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Research Article

Multiparameter Statistical Models from $N^2 \times N^2$ Braid Matrices: Explicit Eigenvalues of Transfer Matrices $\mathbf{T}^{(r)}$, Spin Chains, Factorizable Scatterings for All N

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For a class of multiparameter statistical models based on $N^2 \times N^2$ braid matrices, the eigenvalues of the transfer matrix $\mathbf{T}^{(r)}$ are obtained explicitly for all (r, N). Our formalism yields them as solutions of sets of linear equations with simple constant coefficients. The role of zero-sum multiplets constituted in terms of roots of unity is pointed out, and their origin is traced to circular permutations of the indices in the tensor products of basis states induced by our class of $\mathbf{T}^{(r)}$ matrices. The role of free parameters, increasing as N^2 with N, is emphasized throughout. Spin chain Hamiltonians are constructed and studied for all N. Inverse Cayley transforms of the Yang-Baxter matrices corresponding to our braid matrices are obtained for all N. They provide potentials for factorizable S-matrices. Main results are summarized, and perspectives are indicated in the concluding remarks.

1. Introduction

Statistical models "exact" in the sense of Baxter [1] satisfy "star-triangle" relations leading to transfer matrices commuting for different values of the spectral parameter. Crucial in the study of such models is the spectrum of eigenvalues of these matrices. But even for the extensively studied 6-vertex and 8-vertex models based on 4×4 braid matrices (the braid property guaranteeing star-triangle relations), after the first few simple steps, one has to resort to numerical computations. (We cannot adequately discuss here the vast associated literature but refer to texts citing major sources [2, 3].) But the basic reason for such situation is that, starting with 2×2 blocks T_{ab} of the transfer matrix and constructing $2^r \times 2^r$ blocks via coproduct rules for order r $(r = 1,2,3,\ldots)$, one faces increasingly complicated nonlinear systems of equations to be solved in constructing eigenstates and eigenvalues. Even when a systematic approach is available, such as the Bethe Ansatz for the 6-vertex case, it only means that the relevant nonlinear equations can be written down systematically. The task of solving them remains. For the 8-vertex case one explores analytical properties to extract informations (See [4], e.g.). Also the number of free parameters remains strictly limited for such models, including the multistate generalization of the 6-vertex one [2].

Our class of models exhibits the following properties.

(1) A systematic construction for all dimensions. One starts from $N^2 \times N^2$ braid matrices leading to $N^r \times N^r$ blocks of $\mathbf{T}^{(r)}$, the transfer matrix of order r for $N = 2,3,4,5,\ldots; r = 1,2,3,\ldots$ with no upper limit.

(2) The number of free parameters increases with N as N^2. This is a unique feature of our models.

(3) For all (N,r) one solves sets of linear equations with systematically obtained simple constant coefficients to construct eigenvalues of $\mathbf{T}^{(r)} = \sum_{a=1}^{N} T_{aa}^{(r)}$. This is a consequence of our starting point: braid matrices constructed on a basis of "nested sequence" of projectors [5].

The total dimension of the base space on which such equations have to be solved increases as N^r. This will be seen to break up into subspaces closed under the action of $\mathbf{T}^{(r)}$, thus reducing the work considerably. Evidently one cannot continue to display the results explicitly, as N^r increases. But it is possible to implement a general and particularly efficient approach. This will be illustrated for all N and $r = 1,2,3,4,5$. The generalization to $r > 5$ will be clearly visible. This is the central purpose of our paper. Certain other features will be explored with comments in conclusion.

2. Braid and Transfer Matrices

The roots of our class of braid matrices and the transfer matrices they generate are to be found in the nested sequence of projectors [5]. In [5] it was noted that the 4×4 projectors providing the basis of the 6-vertex and 8-vertex models (i.e., the braid matrices leading to those models) can be generalized to higher dimensions—all higher ones (N odd or even). Braid matrices for odd dimensions were exhaustively constructed on such a basis [6], and related statistical models were studied [7]. Then even dimensional ones were presented [8], and the corresponding braid matrices and statistical models were studied [9]. In previous works the eigenvalues and eigenfunctions of the transfer matrices were presented mostly for the lowest values of N, namely, the 4×4 and 9×9 braid matrices. We present below a general approach for all N.

Let us just mention that systematic study of the exotic bialgebras that arise from the 9×9 unitary braid matrices was constructed in [10]. The dual bialgebra of one of these exotic bialgebras was also presented.

We start now by recapitulating the construction of our nested sequence of projectors, leading to the remarkable properties of our solutions.

2.1. Even Dimensions

Let $N = 2n$ $(n = 1, 2, 3, \ldots)$. Define ((ab) denoting a matrix with unity on row a and column b)

$$P_{ij}^{(\epsilon)} = \frac{1}{2}\left\{(ii) \otimes (jj) + \left(\bar{i}\,\bar{i}\right) \otimes \left(\bar{j}\,\bar{j}\right) + \epsilon\left[\left(i\bar{i}\right) \otimes \left(j\bar{j}\right) + \left(\bar{i}i\right) \otimes \left(\bar{j}j\right)\right]\right\}, \tag{2.1}$$

where $i, j \in \{1, \ldots, n\}$, $\epsilon = \pm$, $\bar{i} = 2n + 1 - i$, $\bar{j} = 2n + 1 - j$. Interchanging $j \rightleftharpoons \bar{j}$ on the right one obtains $P_{i\bar{j}}^{(\epsilon)}$. One thus obtains a complete basis of projectors satisfying (with $P_{ij}^{(\epsilon)} = P_{\bar{i}\,\bar{j}}^{(\epsilon)}$, $P_{i\bar{j}}^{(\epsilon)} = P_{\bar{i}j}^{(\epsilon)}$ by definition)

$$\begin{gathered} P_{ab}^{(\epsilon)} P_{cd}^{(\epsilon')} = \delta_{ac}\delta_{bd}\delta_{\epsilon\epsilon'} P_{ab}^{(\epsilon)}, \qquad a, b, c, d \in \{1, \ldots, N = 2n\}, \\ \sum_{\epsilon=\pm}\sum_{i,j=1}^{n}\left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)}\right) = I_{(2n)^2\times(2n)^2}. \end{gathered} \tag{2.2}$$

Let $m_{ij}^{(\epsilon)}$ be an arbitrary set of parameters satisfying the crucial constraint

$$m_{ij}^{(\epsilon)} = m_{i\bar{j}}^{(\epsilon)}. \tag{2.3}$$

Define the $N^2 \times N^2$ matrix

$$\widehat{R}(\theta) = \sum_{\epsilon}\sum_{i,j} e^{m_{ij}^{(\epsilon)}\theta}\left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)}\right). \tag{2.4}$$

The θ-dependence entering exclusively through the exponentials as coefficients as above. Such construction guarantees [7–9] the braid property

$$\widehat{R}_{12}(\theta)\widehat{R}_{23}(\theta + \theta')\widehat{R}_{12}(\theta') = \widehat{R}_{23}(\theta')\widehat{R}_{12}(\theta + \theta')\widehat{R}_{23}(\theta), \tag{2.5}$$

where, in standard notations, $\widehat{R}_{12} = \widehat{R} \otimes I$ and $\widehat{R}_{23} = I \otimes \widehat{R}$, I denoting the $N \times N$ identity matrix.

Define the permutation matrix $\mathbf{P} = \sum_{a,b}(ab) \otimes (ba)$. Then

$$R(\theta) = \mathbf{P}\widehat{R}(\theta) \tag{2.6}$$

satisfies the Yang-Baxter equation. The monodromy matrix of order 1 $(r = 1)$ is given by

$$T^{(1)}(\theta) = R(\theta). \tag{2.7}$$

The $N \times N$ blocks are (with $a, b \in \{1, \ldots, 2n\}$)

$$T_{ab}^{(1)}(\theta) = \frac{1}{2}\left(e^{m_{ba}^{(+)}\theta} + e^{m_{ba}^{(-)}\theta}\right)(ba) + \frac{1}{2}\left(e^{m_{ba}^{(+)}\theta} - e^{m_{ba}^{(-)}\theta}\right)\left(\overline{b}\overline{a}\right) \equiv f_{ba}^{(+)}(\theta)(ba) + f_{ba}^{(-)}(\theta)\left(\overline{b}\overline{a}\right), \quad (2.8)$$

with

$$m_{ba}^{(\epsilon)} = m_{b\overline{a}}^{(\epsilon)} = m_{\overline{b}a}^{(\epsilon)} = m_{\overline{b}\overline{a}}^{(\epsilon)}. \quad (2.9)$$

Higher orders ($r > 1$) are obtained via coproducts

$$T_{ab}^{(r)}(\theta) = \sum_{c_1, \ldots, c_{r-1}} T_{ac_1}^{(1)}(\theta) \otimes T_{c_1 c_2}^{(1)}(\theta) \otimes \cdots \otimes T_{c_{r-2} c_{r-1}}^{(1)}(\theta) \otimes T_{c_{r-1} b}^{(1)}(\theta). \quad (2.10)$$

For some essential purposes, it is worthwhile to express (2.8) as (with $i, j \in \{1, \ldots, n\}$)

$$\begin{aligned}
T_{ij}^{(1)}(\theta) &= \frac{1}{2}\sum_{\epsilon} e^{m_{ji}^{(\epsilon)}\theta}\left((ji) + \epsilon\left(\overline{j}\,\overline{i}\right)\right), \\
T_{\overline{i}\overline{j}}^{(1)}(\theta) &= \frac{1}{2}\sum_{\epsilon} e^{m_{ji}^{(\epsilon)}\theta}\left(\left(\overline{j}\,\overline{i}\right) + \epsilon(ji)\right), \\
T_{i\overline{j}}^{(1)}(\theta) &= \frac{1}{2}\sum_{\epsilon} e^{m_{ji}^{(\epsilon)}\theta}\left(\left(\overline{j}i\right) + \epsilon\left(j\overline{i}\right)\right), \\
T_{\overline{i}j}^{(1)}(\theta) &= \frac{1}{2}\sum_{\epsilon} e^{m_{ji}^{(\epsilon)}\theta}\left(\left(j\overline{i}\right) + \epsilon\left(\overline{j}i\right)\right).
\end{aligned} \quad (2.11)$$

The transfer matrix is the trace on (a, b):

$$\mathbf{T}^{(r)}(\theta) = \sum_{a} T_{aa}^{(r)}(\theta). \quad (2.12)$$

The foregoing construction assures the crucial commutativity

$$\left[\mathbf{T}^{(r)}(\theta), \mathbf{T}^{(r)}(\theta')\right] = 0. \quad (2.13)$$

This implies that the eigenstates of $\mathbf{T}^{(r)}(\theta)$ are θ-independent (sum of basis vectors with θ-independent, constant relative coefficients).

2.2. Odd Dimensions

From the extensive previous studies [6, 7], we select the essential features. For $N = 2n - 1$ ($n = 2, 3, \ldots$),

$$\overline{n} = N - n + 1 = n \quad (2.14)$$

with the same definition as in (2.1). This is the crucial new feature. For $(i,j) \neq n$, one has the same $P_{ij}^{(\epsilon)}$, $P_{\bar{i}\bar{j}}^{(\epsilon)}$ as before. But now

$$
\begin{aligned}
P_{in}^{(\epsilon)} &= \frac{1}{2}\left\{(ii) + \left(\bar{i}\,\bar{i}\right) + \epsilon\left[\left(i\bar{i}\right) + \left(\bar{i}i\right)\right]\right\} \otimes (nn), \\
P_{ni}^{(\epsilon)} &= \frac{1}{2}(nn) \otimes \left\{(ii) + \left(\bar{i}\,\bar{i}\right) + \epsilon\left[\left(i\bar{i}\right) + \left(\bar{i}i\right)\right]\right\}, \\
P_{nn}^{(\epsilon)} &= (nn) \otimes (nn).
\end{aligned}
\tag{2.15}
$$

Normalizing the coefficient of $P_{nn}^{(\epsilon)}$ to unity, $\widehat{R}(\theta)$ of (2.4) now becomes

$$
\widehat{R}(\theta) = P_{nn}^{(\epsilon)} + \sum_{i,\epsilon}\left(e^{m_{ni}^{(\epsilon)}\theta} P_{ni}^{(\epsilon)} + e^{m_{in}^{(\epsilon)}\theta} P_{in}^{(\epsilon)}\right) + \sum_{i,j\epsilon} e^{m_{ij}^{(\epsilon)}\theta}\left(P_{ij}^{(\epsilon)} + P_{\bar{i}\bar{j}}^{(\epsilon)}\right) \tag{2.16}
$$

($i, j \in \{1, \ldots, n-1\}$, $\bar{i}, \bar{j} \in \{2n-1, \ldots, n+1\}$, $\epsilon = \pm$). There are evident parallel modifications concerning $T_{ab}^{(r)}(\theta)$. Reference [7] contains detailed discussions and results for $N = 3$, $r = 1, 2, 3, 4$. We will not repeat them here but will reconsider them later in the context of the general methods to be implemented below.

2.3. Free Parameters

A unique feature of our constructions is the number of free parameters $m_{ij}^{(\epsilon)}$, increasing as N^2 with N. For our choice of normalizations (apart from a possible altered choice of an overall factor, irrelevant for (2.5)), the exact numbers are

$$
\begin{aligned}
\frac{1}{2}N^2 &= 2n^2 \qquad \text{for} \quad N = 2n, \\
\frac{1}{2}(N+3)(N-1) &= 2\left(n^2 - 1\right) \qquad \text{for} \quad N = 2n-1.
\end{aligned}
\tag{2.17}
$$

This is to be contrasted with the multistate generalization of the 6-vertex model where the parametrization remain fixed at the 6-vertex level ([2] and sources cited here).

One of our principal aims is to display the roles of our free parameters concerning the basic features of our models.

3. Explicit Eigenvalues of Transfer Matrices for All Dimensions N and Orders r

We present below a unified approach for all N and illustrate it in some detail for $r = 1, 2, 3, 4, 5$. For $r > 5$ the extension will be evident.

3.1. Even Dimensions

In Section 5 of [9] it was pointed out that the antidiagonal matrix

$$K = \sum_{a=1}^{2n} (a\overline{a}) = \sum_{i=1}^{n} \left(\left(i\overline{i} \right) + \left(\overline{i}i \right) \right) \tag{3.1}$$

relates the blocks of $T^{(1)}$ as follows:

$$KT^{(1)}_{ab} = T^{(1)}_{\overline{a}b}, \qquad T^{(1)}_{ab} K = T^{(1)}_{a\overline{b}}, \qquad KT^{(1)}_{ab} K = T^{(1)}_{\overline{a}\overline{b}}. \tag{3.2}$$

For $N = 2$ this led to an iterative construction of eigenstates and eigenvalues of the transfer matrix $\mathbf{T}^{(r)}$ for increasing r. This works efficiently only for $N = 2$ (the 4×4 $\widehat{R}(\theta)$-matrix).

For our present approach the crucial ingredient shows up immediately in the structure (arising from those of our projectors and the coproducts rules)

$$\begin{aligned} \mathbf{T}^{(r)} = \frac{1}{2^r} \sum_{a_1, a_2, \dots, a_r} \sum_{\epsilon_{21}, \epsilon_{32}, \dots, \epsilon_{1r}} & e^{(m^{(\epsilon_{21})}_{a_2 a_1} + m^{(\epsilon_{32})}_{a_3 a_2} + \cdots + m^{(\epsilon_{1r})}_{a_1 a_r})\theta} \\ & \times \left[(a_2 a_1) + \epsilon_{21} (\overline{a}_2 \overline{a}_1) \right] \\ & \otimes \left[(a_3 a_2) + \epsilon_{32} (\overline{a}_3 \overline{a}_2) \right] \otimes \cdots \otimes \left[(a_1 a_r) + \epsilon_{1r} (\overline{a}_1 \overline{a}_r) \right], \end{aligned} \tag{3.3}$$

where each ϵ is independently $(\pm)$. From (3.3), one obtains immediately

$$\begin{aligned} \left(\mathbf{T}^{(r)} \right)^{-1} = \frac{1}{2^r} \sum_{a_1, a_2, \dots, a_r} \sum_{\epsilon_{21}, \epsilon_{32}, \dots, \epsilon_{1r}} & e^{-(m^{(\epsilon_{21})}_{a_2 a_1} + m^{(\epsilon_{32})}_{a_3 a_2} + \cdots + m^{(\epsilon_{1r})}_{a_1 a_r})\theta} \\ & \times \left[(a_1 a_2) + \epsilon_{21} (\overline{a}_1 \overline{a}_2) \right] \\ & \otimes \left[(a_2 a_3) + \epsilon_{32} (\overline{a}_2 \overline{a}_3) \right] \otimes \cdots \otimes \left[(a_r a_1) + \epsilon_{1r} (\overline{a}_r \overline{a}_1) \right]. \end{aligned} \tag{3.4}$$

Consider the basis state

$$|b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_r\rangle \equiv |b_1 b_2 \cdots b_r\rangle. \tag{3.5}$$

(1) If any one (even a single one) of the indices $b_i \neq a_{i+1}$ or $\overline{a_{i+1}}$ (cyclic), then the coefficient of $e^{(m^{(\epsilon_{21})}_{a_2 a_1} + \cdots + m^{(\epsilon_{1r})}_{a_1 a_r})\theta}$, namely, $[(a_2 a_1) + \epsilon_{21}(\overline{a}_2 \overline{a}_1)], \otimes \cdots \otimes [(a_1 a_r) + \epsilon_{1r}(\overline{a}_1 \overline{a}_r)]$, will annihilate it. On the other hand (2.9) plays a crucial role. Gathering together all the terms with

$e^{(m_{b_2b_1}^{(\epsilon_{21})}+m_{b_3b_2}^{(\epsilon_{32})}+\cdots+m_{b_rb_{r-1}}^{(\epsilon_{r,r-1})}+m_{b_1b_r}^{(\epsilon_{1r})})\theta}$ as coefficient, one obtains the action of $\mathbf{T}^{(r)}$ on (3.5). All essential features for the general case can be read off the first few simples examples:

$$2^2\mathbf{T}^{(2)}|b_1b_2\rangle = \sum_{\epsilon_{21},\epsilon_{12}} e^{(m_{b_2b_1}^{(\epsilon_{21})}+m_{b_1b_2}^{(\epsilon_{12})})\theta}(1+\epsilon_{21}\epsilon_{12})\left[|b_2b_1\rangle + \epsilon_{21}\left|\overline{b}_2\overline{b}_1\right\rangle\right], \tag{3.6}$$

$$\begin{aligned} 2^3\mathbf{T}^{(3)}|b_1b_2b_3\rangle = &\sum_{\epsilon_{21},\epsilon_{32},\epsilon_{13}} e^{(m_{b_2b_1}^{(\epsilon_{21})}+m_{b_3b_2}^{(\epsilon_{32})}+m_{b_1b_3}^{(\epsilon_{13})})\theta}(1+\epsilon_{21}\epsilon_{32}\epsilon_{13}) \\ &\times\left[|b_2b_3b_1\rangle + \epsilon_{21}\left|\overline{b}_2b_3\overline{b}_1\right\rangle + \epsilon_{32}\left|\overline{b}_2\overline{b}_3b_1\right\rangle + \epsilon_{13}\left|b_2\overline{b}_3\overline{b}_1\right\rangle\right], \end{aligned} \tag{3.7}$$

$$\begin{aligned} 2^4\mathbf{T}^{(4)}|b_1b_2b_3b_4\rangle = &\sum_{\epsilon_{21},\epsilon_{32},\epsilon_{43},\epsilon_{14}} e^{(m_{b_2b_1}^{(\epsilon_{21})}+m_{b_3b_2}^{(\epsilon_{32})}+m_{b_4b_3}^{(\epsilon_{43})}+m_{b_1b_4}^{(\epsilon_{14})})\theta}(1+\epsilon_{21}\epsilon_{32}\epsilon_{43}\epsilon_{14}) \\ &\times\left[|b_2b_3b_4b_1\rangle + \epsilon_{21}\left|\overline{b}_2b_3b_4\overline{b}_1\right\rangle + \epsilon_{32}\left|\overline{b}_2\overline{b}_3b_4b_1\right\rangle\right. \\ &+\epsilon_{43}\left|b_2\overline{b}_3\overline{b}_4b_1\right\rangle + \epsilon_{14}\left|b_2b_3\overline{b}_4\overline{b}_1\right\rangle + \epsilon_{21}\epsilon_{32}\left|b_2\overline{b}_3b_4\overline{b}_1\right\rangle \\ &\left.+\epsilon_{21}\epsilon_{43}\left|\overline{b}_2\overline{b}_3\overline{b}_4\overline{b}_1\right\rangle + \epsilon_{21}\epsilon_{14}\left|\overline{b}_2b_3\overline{b}_4b_1\right\rangle\right]. \end{aligned} \tag{3.8}$$

Note that the bars correspond to the indies of the ϵ's. One has thus $\epsilon_{ij}|\overline{b}_i\overline{b}_jb_kb_l\rangle$, $\epsilon_{ij}\epsilon_{ki}|\overline{\overline{b}}_i\overline{b}_j\overline{b}_kb_l\rangle = \epsilon_{ij}\epsilon_{ki}|b_i\overline{b}_j\overline{b}_kb_l\rangle$ since $\overline{\overline{b}}_i = b_i$. Consequently only even number of additional bars appear on the right. Starting with even or odd number of bars on the left leads to "even" and "odd" closed subspaces.

$$\begin{aligned} 2^5\mathbf{T}^{(5)}|b_1b_2b_3b_4b_5\rangle = &\sum_{\epsilon_{21},\epsilon_{32},\epsilon_{43},\epsilon_{14}} e^{(m_{b_2b_1}^{(\epsilon_{21})}+m_{b_3b_2}^{(\epsilon_{32})}+m_{b_4b_3}^{(\epsilon_{43})}+m_{b_5ba_4}^{(\epsilon_{54})}+m_{b_1b_5}^{(\epsilon_{15})})\theta}(1+\epsilon_{21}\epsilon_{32}\epsilon_{43}\epsilon_{54}\epsilon_{15}) \\ &\times\left[|b_2b_3b_4b_5b_1\rangle + \epsilon_{21}\left|\overline{b}_2b_3b_4b_5\overline{b}_1\right\rangle + \epsilon_{32}\left|\overline{b}_2\overline{b}_3b_4b_5b_1\right\rangle\right. \\ &+\epsilon_{43}\left|b_2\overline{b}_3\overline{b}_4b_5b_1\right\rangle + \epsilon_{54}\left|b_2b_3\overline{b}_4\overline{b}_5b_1\right\rangle + \epsilon_{15}\left|b_2b_3b_4\overline{b}_5\overline{b}_1\right\rangle \\ &+\epsilon_{21}\epsilon_{32}\left|b_2\overline{b}_3b_4b_5\overline{b}_1\right\rangle + \epsilon_{21}\epsilon_{43}\left|\overline{b}_2\overline{b}_3\overline{b}_4b_5\overline{b}_1\right\rangle \\ &+\epsilon_{21}\epsilon_{54}\left|\overline{b}_2b_3\overline{b}_4\overline{b}_5\overline{b}_1\right\rangle + \epsilon_{21}\epsilon_{15}\left|\overline{b}_2b_3b_4\overline{b}_5b_1\right\rangle \\ &+\epsilon_{32}\epsilon_{43}\left|\overline{b}_2b_3\overline{b}_4b_5b_1\right\rangle + \epsilon_{32}\epsilon_{54}\left|\overline{b}_2\overline{b}_3\overline{b}_4\overline{b}_5b_1\right\rangle \\ &+\epsilon_{32}\epsilon_{15}\left|\overline{b}_2\overline{b}_3b_4\overline{b}_5\overline{b}_1\right\rangle + \epsilon_{43}\epsilon_{54}\left|b_2\overline{b}_3b_4\overline{b}_5b_1\right\rangle \\ &\left.+\epsilon_{43}\epsilon_{15}\left|b_2\overline{b}_3\overline{b}_4\overline{b}_5\overline{b}_1\right\rangle + \epsilon_{54}\epsilon_{15}\left|b_2b_3\overline{b}_4b_5\overline{b}_1\right\rangle\right]. \end{aligned} \tag{3.9}$$

(2) It is important to note that the preceding features are independent of N—for a given r they are valid for all N. For $r > N$ all the indices $(b_1, \ldots, b_r)$ cannot be distinct. But

even for $r < N$, some or all of them can coincide. This general validity permits a unified treatment for all N.

(3) Note also the crucial cyclic permutation of the indices under the action of $\mathbf{T}^{(r)}$: $(b_1, b_2, b_3, \ldots, b_{r-1}, b_r) \rightarrow (b_2, b_3, b_4, \ldots, b_r, b_1)$. This is a consequence of the trace condition in (2.12) incorporated in (3.3). It will be seen later how this leads to an essential role of the rth roots of unity $e^{\mathbf{i}((k\cdot 2\pi)/r)}$ $(k = 0, 1, \ldots, r-1)$ in our construction of eigenstates and also of eigenvalues as factors accompanying exponentials of the type appearing in (3.6)–(3.9).

We continue the study of the particular cases $r = 1, 2, 3, 4, 5$ each one for any N. General features will be better understood afterwards.

(i) $r = 1$

$$\mathbf{T}^{(1)} = \frac{1}{2}\sum_{a,\epsilon} e^{m_{aa}^{(\epsilon)}\theta}\left[(aa) + \epsilon(\overline{a}\,\overline{a})\right] = \frac{1}{2}\sum_{i,\epsilon}(1+\epsilon)e^{m_{ii}^{(\epsilon)}\theta}\left[(ii) + \epsilon\left(\bar{i}\,\bar{i}\right)\right], \tag{3.10}$$

where $a \in \{1, \ldots, 2n\}$, $i \in \{1, \ldots, n\}$. This is already in diagonal form. Eigenstates and values are obtained trivially.

(ii) $r = 2$

We start with (3.6). Separating "even" and "odd" spaces, define (for $i \neq j$ when $(i, j, \bar{i}, \bar{j})$ are all distinct)

$$\begin{gathered}
\mathbf{T}^{(2)}\left(c_1|ij\rangle + c_2\left|\bar{i}\,\bar{j}\right\rangle + c_3|ji\rangle + c_4\left|\bar{j}\,\bar{i}\right\rangle\right) = v_e\left(c_1|ij\rangle + c_2\left|\bar{i}\,\bar{j}\right\rangle + c_3|ji\rangle + c_4\left|\bar{j}\,\bar{i}\right\rangle\right), \\
\mathbf{T}^{(2)}\left(d_1\left|i\bar{j}\right\rangle + d_2\left|\bar{i}j\right\rangle + d_3\left|j\bar{i}\right\rangle + d_4\left|\bar{j}i\right\rangle\right) = v_o\left(d_1\left|i\bar{j}\right\rangle + d_2\left|\bar{i}j\right\rangle + d_3\left|j\bar{i}\right\rangle + d_4\left|\bar{j}i\right\rangle\right).
\end{gathered} \tag{3.11}$$

The solutions are, for $(c_1, c_2, c_3, c_4) = (1,1,1,1), (1,1,-1,-1), (1,-1,1,-1), (1,-1,-1,1)$ respectively,

$$v_e = e^{(m_{ij}^{(+)} + m_{ji}^{(+)})\theta}, -e^{(m_{ij}^{(+)} + m_{ji}^{(+)})\theta}, e^{(m_{ij}^{(-)} + m_{ji}^{(-)})\theta}, -e^{(m_{ij}^{(-)} + m_{ji}^{(-)})\theta}, \tag{3.12}$$

and for $(d_1, d_2, d_3, d_4) = (1,1,1,1), (1,1,-1,-1), (1,-1,1,-1), (1,-1,-1,1)$, respectively,

$$v_o = e^{(m_{ij}^{(+)} + m_{ji}^{(+)})\theta}, -e^{(m_{ij}^{(+)} + m_{ji}^{(+)})\theta}, -e^{(m_{ij}^{(-)} + m_{ji}^{(-)})\theta}, e^{(m_{ij}^{(-)} + m_{ji}^{(-)})\theta}. \tag{3.13}$$

For $i = j$, with $\epsilon = \pm$

$$\begin{gathered}
\mathbf{T}^{(2)}\left(|ii\rangle + \epsilon\left|\bar{i}\,\bar{i}\right\rangle\right) = e^{2m_{ii}^{(\epsilon)}\theta}\left(|ii\rangle + \epsilon\left|\bar{i}\,\bar{i}\right\rangle\right), \\
\mathbf{T}^{(2)}\left(\left|i\bar{i}\right\rangle + \epsilon\left|\bar{i}i\right\rangle\right) = \epsilon e^{2m_{ii}^{(\epsilon)}\theta}\left(\left|i\bar{i}\right\rangle + \epsilon\left|\bar{i}i\right\rangle\right).
\end{gathered} \tag{3.14}$$

Summing over all i, consistently with the general rule, one obtains

$$\mathrm{Tr}\left(\mathbf{T}^{(2)}\right) = 2\sum_{i=1}^{n} e^{2m_{ii}^{(+)}\theta} \tag{3.15}$$

contributions coming only for $i = j$. For $r = 2$, the rth roots of unity are ± 1 and their role above is not visible very distinctly due to the presence of ± 1 also from ϵ. The role of the roots of unity will be more evident from $r = 3$ onwards. The simple exercise above gives all eigenvalues and eigenstates for $\mathbf{T}^{(2)}$ for all $N\ (= 2n)$. For $N = 2$, $r = 2$, writing out the 16×16 $\mathbf{T}^{(2)}$ in full and effectively diagonalizing it we have confirmed the results above.

(iii) *r=3*

Now the starting point is (3.7). Define, for the even subset,

$$\begin{aligned} V_\lambda = {} & \left(|i_1 i_2 i_3\rangle + \epsilon_{12}\left|\bar{i}_1\bar{i}_2 i_3\right\rangle + \epsilon_{23}\left|i_1\bar{i}_2\bar{i}_3\right\rangle + \epsilon_{31}\left|\bar{i}_1 i_2\bar{i}_3\right\rangle\right) \\ & + \lambda\left(|i_3 i_1 i_2\rangle + \epsilon_{12}\left|i_3\bar{i}_1\bar{i}_2\right\rangle + \epsilon_{23}\left|\bar{i}_3 i_1\bar{i}_2\right\rangle + \epsilon_{31}\left|\bar{i}_3\bar{i}_1 i_2\right\rangle\right) \\ & + \lambda^2\left(|i_2 i_3 i_1\rangle + \epsilon_{12}\left|\bar{i}_2 i_3\bar{i}_1\right\rangle + \epsilon_{23}\left|\bar{i}_2\bar{i}_3 i_1\right\rangle + \epsilon_{31}\left|i_2\bar{i}_3\bar{i}_1\right\rangle\right), \end{aligned} \tag{3.16}$$

where

$$\lambda = \left(1, e^{\mathbf{i}(2\pi/3)}, e^{\mathbf{i}(4\pi/3)}\right), \tag{3.17}$$

and hence $\lambda^3 = 1$, and also $1 + e^{\mathbf{i}(2\pi/3)} + e^{\mathbf{i}(4\pi/3)} = 0$. Corresponding to $(i_1, i_2, i_3) \rightleftharpoons (\bar{i}_1, \bar{i}_2, \bar{i}_3)$, define for the odd-subset,

$$\begin{aligned} \overline{V_\lambda} = {} & \left(\left|\bar{i}_1\bar{i}_2\bar{i}_3\right\rangle + \epsilon_{12}\left|i_1 i_2\bar{i}_3\right\rangle + \epsilon_{23}\left|\bar{i}_1 i_2 i_3\right\rangle + \epsilon_{31}\left|i_1\bar{i}_2 i_3\right\rangle\right) \\ & + \lambda\left(\left|\bar{i}_3\bar{i}_1\bar{i}_2\right\rangle + \epsilon_{12}\left|\bar{i}_3 i_1 i_2\right\rangle + \epsilon_{23}\left|i_3\bar{i}_1 i_2\right\rangle + \epsilon_{31}\left|i_3 i_1\bar{i}_2\right\rangle\right) \\ & + \lambda^2\left(\left|\bar{i}_2\bar{i}_3\bar{i}_1\right\rangle + \epsilon_{12}\left|i_2\bar{i}_3 i_1\right\rangle + \epsilon_{23}\left|i_2 i_3\bar{i}_1\right\rangle + \epsilon_{31}\left|\bar{i}_2 i_3 i_1\right\rangle\right). \end{aligned} \tag{3.18}$$

The 8 values of the set $(\epsilon_{12}, \epsilon_{23}, \epsilon_{31})$ are restricted (see (3.7)) by $(1 + \epsilon_{12}\epsilon_{23}\epsilon_{31}) = 2\ (\neq 0)$. The remaining 4 possibilities along with 3 for λ (see (3.17)) yield $4 \times 3 = 12$ solutions for V_λ and $\overline{V_\lambda}$ each for distinct indices $i_1 \neq i_2 \neq i_3 \neq i_1$. The eigenvalues are both V_λ and $\overline{V_\lambda}$:

$$e^{(m_{i_1 i_2}^{(\epsilon_{12})} + m_{i_2 i_3}^{(\epsilon_{23})} + m_{i_3 i_1}^{(\epsilon_{31})})\theta}\left(1, e^{\mathbf{i}(2\pi/3)}, e^{\mathbf{i}(4\pi/3)}\right). \tag{3.19}$$

For two of the three indices equal $(i_1 = i_2 \neq i_3, \ldots)$ the preceding results can be carried over without crucial change. But for $i_1 = i_2 = i_3 \equiv i$, say, there is a special situation due to the fact

$$1 + \lambda + \lambda^2 = (3, 0, 0), \tag{3.20}$$

for $\lambda = (1, e^{\mathbf{i}(2\pi/3)}, e^{\mathbf{i}(4\pi/3)})$, respectively. Now

$$V_\lambda = \left(1 + \lambda + \lambda^2\right)|iii\rangle + \left(\epsilon_{12} + \epsilon_{31}\lambda + \epsilon_{23}\lambda^2\right)\left|\bar{i}\,\bar{i}i\right\rangle + \left(\epsilon_{23} + \epsilon_{12}\lambda + \epsilon_{31}\lambda^2\right)\left|i\bar{i}\,\bar{i}\right\rangle + \left(\epsilon_{31} + \epsilon_{23}\lambda + \epsilon_{12}\lambda^2\right)\left|\bar{i}i\bar{i}\right\rangle. \tag{3.21}$$

The eigenfunctions and eigenvalues are (for mutually orthogonal basis states)

$$\begin{aligned}
&\mathbf{T}^{(3)}\left(|iii\rangle + \left|\bar{i}\,\bar{i}i\right\rangle + \left|i\bar{i}\,\bar{i}\right\rangle + \left|\bar{i}i\bar{i}\right\rangle\right) \\
&\quad = e^{3m_{ii}^{(+)}\theta}\left(|iii\rangle + \left|\bar{i}\,\bar{i}i\right\rangle + \left|i\bar{i}\,\bar{i}\right\rangle + \left|\bar{i}i\bar{i}\right\rangle\right), \\
&\mathbf{T}^{(3)}\left(3|iii\rangle - \left|\bar{i}\,\bar{i}i\right\rangle - \left|i\bar{i}\,\bar{i}\right\rangle - \left|\bar{i}i\bar{i}\right\rangle\right) \\
&\quad = e^{(m_{ii}^{(+)}+2m_{ii}^{(-)})\theta}\left(3|iii\rangle - \left|\bar{i}\,\bar{i}i\right\rangle - \left|i\bar{i}\,\bar{i}\right\rangle - \left|\bar{i}i\bar{i}\right\rangle\right), \\
&\mathbf{T}^{(3)}\left(\left|\bar{i}\,\bar{i}i\right\rangle + e^{\mathbf{i}(2\pi/3)}\left|i\bar{i}\,\bar{i}\right\rangle + e^{\mathbf{i}(4\pi/3)}\left|\bar{i}i\bar{i}\right\rangle\right) \\
&\quad = e^{\mathbf{i}(2\pi/3)}e^{(m_{ii}^{(+)}+2m_{ii}^{(-)})\theta}\left(\left|\bar{i}\,\bar{i}i\right\rangle + e^{\mathbf{i}(2\pi/3)}\left|i\bar{i}\,\bar{i}\right\rangle + e^{\mathbf{i}(4\pi/3)}\left|\bar{i}i\bar{i}\right\rangle\right), \\
&\mathbf{T}^{(3)}\left(\left|\bar{i}\,\bar{i}i\right\rangle + e^{\mathbf{i}(4\pi/3)}\left|i\bar{i}\,\bar{i}\right\rangle + e^{\mathbf{i}(2\pi/3)}\left|\bar{i}i\bar{i}\right\rangle\right) \\
&\quad = e^{\mathbf{i}(4\pi/3)}e^{(m_{ii}^{(+)}+2m_{ii}^{(-)})\theta}\left(\left|\bar{i}\,\bar{i}i\right\rangle + e^{\mathbf{i}(4\pi/3)}\left|i\bar{i}\,\bar{i}\right\rangle + e^{\mathbf{i}(2\pi/3)}\left|\bar{i}i\bar{i}\right\rangle\right).
\end{aligned} \tag{3.22}$$

For $\overline{V_\lambda}$ one follows exactly similar steps and obtains the same set of eigenvalues (with $i \rightleftharpoons \bar{i}$ in the eigenstates). Combining all the solutions above, we obtain complete results for $r = 3$ and all even N.

(iv) $r = 4$

The key result (3.8) indicates the following construction (a direct generalization of (3.16)). Define, for the even subset,

$$\begin{aligned}
V_{(b_1b_2b_3b_4)} = {} & |b_1b_2b_3b_4\rangle + \epsilon_{12}\left|\overline{b}_1\,\overline{b}_2b_3b_4\right\rangle + \epsilon_{23}\left|b_1\overline{b}_2\,\overline{b}_3b_4\right\rangle + \epsilon_{34}\left|b_1b_2\overline{b}_3\overline{b}_4\right\rangle \\
& + \epsilon_{41}\left|\overline{b}_1b_2b_3\overline{b}_4\right\rangle + \epsilon_{12}\epsilon_{23}\left|\overline{b}_1b_2\overline{b}_3b_4\right\rangle + \epsilon_{12}\epsilon_{34}\left|\overline{b}_1\,\overline{b}_2\,\overline{b}_3\overline{b}_4\right\rangle + \epsilon_{23}\epsilon_{34}\left|b_1\overline{b}_2b_3\overline{b}_4\right\rangle,
\end{aligned} \tag{3.23}$$

and similarly, implementing circular permutations as in (3.16), $V_{(b_4b_1b_2b_3)}, V_{(b_3b_4b_1b_2)}, V_{(b_2b_3b_4b_1)}$, define

$$V_\lambda = V_{(b_1b_2b_3b_4)} + \lambda V_{(b_4b_1b_2b_3)} + \lambda^2 V_{(b_3b_4b_1b_2)} + \lambda^3 V_{(b_2b_3b_4b_1)}, \tag{3.24}$$

where

$$\lambda = \left(1, e^{\mathbf{i}(2\pi/4)}, e^{\mathbf{i}2\cdot(2\pi/4)}, e^{\mathbf{i}3\cdot(2\pi/4)}\right) = (1, \mathbf{i}, -1, -\mathbf{i}). \tag{3.25}$$

One obtains for distinct (i_1, i_2, i_3, i_4)

$$\mathbf{T}^{(4)} V_\lambda = \lambda e^{(m_{i_1 i_2}^{(\epsilon_{12})} + m_{i_2 i_3}^{(\epsilon_{23})} + m_{i_3 i_4}^{(\epsilon_{34})} + m_{i_4 i_1}^{(\epsilon_{41})})\theta} V_\lambda, \tag{3.26}$$

where for nonzero results $(1 + \epsilon_{12}\epsilon_{23}\epsilon_{34}\epsilon_{41}) = 2$. For the 8 possibilities remaining for the ϵ's and the 4 values of λ, one obtains the full set of 32 eigenstates and the corresponding eigenvalues. For the complementary "odd" subspace formed by

$$\begin{aligned}&\left\{\left|b_1 b_2 b_3 \overline{b}_4\right\rangle, \left|b_1 b_2 \overline{b}_3 b_4\right\rangle, \left|b_1 \overline{b}_2 b_3 b_4\right\rangle, \left|\overline{b}_1 b_2 b_3 b_4\right\rangle,\right.\\ &\left.\left|\overline{b}_1 \overline{b}_2 \overline{b}_3 b_4\right\rangle, \left|\overline{b}_1 \overline{b}_2 b_3 \overline{b}_4\right\rangle, \left|\overline{b}_1 b_2 \overline{b}_3 \overline{b}_4\right\rangle, \left|b_1 \overline{b}_2 \overline{b}_3 \overline{b}_4\right\rangle\right\},\end{aligned} \tag{3.27}$$

and the sets related through circular permutations of the indices, one obtains, entirely in evident analogy of (3.24),

$$\overline{V_\lambda} = \overline{V_{(b_1 b_2 b_3 b_4)}} + \lambda \overline{V_{(b_4 b_1 b_2 b_3)}} + \lambda^2 \overline{V_{(b_3 b_4 b_1 b_2)}} + \lambda^3 \overline{V_{(b_2 b_3 b_4 b_1)}}. \tag{3.28}$$

One obtains, for distinct indices, the same set of eigenvalues as in (3.26). For some equal indices, but not all, the preceding results can be carried over essentially. For all equal indices one obtains the following spectrum of eigenvalues (grouping together the even and the odd subspaces):

$$\begin{gathered} e^{4m_{ii}^{(+)}\theta}(1,1), \\ [\text{3-times}] : e^{2(m_{ii}^{(+)}+m_{ii}^{(-)})\theta}\left(1, e^{\mathbf{i}(2\pi/4)}, e^{\mathbf{i}2\cdot(2\pi/4)}, e^{\mathbf{i}3\cdot(2\pi/4)}\right), \\ e^{4m_{ii}^{(-)}\theta}(1,-1), \end{gathered} \tag{3.29}$$

(In fact these are the same as in (A.13) of [7], since the distinguishing index for odd dimensions signalled in (2.14) and (2.15) is not involved.) We will not present here the 16 orthogonal eigenstates corresponding to (3.29). But a new feature, as compared to (3.22), should be pointed out. For $r = 3$, considering together even and odd spaces, one has for identical indices the eigenvalues

$$\begin{gathered} e^{3m_{ii}^{(+)}\theta}(1,1), \\ [\text{2-times}] : e^{(m_{ii}^{(+)}+2m_{ii}^{(-)})\theta}\left(1, e^{\mathbf{i}(2\pi/3)}, e^{\mathbf{i}2\cdot(2\pi/3)}\right). \end{gathered} \tag{3.30}$$

In fact, for each r one obtains, for $\epsilon = 1$, $e^{rm_{ii}^{(+)}\theta}(1,1)$ giving

$$\operatorname{Tr}\left(\mathbf{T}^{(r)}\right) = 2\sum_{i=1}^{n} e^{rm_{ii}^{(\epsilon)}\theta}. \tag{3.31}$$

All other multiplets have each one zero sum. Concerning the zero sum multiplets, comparing (3.30) and (3.29), we note that

(1) for $r = 3$, one has only triplets;

(2) for $r = 4$, one has quadruplets and also a doublet $(1, -1)$. This last enters due to factorizability of r $(4 = 2 \times 2)$.

Such a feature is worth signalling since it generalizes.

When r is a prime number, the zero-sum multiplets appear only as r-plets (with rth roots of unity). How they consistently cover the whole base space (along with the doublets giving the trace as in (3.31)) has been amply discussed in our previous papers [7, 9] under the heading "An encounter with a theorem of Fermat." When r has many factors the multiplicity of submultiplets corresponding to each one is difficult to formulate giving, say, an explicit general prescription.

(v) $r \geq 5$

Another problem concerning systematic, complete enumeration one encounters already for $r = 5$. Starting with (3.9) the generalizations of (3.23)–(3.26) are quite evident. We will not present them here. But as compared to (3.29) one has now for the 32-dimensional subspace (for $i_1 = i_2 = \cdots = i_5$)

$$\begin{aligned}
&e^{5m_{ii}^{(+)}\theta}(1,1),\\
&[n_2\text{-times}] : e^{(3m_{ii}^{(+)}+2m_{ii}^{(-)})\theta}\left(1, e^{\mathbf{i}(2\pi/5)}, e^{\mathbf{i}2\cdot(2\pi/5)}, e^{\mathbf{i}3\cdot(2\pi/5)}, e^{\mathbf{i}4\cdot(2\pi/5)}\right),\\
&[n_4\text{-times}] : e^{(m_{ii}^{(+)}+4m_{ii}^{(-)})\theta}\left(1, e^{\mathbf{i}(2\pi/5)}, e^{\mathbf{i}2\cdot(2\pi/5)}, e^{\mathbf{i}3\cdot(2\pi/5)}, e^{\mathbf{i}4\cdot(2\pi/5)}\right),
\end{aligned} \tag{3.32}$$

where $n_2 + n_4 = 6$. At this stage it is not difficult to present complete construction. But as r increases, for prime numbers, one has

$$\begin{aligned}
&e^{rm_{ii}^{(+)}\theta}(1,1),\\
&[n_2\text{-times}] : e^{((r-2)m_{ii}^{(+)}+2m_{ii}^{(-)})\theta}\left(1, e^{\mathbf{i}(2\pi/r)}, \dots, e^{\mathbf{i}(r-1)\cdot(2\pi/r)}\right),\\
&[n_4\text{-times}] : \ e^{((r-4)m_{ii}^{(+)}+4m_{ii}^{(-)})\theta}\left(1, e^{\mathbf{i}(2\pi/r)}, \dots, e^{\mathbf{i}(r-1)\cdot(2\pi/r)}\right),\\
&\quad\vdots \qquad\qquad\qquad\qquad \vdots\\
&[n_{r-1}\text{-times}] : e^{(m_{ii}^{(+)}+(r-1)m_{ii}^{(-)})\theta}\left(1, e^{\mathbf{i}(2\pi/r)}, \dots, e^{\mathbf{i}(r-1)\cdot(2\pi/r)}\right),
\end{aligned} \tag{3.33}$$

the sum of the multiplicities of such r-plets satisfying

$$n_2 + n_4 + \cdots + n_{r-1} = \frac{1}{r}(2^r - 2). \tag{3.34}$$

(That the right-hand side is an integer for r a prime number is guaranteed by a theorem of Fermat, as has been discussed before.) A general explicit prescription for the sequence $(n_2, n_4, \ldots, n_{r-1})$ is beyond the scope of this paper.

The situation is as above for prime numbers r. For factorizable r the presence, in addition, of submultiplets has already been pointed out. For $r = 4$, only such submultiplets were $(1, -1)$. For $r = 6$, one can have $(1, -1)$, $(1, e^{\mathrm{i}(2\pi/3)}, e^{\mathrm{i}2\cdot(2\pi/3)})$ corresponding to the factors $6 = 2 \times 3$, respectively. For $r = 30 = 2 \times 3 \times 5$ one can have also submultiplets corresponding to 5. And so on.

But apart from the above-mentioned limitations concerning multiplicities of zero-sum ("roots of unity") multiplets and submultiplets we can claim to have elucidated the spectrum of eigenvalues for all (N, r). For the generic case with distinct indices for any (N, r), one has eigenvalues

$$\lambda_k e^{(m_{i_1 i_2}^{(\epsilon_{12})} + m_{i_2 i_3}^{(\epsilon_{23})} + \cdots + m_{i_{r-1} i_r}^{(\epsilon_{r-1,r})} + m_{i_1 i_r}^{(\epsilon_{1r})})\theta}, \tag{3.35}$$

where $(1 + \epsilon_{12}\epsilon_{23}\cdots\epsilon_{r-1,r}\epsilon_{1,r}) = 2$ and $\lambda_k = e^{\mathrm{i}k\cdot(2\pi/r)}$ $(k = 0, 1, 2, \ldots, r-1)$. Here each $m_{i_k i_l}^{(\epsilon_{kl})}$ is a free parameter. We solve only sets of linear equations with quite simple constant coefficients to obtain the eigenstates and the eigenvalues.

3.2. *Odd Dimensions*

We refer back to (2.14)–(2.16). When the special index n $(n = \overline{n})$ for $N = 2n - 1$ is not present in the basis state $|b_1 b_2 \cdots b_r\rangle$ and hence in the subspace it generates via cyclic permutations as in the forgoing examples, the foregoing constructions can be taken over wholesale. When n is present, the modifications are not difficult to take into account. The trace is now $\mathrm{Tr}\,(\mathbf{T}^{(r)}) = 2\sum_{i=1}^{n-1} e^{rm_{ii}^{(\epsilon)}\theta} + 1$, since the central state $|n\rangle \otimes |n\rangle \otimes \cdots |n\rangle \otimes \equiv |nn\cdots n\rangle$ contributes with our normalization 1 to the trace. Various aspects have been studied in considerable detail in our previous papers [6, 7] on odd N. Here we just refer to them.

4. Spin Chain Hamiltonians

Spin chains corresponding to our braid matrices have already been studied in our previous papers [7, 8]. Here we formulate a unified approach for all dimensions $(N = 2n - 1, 2n)$.

The basic formula (see sources cited in [7, 8]) is

$$H = \sum_{k=1}^{r} I \otimes \cdots \otimes I \otimes \dot{\hat{R}}_{k,k+1}(0) \otimes I \otimes \cdots \otimes I, \tag{4.1}$$

where for circular boundary conditions , $k+1 = r+1 \approx 1$. For even N ($N = 2n$), from (2.4),

$$\dot{\widehat{R}}(0) = \frac{d}{d\theta}\widehat{R}(\theta)\Big|_{\theta=0} = \sum_{\epsilon,i,j} m_{ij}^{(\epsilon)}\left(P_{ij}^{(\epsilon)} + P_{\bar{i}\bar{j}}^{(\epsilon)}\right) \tag{4.2}$$

the projectors being given by (2.1) and the remark below (2.1). For N odd ($N = 2n-1$; $n = 2, 3, \ldots$) we introduce a modified overall normalization factor to start with. Such a factor is trivial concerning the braid equation, but not for the Hamiltonian (since a derivative is involved) if the factor is θ-dependent. Multiply (2.16) by $e^{m\theta}$ and redefine

$$\left(\left(m + m_{ij}^{(\epsilon)}\right), \left(m + m_{ni}^{(\epsilon)}\right), \left(m + m_{in}^{(\epsilon)}\right)\right) \longrightarrow \left(m_{ij}^{(\epsilon)}, m_{ni}^{(\epsilon)}, m_{in}^{(\epsilon)}\right) \tag{4.3}$$

since $(m_{ij}^{(\epsilon)}, \ldots)$ are arbitrary to start with. Now for odd N (with the ranges of (i,j), $(\bar{i},\bar{j})$ of (2.16)),

$$\dot{\widehat{R}}(0) = mP_{nn} + \sum_{\epsilon,i}\left(m_{ni}^{(\epsilon)}P_{ni}^{(\epsilon)} + m_{in}^{(\epsilon)}P_{in}^{(\epsilon)}\right) + \sum_{\epsilon,i,j} m_{ij}^{(\epsilon)}\left(P_{ij}^{(\epsilon)} + P_{\bar{i}\bar{j}}^{(\epsilon)}\right). \tag{4.4}$$

This extends the $m = 0$ case by including the presence of P_{nn}.

In H, $\dot{\widehat{R}}(0)_{k,k+1}$ acts on the basis $|V\rangle_{(k)} \otimes |V\rangle_{(k+1)}$. One can denote, using standard ordering of spin components for each k,

$$|V\rangle_{(k)} = \left| \begin{array}{c} |n-1/2\rangle_k \\ \vdots \\ |1/2\rangle_k \\ |-1/2\rangle_k \\ \vdots \\ |-n+1/2\rangle_k \end{array} \right\rangle, \qquad |V\rangle_{(k)} = \left| \begin{array}{c} |n-1\rangle_k \\ \vdots \\ |1\rangle_k \\ |0\rangle_k \\ |-1\rangle_k \\ \vdots \\ |-n+1\rangle_k \end{array} \right\rangle, \tag{4.5}$$

for $N = 2n, 2n-1$, respectively. Without being restricted to spin, one can consider more generally any system with N orthogonal states. We will continue, however, to use the terminology of spin. At each site one can consider a superposition at each level such as $(\sum_l c_{il}^{(k)}|l\rangle_k)$. To keep the notation tractable, we will just denote

$$|V\rangle_{(k)} = \left| \begin{array}{c} |1\rangle_k \\ |2\rangle_k \\ \vdots \\ |\bar{2}\rangle_k \\ |\bar{1}\rangle_k \end{array} \right\rangle, \tag{4.6}$$

and keep possible significances of $|i\rangle_k$ in mind. Each index k is acted upon twice by H, namely, by $\dot{\hat{R}}_{k-1,k}(0)$, $\dot{\hat{R}}_{k,k+1}(0)$, and, for closed chains, the index 1 is also thus involved in $\dot{\hat{R}}_{12}(0)$ and $\dot{\hat{R}}_{r1}(0)$. Some simple examples are

(i) $N = 2$

$$\dot{\hat{R}}(0) = \begin{vmatrix} \hat{a}_+ & 0 & 0 & \hat{a}_- \\ 0 & \hat{a}_+ & \hat{a}_- & 0 \\ 0 & \hat{a}_- & \hat{a}_+ & 0 \\ \hat{a}_- & 0 & 0 & \hat{a}_+ \end{vmatrix}, \tag{4.7}$$

where $\hat{a}_\pm = (1/2)(m_{11}^{(+)} \pm m_{11}^{(-)})$ and hence in evident notations

$$\dot{\hat{R}}(0)|V\rangle_{(k)} \otimes |V\rangle_{(k+1)} = \left| \begin{matrix} \hat{a}_+|11\rangle + \hat{a}_-\left|\bar{1}\bar{1}\right\rangle \\ \hat{a}_+\left|1\bar{1}\right\rangle + \hat{a}_-\left|\bar{1}1\right\rangle \\ \hat{a}_-\left|1\bar{1}\right\rangle + \hat{a}_+\left|\bar{1}1\right\rangle \\ \hat{a}_-|11\rangle + \hat{a}_+\left|\bar{1}\bar{1}\right\rangle \end{matrix} \right\rangle_{(k,k+1)}, \tag{4.8}$$

(ii) $N = 4$

In the notation of Section 7 of [9]

$$\hat{R}(\theta) = \begin{vmatrix} \hat{D}_{11} & 0 & 0 & \hat{A}_{1\bar{1}} \\ 0 & \hat{D}_{22} & \hat{A}_{2\bar{2}} & 0 \\ 0 & \hat{A}_{\bar{2}2} & \hat{D}_{\bar{2}\bar{2}} & 0 \\ \hat{A}_{\bar{1}1} & 0 & 0 & \hat{D}_{\bar{1}\bar{1}} \end{vmatrix}, \tag{4.9}$$

where

$$\hat{D}_{11} = \hat{D}_{\bar{1}\bar{1}} = \begin{pmatrix} \hat{a}_+ & 0 & 0 & 0 \\ 0 & \hat{b}_+ & 0 & 0 \\ 0 & 0 & \hat{b}_+ & 0 \\ 0 & 0 & 0 & \hat{a}_+ \end{pmatrix}, \qquad \hat{D}_{22} = \hat{D}_{\bar{2}\bar{2}} = \begin{pmatrix} \hat{c}_+ & 0 & 0 & 0 \\ 0 & \hat{d}_+ & 0 & 0 \\ 0 & 0 & \hat{d}_+ & 0 \\ 0 & 0 & 0 & \hat{c}_+ \end{pmatrix},$$
$$\hat{A}_{1\bar{1}} = \hat{A}_{\bar{1}1} = \begin{pmatrix} 0 & 0 & 0 & \hat{a}_- \\ 0 & 0 & \hat{b}_- & 0 \\ 0 & \hat{b}_- & 0 & 0 \\ \hat{a}_- & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{A}_{2\bar{2}} = \hat{A}_{\bar{2}2} = \begin{pmatrix} 0 & 0 & 0 & \hat{c}_- \\ 0 & 0 & \hat{d}_- & 0 \\ 0 & \hat{d}_- & 0 & 0 \\ \hat{c}_- & 0 & 0 & 0 \end{pmatrix}, \tag{4.10}$$
$$\hat{a}_\pm = \frac{1}{2}\left(m_{11}^{(+)} \pm m_{11}^{(-)}\right), \qquad \hat{b}_\pm = \frac{1}{2}\left(m_{12}^{(+)} \pm m_{12}^{(-)}\right),$$
$$\hat{c}_\pm = \frac{1}{2}\left(m_{21}^{(+)} \pm m_{21}^{(-)}\right), \qquad \hat{d}_\pm = \frac{1}{2}\left(m_{22}^{(+)} \pm m_{22}^{(-)}\right).$$

Hence,

$$\dot{\hat{R}}(0)|V\rangle_{(k)} \otimes |V\rangle_{(k+1)} = \left| \begin{array}{l} |1\rangle \otimes \left(\hat{D}_{11}|V\rangle\right) + \left|\overline{1}\right\rangle \otimes \left(\hat{A}_{1\overline{1}}|V\rangle\right) \\ |2\rangle \otimes \left(\hat{D}_{22}|V\rangle\right) + \left|\overline{2}\right\rangle \otimes \left(\hat{A}_{2\overline{2}}|V\rangle\right) \\ |2\rangle \otimes \left(\hat{A}_{2\overline{2}}|V\rangle\right) + \left|\overline{2}\right\rangle \otimes \left(\hat{D}_{22}|V\rangle\right) \\ |1\rangle \otimes \left(\hat{A}_{1\overline{1}}|V\rangle\right) + \left|\overline{1}\right\rangle \otimes \left(\hat{D}_{11}|V\rangle\right) \end{array} \right\rangle_{(k,k+1)}, \tag{4.11}$$

where for each k, $|V\rangle = \left| \begin{array}{l} |1\rangle \\ |2\rangle \\ |\overline{2}\rangle \\ |\overline{1}\rangle \end{array} \right\rangle$

(iii) $N = 3$

Adapting the results of Section 1 of [7] and Section 11 of [8] to notations analogous to the cases above, one can write

$$\hat{R}(\theta) = \begin{vmatrix} D & 0 & A \\ 0 & C & 0 \\ A & 0 & D \end{vmatrix}, \tag{4.12}$$

where

$$D = \begin{pmatrix} \hat{a}_+ & 0 & 0 \\ 0 & \hat{b}_+ & 0 \\ 0 & 0 & \hat{a}_+ \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & \hat{a}_- \\ 0 & \hat{b}_- & 0 \\ \hat{a}_- & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \hat{c}_+ & 0 & \hat{c}_- \\ 0 & m & 0 \\ \hat{c}_- & 0 & \hat{c}_+ \end{pmatrix}, \tag{4.13}$$

the central element m corresponding to mP_{nn} of (4.4) and

$$\hat{a}_\pm = \frac{1}{2}\left(m_{11}^{(+)} \pm m_{11}^{(-)}\right), \qquad \hat{b}_\pm = \frac{1}{2}\left(m_{12}^{(+)} \pm m_{12}^{(-)}\right), \qquad \hat{c}_\pm = \frac{1}{2}\left(m_{21}^{(+)} \pm m_{21}^{(-)}\right). \tag{4.14}$$

Denote the basis factors for each k as, $|V\rangle_{(k)} = \left| \begin{array}{l} |1\rangle \\ |2\rangle \\ |\overline{1}\rangle \end{array} \right\rangle_{(k)}$ and

$$|V\rangle_{(k)} \otimes |V\rangle_{(k+1)} = \left| \begin{array}{l} |1\rangle \otimes |V\rangle \\ |2\rangle \otimes |V\rangle \\ \left|\overline{1}\right\rangle \otimes |V\rangle \end{array} \right\rangle_{(k,k+1)}. \tag{4.15}$$

One obtains

$$\dot{\hat{R}}(0)|V\rangle_{(k)} \otimes |V\rangle_{(k+1)} = \left|\begin{matrix} D & 0 & A \\ 0 & C & 0 \\ A & 0 & D \end{matrix}\right| \left| \begin{matrix} |1\rangle \otimes |V\rangle \\ |2\rangle \otimes |V\rangle \\ \left|\bar{1}\right\rangle \otimes |V\rangle \end{matrix} \right\rangle_{(k,k+1)}$$
$$= \left| \begin{matrix} |1\rangle \otimes (D|V\rangle) + \left|\bar{1}\right\rangle \otimes (A|V\rangle) \\ |2\rangle \otimes (C|V\rangle) \\ |1\rangle \otimes (A|V\rangle) + \left|\bar{1}\right\rangle \otimes (D|V\rangle) \end{matrix} \right\rangle_{(k,k+1)}. \tag{4.16}$$

For $N = 5, 6, \ldots$, one can generate such results systematically. They, considering the action of all the terms of (4.1), furnish the transition matrix elements and expectation values for possible states of the chain and permit a study of correlations.

One can also consider higher-order conserved quantities (Section 1.5 of [10]) given by

$$H_l = \frac{d^l}{d\theta^l} \log \mathbf{T}^{(r)}(\theta)\bigg|_{\theta=0}. \tag{4.17}$$

For $l = 2$, (4.1) is generalized by the appearance of factors of the type (apart from non-overlapping derivatives)

$$\dot{\hat{R}}(0)_{k-1,k} \otimes \dot{\hat{R}}(0)_{k,k+1}, \qquad \ddot{\hat{R}}(0)_{k,k+1}. \tag{4.18}$$

For $l > 2$ this generalizes in an evident fashion. A study of spin chains for the "exotic" SØ3 can be found in [11]. It is interesting to compare it with the 4×4 (for $N = 2$) case briefly presented above by exploring the latter case in comparable detail.

One may note that for our class of braid matrices, for even N as compared to (4.2), are

$$\frac{d^l}{d\theta^l}\hat{R}(\theta)\bigg|_{\theta=0} = \sum_{\epsilon}\sum_{i,j}\left(m_{ij}^{(\epsilon)}\right)^l \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)}\right), \tag{4.19}$$

and for odd N, as compared to (4.4),

$$\frac{d^l}{d\theta^l}\hat{R}(\theta)\bigg|_{\theta=0} = m^l P_{nn} + \sum_{\epsilon,i}\left(\left(m_{ni}^{(\epsilon)}\right)^l P_{ni}^{(\epsilon)} + \left(m_{in}^{(\epsilon)}\right)^l P_{in}^{(\epsilon)}\right) + \sum_{\epsilon,i,j}\left(m_{ij}^{(\epsilon)}\right)^l \left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)}\right). \tag{4.20}$$

5. Potentials for Factorizable S-Matrices

Such potentials can be obtained as inverse Cayley transforms of the Yang-Baxter matrices of appropriate dimensions (see, for example, Section 3 of [12] and Section 1 of [2]). Starting with the Yang-Baxter matrix $R(\theta) = \mathbf{P}\hat{R}(\theta)$, the required potential $\mathbf{V}(\theta)$ is given by

$$-\mathbf{i}\mathbf{V}(\theta) = (R(\theta) - \lambda(\theta)I)^{-1}(R(\theta) + \lambda(\theta)I). \tag{5.1}$$

We have emphasized in our previous studies (Section 5 of [7], Section 8 of [8]) that $\lambda(\theta)$ cannot be arbitrary, a set of values must be excluded for the inverse (5.1) to be well defined. Here we generalize our previous results to all N. As compared to other well-known studies [13] of fields corresponding to factorizable scatterings, here our construction starts with the braid matrices (2.4) and (2.16).

Define

$$X(\theta) = (R(\theta) - \lambda(\theta)I)^{-1} \tag{5.2}$$

when

$$-\mathbf{i}\mathbf{V}(\theta) = I + 2\lambda(\theta)X(\theta). \tag{5.3}$$

We give below immediately the general solution and then explain the notations more precisely. The solution was obtained via formal series expansions. But the final closed form can be verified directly. The solution is

$$\begin{aligned} X(\theta) = -\frac{1}{2}\sum_{\epsilon=\pm}\sum_{a,b=1}^{N}\frac{1}{\lambda^2(\theta) - e^{(m_{ab}^{(\epsilon)}+m_{ba}^{(\epsilon)})\theta}}&\Big\{\lambda(\theta)\Big[(aa)\otimes(bb) + \epsilon(a\overline{a})\otimes\Big(b\overline{b}\Big)\Big] \\ &+e^{m_{ba}^{(\epsilon)}\theta}\Big[(ab)\otimes(ba) + \epsilon\Big(a\overline{b}\Big)\otimes(b\overline{a})\Big]\Big\}, \end{aligned} \tag{5.4}$$

where

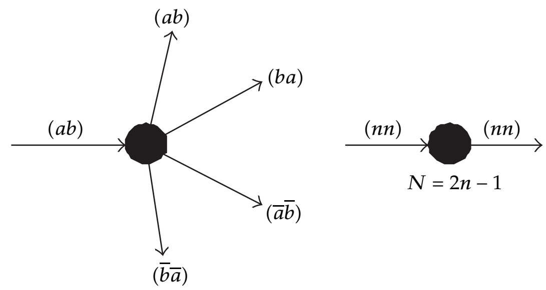
$$\lambda(\theta) \neq \pm e^{(1/2)(m_{ab}^{(\epsilon)}+m_{ba}^{(\epsilon)})\theta}. \tag{5.5}$$

For $N = 2n$, (2.9) is implicit in this result. For $N = 2n-1$, when $\overline{n} = n$, the conventions (2.16) (or (4.4)) are to be implemented when a or b or both (a,b) are n. From (5.3) and (5.4),

$$\begin{aligned} \mathbf{V}(\theta) &= -\frac{\mathbf{i}}{2}\sum_{\epsilon=\pm}\sum_{a,b=1}^{N}\left\{\frac{\lambda^2(\theta) + e^{(m_{ab}^{(\epsilon)}+m_{ba}^{(\epsilon)})\theta}}{\lambda^2(\theta) - e^{[m_{ab}^{(\epsilon)}+m_{ba}^{(\epsilon)}]\theta}}\Big[(aa)\otimes(bb) + \epsilon(a\overline{a})\otimes\Big(b\overline{b}\Big)\Big]\right. \\ &\qquad\left. + \frac{e^{m_{ab}^{(\epsilon)}\theta}}{\lambda^2(\theta) - e^{(m_{ab}^{(\epsilon)}+m_{ba}^{(\epsilon)})\theta}}\Big[(ba)\otimes(ab) + \epsilon(b\overline{a})\otimes\Big(a\overline{b}\Big)\Big]\right\} \\ &\equiv \sum_{ab,cd}\mathbf{V}(\theta)_{(ab,cd)}(ab)\otimes(cd). \end{aligned} \tag{5.6}$$

Now one can write down the Lagrangians for scalar and spinor fields. For the spinor case, for example,

$$\mathcal{L} = \int dx\big[\mathbf{i}\overline{\psi}_a\gamma_\nu\partial_\nu\psi_a - \mathrm{g}(\overline{\psi}_a\gamma_\nu\psi_c)\mathbf{V}_{ab,cd}(\overline{\psi}_b\gamma_\nu\psi_d)\big]. \tag{5.7}$$



$a, b = 1, \ldots, N$ and $(a, b) \neq (n, n)$ if $N = 2n - 1$

Figure 1: Scattering process.

The simpler scalar case can be written analogously. The scalar Lagrangian has an interaction term of the form $(\overline{\phi}_a\overline{\phi}_c)\mathbf{V}_{ab,cd}(\phi_b\phi_d)$. For $N = 2n$, one obtains the nonvanishing elements of $\mathbf{V}$ as

$$\mathbf{V}_{bb,dd} = -\frac{\mathbf{i}}{2}\sum_{\epsilon}\frac{\lambda^2(\theta) + e^{(m_{bd}^{(\epsilon)}+m_{db}^{(\epsilon)})\theta}}{\lambda^2(\theta) - e^{(m_{bd}^{(\epsilon)}+m_{db}^{(\epsilon)})\theta}},$$
$$\mathbf{V}_{\overline{b}b,\overline{d}d} = -\frac{\mathbf{i}}{2}\sum_{\epsilon}\epsilon\frac{\lambda^2(\theta) + e^{(m_{bd}^{(\epsilon)}+m_{db}^{(\epsilon)})\theta}}{\lambda^2(\theta) - e^{(m_{bd}^{(\epsilon)}+m_{db}^{(\epsilon)})\theta}},$$
$$\mathbf{V}_{db,bd} = -\frac{\mathbf{i}}{2}\sum_{\epsilon}\frac{e^{m_{bd}^{(\epsilon)}\theta}}{\lambda^2(\theta) - e^{(m_{bd}^{(\epsilon)}+m_{db}^{(\epsilon)})\theta}},$$
$$\mathbf{V}_{\overline{d}b,\overline{b}d} = -\frac{\mathbf{i}}{2}\sum_{\epsilon}\epsilon\frac{e^{m_{bd}^{(\epsilon)}\theta}}{\lambda^2(\theta) - e^{(m_{bd}^{(\epsilon)}+m_{db}^{(\epsilon)})\theta}}, \tag{5.8}$$

where $a, b \in \{1, \ldots, N\}$ and $(a, b) \neq (n, n)$ if $N = 2n - 1$. When N is odd $(N = 2n - 1)$, one has, for the special index n,

$$\mathbf{V}_{nn,nn} = -\mathbf{i}\left(\frac{\lambda^2(\theta) + 2}{\lambda^2(\theta) - 1}\right), \tag{5.9}$$

where $m_{nn}^{(\epsilon)}$ are taken to be zero. For our models, the scattering process can be schematically, presented as (see in Figure 1)

6. Remarks

We have obtained explicit eigenvalues of the transfer matrix $\mathbf{T}^{(r)}$ corresponding to our class of $N^2 \times N^2$ braid matrices for all (r, N). Starting with our nested sequence of projectors, we obtained a very specific structure of $\mathbf{T}^{(r)}$. Exploiting this structure fully, we obtained the eigenvalues and eigenstates. The zero-sum multiplets parametrized by sets of roots of unity arose from the circular permutations of the indices in the tensor products of the basis states under the action of $\mathbf{T}^{(r)}$. The same structure led to systematic constructions of multiparameter Hamiltonians of spin chains related to our class of braid matrices for all N. Finally we constructed the inverse Cayley transformation for the general case (any N) giving potentials compatible with factorizability of S-matrices. The contents of such matrices (generalizing Section 5 of [7]) will be further studied elsewhere. In the treatment of each aspect we emphasized the role of a remarkable feature of our formalism—the presence of free parameters whose number increase as N^2 with N. A single class of constraints ($\theta \geq 0$, $m_{ab}^{(+)} > m_{ab}^{(-)}$, $a, b \in \{1, \ldots, N\}$) assures nonnegative Boltzmann weights in the statistical models. But our constructions also furnish directly some important properties of such models. Thus $\operatorname{Tr}\,(\mathbf{T}^{(r)}) = 2\sum_{i=1}^{n} e^{rm_{ii}^{(+)}\theta}$ ($N = 2n$) and $\operatorname{Tr}\,(\mathbf{T}^{(r)}) = 2\sum_{i=1}^{n-1} e^{rm_{ii}^{(+)}\theta} + 1$ ($N = 2n-1$) (for the normalization (2.16)). The eigenvalues can be ordered in magnitude by choosing the order of values of the parameters. Thus choosing $m_{11}^{(+)} > m_{22}^{(+)} > \cdots$, the largest eigenvalue of $\mathbf{T}^{(r)}$ is for $\theta > 0$, say $e^{rm_{11}^{(+)}\theta}$, the next largest is $e^{rm_{22}^{(+)}\theta}$ and so on. We intend to study elsewhere the properties of our models more thoroughly.

In previous papers [8, 14] we pointed out that for purely imaginary parameters ($\mathbf{i}m_{ab}^{(\pm)}$ with $m_{ab}^{(\pm)}$ real) our braid matrices are all unitary, providing an entire class with free parameters for all N. Here we have not repeated this discussion. But the fact that they generate parametrized entangled states is indeed of interest.

Unitary matrices provide valid transformations of a basis (of corresponding dimension) of quantum states. One may ask the following question. If such a matrix, apart from being unitary, also satisfies the braid equation what consequence might be implied? Link between quantum and topological entanglements has been discussed by several authors [15, 16] cited in our previous papers [8, 14]. For the braid property to be relevant, a triple tensor product ($V \otimes V \otimes V$) of basis space is essential. We hope to explore elsewhere our multiparameter unitary matrices in such a context.

In a following paper we will present a quite different aspect of our multiparameter braid matrices. It will be shown how for imaginary parameters (when the matrices are unitary) they can be implemented to generate quantum entanglements. Topological and quantum entanglements will thus be brought together in the two papers.

Note Added

Professor J. H. H. Perk has kindly pointed out that a class of multiparameter generalization of the 6-vertex model is provided by $sl(m \mid n)$ ones [17, 18]. As the Perk-Schultz models they have been studied by many authors and have led to various important applications. (Relevant references can be easily found via ARXIV.) In this class the sources of parameters are multi-component rapidities. The study of eigenvectors and eigenvalues was pioneered in [18] the most recent followup being [19], where other references can be found. Such studies may be compared to our systematic explicit constructions for all (r, N).

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