(33)}$	$-14.310\,287\,343\,156_{(28)}$	$-4.206\,192\,073\,796_{(31)}$

where $t \in (0, 1)$. The AIM sequence $\lambda_n(x)$ and $s_n(x)$ can be calculated iteratively using the iterative sequences (2.22). The energy eigenvalues of the quantum nonlinear isotonic potential (2.1) are obtained from the roots of the termination condition (2.21). According to the asymptotic iteration method, in particular, the study of Champion et al. [5], unless the differential equation is exactly solvable, the termination condition (2.21) produces for each iteration an expression that depends on both t and E (for given values of the parameters wa^2, g, and l). In such a case, one faces the problem of finding the best possible starting value $t = t_0$ that stabilizes the AIM process [5]. Fortunately, since $t \in (0, 1)$, the starting value t_0 does not represent a serious issue in our eigenvalue calculation using (4.1) and the termination condition (2.21) in contrast to the case of computing the eigenvalues using $\lambda_0(x)$ and $s_0(x)$ as given by, for example, (2.3), where $x \in (0, \infty)$. In Table 4, we report our numerical results for energies of the four lowest states of the generalized isotonic oscillator of parameters w and a such that $wa^2 = 2$ and for different values of g. In this table, we set $l = -1$ for computing the energies $E_0 a^2$ and $E_2 a^2$, while we put $l = 0$ for computing the energies $E_1 a^2$ and $E_3 a^2$, respectively. For most of these values, the starting value of t is $t_0 = 0.5$ and is shifted towards zero as g gets larger in value. For the values of g that admit a quasipolynomial solution, the number of iteration does not exceed *three*. For most of the other values of g, the total number of iteration did not exceed 65. We found that for $wa^2 = 2$ and the values of g reported in Table 4, the number of iteration is relatively small compared to the case of $wa^2 = 1/2$ and a large value of the parameter g. The numerical computations in the present work were done using *Maple* version 13 running on an IBM architecture personal computer in a high-precision environment. In order to accelerate our computation, we have written our own code for a root-finding algorithm instead of using the default procedure Solve of *Maple 13*. These numerical results are accurate to the number of decimals reported.
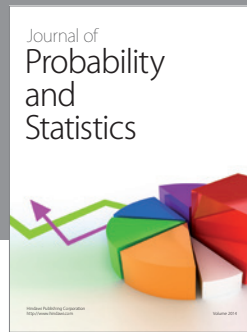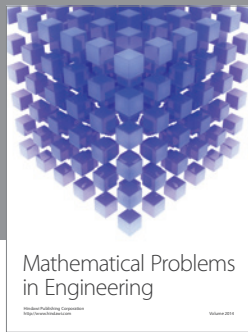
5. Conclusion

We have provided a detailed solution of the eigenproblem posed by Schrödiger's equation with a generalized nonlinear isotonic oscillator potential. We have presented a method for computing the quasipolynomial solutions in cases, where the potential parameters satisfy certain conditions. In other more general cases we have used the asymptotic iteration method to find accurate numerical solutions for arbitrary values of the potential parameters g, w, and a.

Acknowledgment

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