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Research Article

The Entanglement of Independent Quantum Systems

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The entanglement of states on n-independent C^* subalgebras is considered, and equivalent conditions are given for C^* subalgebras to be independent.

1. Introduction

Quantum correlations have been one of the most hottest subjects in the last two decades, many scholars devoted to the study [1–11]. In this paper, the most special correlation between quantum systems is investigated, that is "independence," which is closely related to the entanglement of the states.

In quantum mechanics, the state entanglement is the property of two particles with a common origin whereby a measurement on one of the particles determines not only its quantum state but also the quantum state of the other particle as well, which is characterized as follows.

Definition 1.1. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, and a state ϕ of $B(\mathscr{H}_1 \otimes \mathscr{H}_2) = B(\mathscr{H}_1) \otimes B(\mathscr{H}_2)$ is called to be separable if it is a convex combination of product states $\phi_i^1 \otimes \phi_i^2$, that is

$$\phi(A \otimes B) = \sum_{i=1}^{n} p_i \phi_i^1(A) \phi_i^2(B), \quad \text{where } \sum_{i=1}^{n} p_i = 1,\ p_i \geq 0. \tag{1.1}$$

Otherwise, ϕ is called entanglement.

In the algebraic quantum theory, the observable is represented by an adjoint operator in a C^* algebra. Naturally an interesting problem is raised.

Problem 1. In a C^* algebra $\mathcal{A}$, what is the condition of the commuting C^* subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$, under which the entanglement of states can be considered, where $\mathcal{A}_i$ is the C^* algebra generated by an observable A_i.

Notice that the problem is to seek the condition of commuting C^* algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, such that $C^*(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{A}_1 \otimes \mathcal{A}_2$ holds true.

In fact the condition is that $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^* independent.

Haag and Kastler [12] introduced a notion called statistical independence. If $\mathcal{A}$ and $\mathcal{B}$ represent the algebras generated by the observables associated with two quantum subsystems, the statistical independence of $\mathcal{A}$ and $\mathcal{B}$ can be construed as follows: any two partial states on the two subsystems can be realized by the same preparation procedure. The statistical independence of $\mathcal{A}$ and $\mathcal{B}$ in the category of C^* algebras is called C^* independence, which is defined as follows: if for any state ϕ_1 on $\mathcal{A}$ and ϕ_2 on $\mathcal{B}$, there is a state ϕ on $C^*(\mathcal{A}, \mathcal{B})$, such that $\phi|_{\mathcal{A}} = \phi_1$ and $\phi|_{\mathcal{B}} = \phi_2$, where $C^*(\mathcal{A}, \mathcal{B})$ denotes the unital C^* algebra generated by $\mathcal{A}, \mathcal{B}$.

Roos [13] gave a characterization of C^* independence. He showed if $\mathcal{A}$ and $\mathcal{B}$ are commuting subalgebras of a C^* algebra, then $\mathcal{A}$ and $\mathcal{B}$ are C^* independent, if and only if $0 \neq A \in \mathcal{A}$ and $0 \neq B \in \mathcal{B}$ imply that $AB \neq 0$. Later, Florig and Summers [14] studied the relation between C^* independence and W^* independence and showed if $\mathcal{A}$ and $\mathcal{B}$ are commuting subalgebras of a σ-finite W^* algebra $\mathcal{W}$, then they are C^* independent if and only if they are W^* independent, where $\mathcal{A}$ and $\mathcal{B}$ are W^* independent if for every normal state ϕ_1 on $\mathcal{A}$ and every normal state ϕ_2 on $\mathcal{B}$, there exists a normal state ϕ on $\mathcal{W}$, such that $\phi|_{\mathcal{A}} = \phi_1$ and $\phi|_{\mathcal{B}} = \phi_2$. Bunce and Hamhalter [15] gave some equivalent conditions by the faithfulness of the state on the C^* algebras. Jin et al. in [16–18] proved if $\mathcal{A}$ and $\mathcal{B}$ are C^* subalgebras of $\mathcal{A}$, they are C^* independent if and only if for any observables $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $W(A, B) = W(A) \times W(B)$, where $W(A, B)$ is the joint numerical range of operator tuple (A, B), if and only if for any observables $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\sigma(A, B) = \sigma(A) \times \sigma(B)$, where $\sigma(A, B)$ is the joint spectrum. By the theorem, we know the states on $C^*(\mathcal{A}, \mathcal{B})$ are "almost" separable (i.e., not entangled), which will be characterized and more detailed in the following.

The entanglement of $n\,(n \geq 3)$ particles is very important, and many problems are different from the case of two particles, such as "the maximal entangled pure states" [1]. To consider the entanglement of n particles in a C^* algebra, we introduce the independence of n C^* subalgebras as follows.

Definition 1.2. Let $(\mathcal{A}_i)_{i=1}^n$ be commuting unital C^* subalgebras of a C^* algebra $\mathcal{A}$, where $\mathcal{A}_i$ is generated by the observable A_i, and they are C^* independent if for any state ϕ_i on $\mathcal{A}_i$, there is a state ϕ on $\mathcal{A}$, such that $\phi|_{\mathcal{A}_i} = \phi_i$.

To study the entanglement of independent quantum systems, equivalent conditions are given for n quantum systems to be independent.

Theorem 1.3. *Let $(\mathcal{A}_i)_{i=1}^n$ be commuting unital C^* subalgebras of a C^* algebra $\mathcal{A}$, where $\mathcal{A}_i$ is generated by the observable A_i, then the following statements are equivalent.*

(1) $(\mathscr{A}_i)_{i=1}^n$ *are* C^* *independent.*

(2) *For any* $A_i \in \mathscr{A}_i$,

$$W(A_1,\ldots,A_n) = W(A_1) \times \cdots \times W(A_n), \tag{1.2}$$

where $W(A_1,\ldots,A_n)$ *denotes the joint numerical range.*

(3) *For any* $A_i \in \mathscr{A}_i$, $A_i \neq 0$ $(i = 1,\ldots,n)$ *imply that* $\prod_{i=1}^n A_i \neq 0$.

(4) *For any* $A_i \in \mathscr{A}_i$,

$$\sigma(A_1,\ldots,A_n) = \sigma(A_1) \times \cdots \times \sigma(A_n), \tag{1.3}$$

where $\sigma(A_1,\ldots,A_n)$ *denotes the joint spectrum.*

Remark 1.4. By the theorem, it is seen that the separable states on $C^*(A_1,\ldots,A_n)$ are dense in the state spaces, since the set of pure states corresponding to points of $\sigma(A_1) \times \cdots \times \sigma(A_n)$ is a dense subset of the state spaces. In particular, if $C^*(A_1,\ldots,A_n)$ is finite dimensions as a Banach space, every state of

$$C(\sigma(A_1,\ldots,A_n)) = C(\sigma(A_1)) \otimes \cdots \otimes C(\sigma(A_n)) \tag{1.4}$$

is a convex combination of pure states, pure states correspond to points of $\sigma(A_1) \times \cdots \times \sigma(A_n)$, and point masses are pure product states; thus there is not any entangled states on $C^*(A_1,\ldots,A_n)$, which gave us some hints that the entangled states are caused by those nonindependent and noncommutative observables.

2. Some Lemmas and Proof of the Theorem

It is a well-known result by Gelfand and Naimark that if $\mathscr{A}$ is a unital commutative C^* algebra and $M_{\mathscr{A}}$ is its maximal ideal space, then $\mathscr{A}$ is isometrical $*$-isomorphism with $C(M_{\mathscr{A}})$. In case of $(A_i)_{i=1}^n$ being commutative normal operators, denote by $M_{C^*(A)}$ the maximal ideal space of $C^*(A)$, and the joint spectrum of $A = (A_1,\ldots,A_n)$ is the set

$$\sigma_{C^*(A)}(A_1,\ldots,A_n) = \left\{ \left(f(A_1),\ldots,f(A_n)\right) \in \mathbb{C}^n;\ f \in M_{C^*(A)} \right\}. \tag{2.1}$$

The joint numerical range of $A = (A_1,\ldots,A_n)$ is the set

$$W(A_1,\ldots,A_n) = \left\{ \left(\phi(A_1),\ldots,\phi(A_n)\right) \in \mathbb{C}^n;\ \phi \text{ is a state on } \mathscr{A} \right\}. \tag{2.2}$$

Notice that the joint numerical range of $A = (A_1,\ldots,A_n)$ is the joint measurement of commuting observables A_i. It was shown in [19] that $\sigma(A_1,\ldots,A_n)$ is a nonempty compact set in $\mathbb{C}^n$.

Lemma 2.1 (see [20]). *Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple of adjoint operators in a unital C^* algebra $\mathcal{A}$, then*

$$\mathrm{Ex}(W(A_1, \dots, A_n)) \subseteq \sigma(A_1, \dots, A_n), \tag{2.3}$$

where $\mathrm{Ex}(X)$ *are the extreme points of the set* X.

Proof of Theorem 1.3. (1) ⇒(2) Let $A_i \in \mathcal{A}_i$, and it suffices to prove

$$W(A_1, \dots, A_n) \supseteq W(A_1) \times \cdots \times W(A_n). \tag{2.4}$$

By the definition,

$$W(A_i) = \{\phi_i(A_i);\ \phi_i \text{ is a state on } \mathcal{A}_i\}. \tag{2.5}$$

It follows by the condition of $\mathcal{A}_i$ being C^* independent that there is a state on $\mathcal{A}$, such that $\phi|_{\mathcal{A}_i} = \phi_i$, that is $\phi_i(A_i) = \phi(A_i)$, so

$$(\phi_1(A_1), \dots, \phi_n(A_n)) = (\phi(A_1), \dots, \phi(A_n)) \in W(A_1, \dots, A_n), \tag{2.6}$$

thus

$$W(A_1, \dots, A_n) \supseteq W(A_1) \times \cdots \times W(A_n). \tag{2.7}$$

(2) ⇒(3) Let $A_i \in \mathcal{A}_i$ be any nonzero observables and $\lambda_i = \|A_i\|$, then $\lambda_i \in \mathrm{Ex}(W(A_i))$, so

$$(\lambda_1, \dots, \lambda_n) \in \mathrm{Ex}(W(A_1, \dots, A_n)). \tag{2.8}$$

By Lemma 2.1,

$$(\lambda_1, \dots, \lambda_n) \in \sigma(A_1, \dots, A_n), \tag{2.9}$$

so there is a multiplicative linear functional ϕ on $C^*\,(A_1, \dots, A_n)$, such that

$$\phi(A_i) = \lambda_i, \tag{2.10}$$

therefore

$$\phi(A_1 \cdots A_n) = \phi(A_1) \cdots \phi(A_n), \tag{2.11}$$

thus

$$\left\|\prod_{i=1}^{n} A_i\right\| \geq \phi\left(\prod_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \phi(A_i) \neq 0, \tag{2.12}$$

that is $\prod_{i=1}^{n} A_i \neq 0$.

(3) ⇒(4) This easily follows by Urysohn's lemma.

(4) ⇒(1) By the Gelfand-Naimark theorem, the C^* algebra generated by the commuting observables A_i is isometrical $*$-isomorphism with $C(\sigma(A_1, \ldots, A_n))$,

$$\gamma : C^*(A_1, \ldots, A_n) \longrightarrow C(\sigma(A_1, \ldots, A_n)), \tag{2.13}$$

by condition (4), it has

$$C(\sigma(A_1, \ldots, A_n)) = C(\sigma(A_1)) \otimes \cdots \otimes C(\sigma(A_n)), \tag{2.14}$$

where the tensor product is the minimal C^* cross-norm.

Let ϕ_i be any state on $\mathcal{A}_i$, then $\phi_i \cdot \gamma^{-1} = \psi_i$ is a state on $C(\sigma(A_i))$, and it is seen that

$$\psi = \psi_1 \otimes \cdots \otimes \psi_n \tag{2.15}$$

is a state on $C(\sigma(A_1)) \otimes \cdots \otimes C(\sigma(A_n)) = C(\sigma(A_1, \ldots, A_n))$; thus the state $\phi = (\psi_1 \otimes \cdots \otimes \psi_n) \cdot \gamma$ is a state on $C^*(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, which satisfies $\phi|_{\mathcal{A}_i} = \phi_i$. □

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