Research Article

Nonlinear Bi-Integrable Couplings of Multicomponent Guo Hierarchy with Self-Consistent Sources

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Based on a well-known Lie algebra, the multicomponent Guo hierarchy with self-consistent sources is proposed. With the help of a set of non-semisimple Lie algebra, the nonlinear bi-integrable couplings of the multicomponent Guo hierarchy with self-consistent sources are obtained. It enriches the content of the integrable couplings of hierarchies with self-consistent sources. Finally, the Hamiltonian structures are worked out by employing the variational identity.

1. Introduction

Since the notion of integrable couplings was proposed in view of Virasoro symmetric algebras [1, 2] and the soliton theory, considerable research has been reported on the integrable couplings and associated properties of some known interesting integrable hierarchies, such as the AKNS hierarchy, the KN hierarchy, and the Burger hierarchy [3–6]. In order to produce multicomponent integrable systems, a type of multicomponent loop algebra $\tilde{G}_M$ was structured, and it followed that some multicomponent integrable systems were given and Hamiltonian structures of multicomponent systems were constructed through the component-trace identity in [7–13] and a general structure between Lie algebras and integrable coupling was given in [14, 15]. Here it is necessary to point out that the above-mentioned integrable couplings are linear for the supplementary variable, so they are called linear integrable couplings. Recently, Ma proposed the notion of nonlinear integrable couplings and gave the general scheme to construct the nonlinear integrable couplings of

hierarchies [16]. In [17, 18], based on the general scheme of constructing nonlinear integrable couplings, Zhang introduced some new explicit Lie algebras and obtained the nonlinear integrable couplings of the GJ hierarchy, the Yang hierarchy, and the CBB hierarchy. In [19], Ma employed a class of non-semisimple matrix loop algebras to generate bi-integrable couplings of soliton equation from the zero curvature equations.

On the other hand, with the development of soliton theory, people began to focus on the soliton equation with self-consistent sources [20, 21]. The soliton equations with self-consistent sources have important physical meaning and are often used to express interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state physics, plasma physics, and so on [22, 23]. In [24–26], Yu and Xia discussed the integrable couplings of hierarchies with self-consistent sources and obtained a series of integrable hierarchies with self-consistent sources and their couplings systems as well as Hamiltonian structures. In these papers, the integrable couplings are linear for supplementary variables.

As mentioned above, we will find that there are very few researches on the nonlinear integrable couplings of hierarchies with self-consistent sources available up till now, especially on the nonlinear bi-integrable couplings of multicomponent hierarchy. In our work, we will derive the nonlinear bi-integrable couplings of multicomponent Guo hierarchy with self-consistent sources and their Hamiltonian structures. It will enrich the content of the integrable couplings of hierarchies with self-consistent sources. The plan of the paper is as follows. In Section 2, the multicomponent Guo hierarchy with self-consistent sources is given. By employing the variational identity, the Hamiltonian structure of the above hierarchy is obtained. The notion of bi-integrable couplings is introduced and the nonlinear bi-integrable couplings of the multicomponent Guo hierarchy with self-consistent sources and their Hamiltonian structures are derived in Section 3. Finally, some conclusions are given in Section 4.

2. Multicomponent Guo Hierarchy with Self-Consistent Sources

In this section, we recall briefly some basic notions from the theory of the multicomponent integrable system. If $F_M = \{a = (a_{ij})_{M \times 3} = (a_1, a_2, a_3)\}$ denotes a class of matrices where a_i $(i = 1, 2, 3)$ is the ith column of the matrix a, M stands for an arbitrary positive integer, then F_M is a linear space.

Definition 2.1 (see [7]). Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_M)^T$, their product is defined as

$$\alpha * \beta = \beta * \alpha = (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_M \beta_M)^T, \qquad \frac{\alpha}{\beta} = \left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \ldots, \frac{\alpha_M}{\beta_M}\right)^T. \tag{2.1}$$

Definition 2.2 (see [7]). Set $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in F_M$, a commutation operation is defined as

$$[a, b] = (a_2 * b_3 - a_3 * b_2, \ 2(a_1 * b_2 - a_2 * b_1), \ 2(a_3 * b_1 - a_1 * b_3))^T. \tag{2.2}$$

It is easy to verify that F_M is a Lie algebra. Its corresponding loop algebra $\widetilde{F}_M$ is

$$\widetilde{F}_M = \{a\lambda^n, \ a \in F_M, \ n = 0, \pm 1, \pm 2, \dots\}, \tag{2.3}$$

equipped with a commutation operation as follows:

$$[a\lambda^m, b\lambda^n] = [a,b]\lambda^{m+n}, \quad \forall a, b \in F_M. \tag{2.4}$$

Consider the following isospectral problem:

$$\varphi_x = [U, \varphi], \qquad \lambda_t = 0, \qquad U = \left(\frac{\lambda I_M}{2}, \frac{(u_1 + u_2)}{2}, \frac{\lambda(u_1 - u_2)}{2}\right)^T, \tag{2.5}$$

where u_1, u_2 are potentials, λ is the spectral parameter, $u_i = (u_{i1}, u_{i2}, \dots, u_{iM})^T$, $i = 1, 2$, and $I_M = (\underbrace{1, 1, \dots, 1}_{M})^T$. By means of constructing a proper time evolution

$$\varphi_t = V\varphi, \qquad V = (\lambda v_1, v_2 + v_3, \lambda(v_2 - v_3))^T, \tag{2.6}$$

where $v_k = \sum_{m=0}^{\infty} (v_{k_1,m}, v_{k_2,m}, \dots, v_{k_M,m})^T \lambda^{-m}$, $k = 1, 2, 3$. Then solving stationary zero curvature equation $V_x = [U, V]$, one arrives at the recursion relation as follows:

$$
\begin{aligned}
&v_{1,mx} = u_2 * v_{2,m} - u_1 * v_{3,m}, \\
&v_{2,mx} = v_{3,m+1} - u_2 * v_{1,m+1}, \\
&v_{3,mx} = v_{2,m+1} - u_1 * v_{1,m+1}, \\
&v_{1,0} = 0 I_M, \qquad v_{2,0} = u_1, \qquad v_{3,0} = u_2, \qquad v_{1,1} = \frac{(u_2 * u_2 - u_1 * u_1)}{2}, \\
&v_{2,1} = u_{2,x} + u_1 * \frac{(u_2 * u_2 - u_1 * u_1)}{2}, \qquad v_{3,1} = u_{1,x} + u_2 * \frac{(u_2 * u_2 - u_1 * u_1)}{2}.
\end{aligned}
\tag{2.7}
$$

For an arbitrary natural number n, denoting

$$V_+^{(n)} = \sum_{m=0}^{n} (\lambda v_{1,m}, v_{2,m} + v_{3,m}, \lambda(v_{2,m} - v_{3,m}))^T \lambda^{n-m} = \lambda^n V - V_-^{(n)}, \tag{2.8}$$

then the stationary zero curvature equation can be written as

$$-V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] = V_{-x}^{(n)} - \left[U, V_-^{(n)}\right], \tag{2.9}$$

a direct calculation reads

$$-V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] = -\left(0, \frac{(v_{2,nx} + v_{3,nx})}{2}, \frac{\lambda(v_{2,nx} - v_{3,nx})}{2}\right). \tag{2.10}$$

Then the zero curvature equation

$$U_t - V_x^{(n)} + \left[U, V^{(n)}\right] = 0 \tag{2.11}$$

gives the integrable system

$$U_t = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} v_{2,nx} \\ v_{3,nx} \end{pmatrix} = \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix} \begin{pmatrix} v_{2,n} \\ -v_{3,n} \end{pmatrix} = J_1 L_1^n \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{2.12}$$

where J_1 is a Hamiltonian operator, and the recurrence operator L_1 is given from (2.7) by

$$L_1 = \begin{pmatrix} -u_1 * \partial^{-1} u_1 * \partial & -\partial - u_1 * \partial^{-1} u_2 * \partial \\ -\partial + u_2 * \partial^{-1} u_1 * \partial & u_2 * \partial^{-1} u_2 * \partial \end{pmatrix}. \tag{2.13}$$

Accordingly, when $M = 1$, the system (2.12) reduces to the Guo hierarchy in [5]; when $M > 1$, the system (2.12) is a multicomponent Guo hierarchy. We observe that $J_1 L_1 = L_1^* J_1$, the Hamiltonian functions H_n ($n \geq 1$) are involutive with each other, and each H_n is a common conserved density of the expression (2.12). Therefore, the multicomponent Guo hierarchy (2.12) is Liouville integrable.

In order to deduce the Hamiltonian structure of (2.12), we rewrite (2.2) as follows:

$$[a, b] = a^T * R_1(b), \tag{2.14}$$

where

$$R_1(b) = \begin{pmatrix} 0 & 2b_2 & -2b_3 \\ b_3 & -2b_1 & 0 \\ -b_2 & 0 & 2b_1 \end{pmatrix}. \tag{2.15}$$

Solving the matrix equation for the constant matrix F: $R(b) * F = -(R(b) * F)^T$, $F^T = F$, presents

$$F = \begin{pmatrix} 2I_M & 0 & 0 \\ 0 & 0 & I_M \\ 0 & I_M & 0 \end{pmatrix}. \tag{2.16}$$

Then in terms of F, define a linear functional:

$$\{a, b\} = a^T * F * b = 2a_1 * b_1 + a_3 * b_2 + a_2 * b_3. \tag{2.17}$$

It is easy to find that $\{a, b\}$ satisfies the variational identity

$$\frac{\delta}{\delta u}\{V, U_\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^\gamma\{V, U_u\}. \tag{2.18}$$

By using the above linear functional (2.17), according the variational identity, we have

$$\frac{\delta}{\delta u}\left(v_1 + (u_1 - u_2) * \frac{(v_2 + v_3)}{2}\right) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^\gamma(v_2, -v_3)^T. \tag{2.19}$$

Comparing the coefficients of λ^{-n-1} yields

$$\frac{\delta}{\delta u}\left(v_{1,n+1} + (u_1 - u_2) * \frac{(v_{2,n} + v_{3,n})}{2}\right) = (\gamma - n)(v_{2,n}, -v_{3,n})^T. \tag{2.20}$$

Taking $n = 0$ in the above equation gives $\gamma = 0$.

Hence, the Guo hierarchy (2.12) can be written as a Hamiltonian form

$$U_t = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = J_1 L_1^n \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = J_1 \begin{pmatrix} \dfrac{\delta}{\delta u_1} \\ \dfrac{\delta}{\delta u_2} \end{pmatrix} H_n = J_1 \frac{\delta H_n}{\delta u}, \tag{2.21}$$

where $H_n = -(1/n)[v_{1,n+1} + (u_1 - u_2) * (v_{2,n} + v_{3,n})/2]$.

Next, we will derive the Guo hierarchy with self-consistent sources. Consider the linear problem

$$\begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix}_x = U(u, \lambda_j)\begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix}_t = V(u, \lambda_j)\begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix}. \tag{2.22}$$

The following equation was given in [21]

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^N \alpha_j \frac{\delta \lambda_j}{\delta u} = 0, \tag{2.23}$$

where α_j are constants, and $\delta H_k/\delta u$ determines a finite-dimensional invariant set for the flow (2.21). Here we get [27]

$$\frac{\delta \lambda_j}{\delta u} = \frac{1}{2}\left\{\Psi_j, \frac{\partial U(u, \lambda_j)}{\partial u}\right\}, \quad \Psi_j = (\phi_{1,j} * \phi_{2,j}, -\phi_{1,j} * \phi_{1,j}, \phi_{2,j} * \phi_{2,j}), \tag{2.24}$$

$$\sum_{j=1}^N \frac{\delta \lambda_j}{\delta u} = \frac{1}{4}\begin{pmatrix} -\wedge\langle\Phi_1, \Phi_1\rangle + \langle\Phi_2, \Phi_2\rangle \\ \wedge\langle\Phi_1, \Phi_1\rangle + \langle\Phi_2, \Phi_2\rangle \end{pmatrix}, \tag{2.25}$$

where $\phi_{i,j} = (\phi_{i_1,j}, \phi_{i_2,j}, \ldots, \phi_{i_M,j})^T$, $\wedge = \text{diag}(\lambda_1, \ldots, \lambda_N)$, $\Phi_i = (\phi_{i,1}, \ldots, \phi_{i,N})^T$, $i = 1, 2; j = 1, \ldots, N; \langle\cdot, \cdot\rangle$ stands for the inner product in R^N.

Then, according to (2.23) and taking $\alpha_i = -4$, the Guo hierarchy with self-consistent sources can be rewritten as follows:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = J_1 \frac{\delta H_n}{\delta u} + J_1 \begin{pmatrix} -\wedge \langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle \\ \wedge \langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle \end{pmatrix}. \tag{2.26}$$

Taking $n = 1$, we obtain the following multicomponent equations with self-consistent sources

$$u_{1,t} - u_{2,x} - u_1 * \frac{(u_2 * u_2 - u_1 * u_1)}{2} = -\partial \sum_{j=1}^{N} \lambda_j \phi_{1j}^2 + \partial \sum_{j=1}^{N} \phi_{2j}^2,$$

$$u_{2,t} - u_{1,x} - u_2 * \frac{(u_2 * u_2 - u_1 * u_1)}{2} = -\partial \sum_{j=1}^{N} \lambda_j \phi_{1j}^2 - \partial \sum_{j=1}^{N} \phi_{2j}^2. \tag{2.27}$$

3. Nonlinear Bi-Integrable Couplings of Multicomponent Guo Hierarchy with Self-Consistent Sources

The multicomponent Guo hierarchy has been given in the above. In the section, we will construct the nonlinear bi-integrable couplings of multicomponent Guo hierarchy with self-consistent sources by following the way in [19], which is introduced as follows.

Given an integrable equation

$$u_t = K(u), \tag{3.1}$$

which possesses a zero curvature representation

$$U_t - V_x + [U, V] = 0, \tag{3.2}$$

where $U = U(u, \lambda)$, $V = V(u, \lambda)$, and taking $\overline{U} = \overline{U}(\overline{u}, \lambda) = (U(u, \lambda), U_1(u_1, \lambda), U_2(u_2, \lambda))$, $\overline{V} = \overline{V}(\overline{u}, \lambda) = (V(u, \lambda), V_1(u_1, \lambda), V_2(u_2, \lambda))$, then the enlarged zero-curvature equation

$$\overline{U}_t - \overline{V}_x + \left[\overline{U}, \overline{V} \right] = 0 \tag{3.3}$$

equivalently yields

$$U_t - V_x + [U, V] = 0,$$

$$U_{1,t} - V_{1,x} + [U, V_1] + [U_1, V] + [U_1, V_1] = 0, \tag{3.4}$$

$$U_{2,t} - V_{2,x} + [U, V_2] + [U_2, V] + [U_1, V_1] + [U_1, V_2] + [U_2, V_1] = 0.$$

We call (3.4) bi-integrable couplings of (3.1). Note that it is nonlinear with respect to the two supplementary variables u_1, u_2, thereby providing candidates for nonlinear bi-integrable couplings.

Next, we will construct an expanding Lie algebra Q_M of the Lie algebra F_M and its corresponding loop algebra. Consider a vector space $Q_M = \text{span}\{a = (a_{ij})_{M \times 9} = (a_1, \ldots, a_9)\}$, where a_i $(i = 1, \ldots, 9)$ is the ith column of the matrix a.

Definition 3.1. Set a, b as two elements of Q_M, their commutation operation is defined as

$$
\begin{aligned}
[a, b] = &(a_2 * b_3 - a_3 * b_2, 2(a_1 * b_2 - a_2 * b_1), 2(a_3 * b_1 - a_1 * b_3), a_2 * b_6 - a_6 * b_2 + a_5 * b_3 \\
&- a_3 * b_5 + a_5 * b_6 - a_6 * b_5, 2(a_1 * b_5 - a_5 * b_1 + a_4 * b_2 - a_2 * b_4 + a_4 * b_5 - a_5 * b_4), \\
&2(a_6 * b_1 - a_1 * b_6 + a_3 * b_4 - a_4 * b_3 + a_6 * b_4 - a_4 * b_6), a_2 * b_9 - a_9 * b_2 + a_8 * b_3 \\
&- a_3 * b_8 + a_5 * b_6 - a_6 * b_5 + a_5 * b_9 - a_9 * b_5 + a_8 * b_6 - a_6 * b_8, \\
&2(a_1 * b_8 - a_8 * b_1 + a_7 * b_2 - a_2 * b_7 + a_4 * b_5 - a_5 * b_4 + a_4 * b_8 - a_8 * b_4 \\
&+ a_7 * b_5 - a_5 * b_7), \\
&2(a_9 * b_1 - a_1 * b_9 + a_3 * b_7 - a_7 * b_3 + a_6 * b_4 - a_4 * b_6 + a_9 * b_4 - a_4 * b_9 \\
&+ a_6 * b_7 - a_7 * b_6))^T.
\end{aligned}
\tag{3.5}
$$

It is easy to verify that Q_M consists of a Lie algebra along with (3.5). Its corresponding loop algebra $\widetilde{Q}_M$ is given by

$$
\widetilde{Q}_M = \{a\lambda^n, \ a \in Q_M, \ n = 0, \pm 1, \pm 2, \ldots\},
\tag{3.6}
$$

equipped with a commutation operation as follows:

$$
[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, \quad \forall a, b \in Q_M.
\tag{3.7}
$$

Denoting $\widetilde{Q}_M(1) = \{(a_1, a_2, a_3, 0, 0, 0, 0, 0, 0)\lambda^n\}$ and $\widetilde{Q}_M(2) = \{(0, 0, 0, a_4, a_5, a_6, a_7, a_8, a_9)\lambda^n\}$, then we have

$$
\widetilde{Q}_M = \widetilde{Q}_M(1) \oplus \widetilde{Q}_M(2), \qquad \widetilde{Q}_M(1) \cong \widetilde{F}_M, \qquad \left[\widetilde{Q}_M(1), \widetilde{Q}_M(2)\right] \subset \widetilde{Q}_M(2).
\tag{3.8}
$$

Here we note that the subalgebras $\widetilde{Q}_M(1)$ and $\widetilde{Q}_M(2)$ are both non-semisimple, which is very important for deriving the nonlinear integrable couplings of hierarchy.

In terms of the loop algebra $\widetilde{Q}_M$, consider an isospectral problem as follows:

$$
\psi_x = U\psi, \qquad U = \frac{1}{2}(\lambda I_M, u_1 + u_2, \lambda(u_1 - u_2), 0, u_3 + u_4, \lambda(u_3 - u_4), 0, u_5 + u_6, \lambda(u_5 - u_6))^T.
\tag{3.9}
$$

Set

$$
V = (\lambda v_1, v_2 + v_3, \lambda(v_2 - v_3), \lambda v_4, v_5 + v_6, \lambda(v_5 - v_6), \lambda v_7, v_8 + v_9, \lambda(v_8 - v_9))^T,
\tag{3.10}
$$

where $v_k = \sum_{m=0}^{\infty} (v_{k_1,m}, v_{k_2,m}, \ldots, v_{k_M,m})^T \lambda^{-m}$, $k = 1, \ldots, 9$. A solution to $V_x = [U, V]$ gives

$$v_{1,mx} = u_2 * v_{2,m} - u_1 * v_{3,m},$$

$$v_{2,mx} = v_{3,m+1} - u_2 * v_{1,m+1},$$

$$v_{3,mx} = v_{2,m+1} - u_1 * v_{1,m+1},$$

$$v_{4,mx} = u_2 * v_{5,m} - u_1 * v_{6,m} + u_4 * v_{2,m} - u_3 * v_{3,m} + u_4 * v_{5,m} - u_3 * v_{6,m},$$

$$v_{5,mx} = 2v_{6,m+1} - u_2 * v_{4,m+1} + v_{3,m+1} - u_4 * v_{1,m+1} - u_4 * v_{4,m+1},$$

$$v_{6,mx} = 2v_{5,m+1} - u_1 * v_{4,m+1} + v_{2,m+1} - u_3 * v_{1,m+1} - u_3 * v_{4,m+1},$$

$$v_{7,mx} = u_2 * v_{8,m} - u_1 * v_{9,m} + u_6 * v_{2,m} - u_5 * v_{3,m} + u_4 * v_{5,m} - u_3 * v_{6,m}$$
$$+ u_4 * v_{8,m} - u_3 * v_{9,m} + u_6 * v_{5,m} - u_5 * v_{6,m},$$

$$v_{8,mx} = 2v_{9,m+1} - u_2 * v_{7,m+1} + v_{3,m+1} - u_6 * v_{1,m+1} + 2v_{6,m+1} - u_4 * v_{4,m+1}$$
$$- u_4 * v_{7,m+1} - u_6 * v_{4,m+1},$$

$$v_{9,mx} = 2v_{8,m+1} - u_1 * v_{7,m+1} + v_{2,m+1} - u_5 * v_{1,m+1} + 2v_{5,m+1} - u_3 * v_{4,m+1}$$
$$- u_3 * v_{7,m+1} - u_5 * v_{4,m+1},$$

$$v_{1,0} = 0I_M, \qquad v_{2,0} = u_1, \qquad v_{3,0} = u_2, \qquad v_{4,0} = 0I_M,$$

$$v_{5,0} = u_1 + 2u_3, \qquad v_{6,0} = u_2 + 2u_4, \qquad v_{7,0} = 0I_M,$$

$$v_{8,0} = u_1 + 2u_3 + 2u_5, \qquad v_{9,0} = u_2 + 2u_4 + 2u_6, \qquad v_{1,1} = \frac{(u_2 * u_2 - u_1 * u_1)}{2},$$

$$v_{2,1} = u_{2,x} + u_1 * \frac{(u_2 * u_2 - u_1 * u_1)}{2}, \qquad v_{3,1} = u_{1,x} + u_2 * \frac{(u_2 * u_2 - u_1 * u_1)}{2},$$

$$v_{4,1} = \frac{[(u_2 + u_4) * (u_2 + u_4) - (u_1 + u_3) * (u_1 + u_3)]}{2} - \frac{(u_2 * u_2 - u_1 * u_1)}{2},$$

$$v_{5,1} = u_{4,x} - \frac{u_1 * (u_2 * u_2 - u_1 * u_1)}{2} + \frac{(u_1 + u_3) * [(u_2 + u_4) * (u_2 + u_4) - (u_1 + u_3) * (u_1 + u_3)]}{4},$$

$$v_{6,1} = u_{3,x} - \frac{u_2 * (u_2 * u_2 - u_1 * u_1)}{2} + \frac{(u_2 + u_4) * [(u_2 + u_4) * (u_2 + u_4) - (u_1 + u_3) * (u_1 + u_3)]}{4},$$

$$v_{7,1} = \frac{(u_4 * u_4 - u_3 * u_3)}{2} + [(u_2 + u_4) * u_6 - (u_1 + u_3) * u_5],$$

$$v_{8,1} = u_{6,x} + \frac{(u_5 - u_1) * [(u_2 + u_4) * (u_2 + u_4) - (u_1 + u_3) * (u_1 + u_3)]}{4}$$
$$+ \frac{(u_1 - u_3) * (u_2 * u_2 - u_1 * u_1)}{4} + \frac{(u_1 + u_3) * (u_4 * u_4 - u_3 * u_3)}{4}$$
$$+ \frac{(u_1 + u_3) * [(u_2 + u_4) * u_6 - (u_1 + u_3) * u_5]}{2},$$

$$v_{9,1} = u_{5,x} + \frac{(u_6 - u_2) * [(u_2 + u_4) * (u_2 + u_4) - (u_1 + u_3) * (u_1 + u_3)]}{4}$$
$$+ \frac{(u_2 - u_4) * (u_2 * u_2 - u_1 * u_1)}{4} + \frac{(u_2 + u_4) * (u_4 * u_4 - u_3 * u_3)}{4}$$
$$+ \frac{(u_2 + u_4) * [(u_2 + u_4) * u_6 - (u_1 + u_3) * u_5]}{2}.$$

$$(3.11)$$

Similar to (2.9), we obtain

$$-V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] = \left(0, \frac{(v_{2,nx} + v_{3,nx})}{2}, \frac{\lambda(v_{2,nx} - v_{3,nx})}{2}, 0, v_{5,nx} + v_{6,nx} + \frac{(v_{2,nx} + v_{3,nx})}{2},\right.$$

$$\lambda(v_{5,nx} - v_{6,nx}) + \frac{\lambda(v_{2,nx} - v_{3,nx})}{2}, 0, (v_{8,nx} + v_{9,nx} + v_{5,nx} + v_{6,nx})$$

$$\left.+ \frac{(v_{2,nx} + v_{3,nx})}{2}, (v_{8,nx} - v_{9,nx} + v_{5,nx} - v_{6,nx}) + \frac{(v_{2,nx} - v_{3,nx})}{2}\right). \tag{3.12}$$

Therefore, the zero-curvature equation

$$U_t - V_x^{(n)} + \left[U, V^{(n)}\right] = 0 \tag{3.13}$$

admits the following bigger integrable system:

$$U_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_t = \begin{pmatrix} \partial & 0 & -\partial & 0 & \partial & 0 \\ 0 & -\partial & 0 & \partial & 0 & -\partial \\ -\partial & 0 & \partial & 0 & \partial & 0 \\ 0 & \partial & 0 & -\partial & 0 & -\partial \\ \partial & 0 & \partial & 0 & -\partial & 0 \\ 0 & -\partial & 0 & -\partial & 0 & \partial \end{pmatrix} \begin{pmatrix} v_{2,n} + v_{5,n} + v_{8,n} \\ -v_{3,n} - v_{6,n} - v_{9,n} \\ v_{2,n} + 2v_{5,n} + v_{8,n} \\ -v_{3,n} - 2v_{6,n} - v_{9,n} \\ v_{2,n} + v_{5,n} \\ -v_{3,n} - v_{6,n} \end{pmatrix}$$

$$= J_2 P_n = J_2 L_2^n \begin{pmatrix} v_{2,0} + v_{5,0} + v_{8,0} \\ -v_{3,0} - v_{6,0} - v_{9,0} \\ v_{2,0} + 2v_{5,0} + v_{8,0} \\ -v_{3,0} - 2v_{6,0} - v_{9,0} \\ v_{2,0} + v_{5,0} \\ -v_{3,0} - v_{6,0} \end{pmatrix}, \tag{3.14}$$

where J_2 is the Hamiltonian operator. Similar to (2.13), by employing (3.11), after tedious calculation, the recurrence operator L_2 can be obtained and here we omit it.

Taking $u_3 = u_4 = 0$ in (3.14) reduces to an integrable coupling of the Guo hierarchy; taking $u_5 = u_6 = 0$ in (3.14) reduces to another integrable coupling of the Guo hierarchy. Here we note that except the first variables u_1, u_2, all integrable couplings presented above are nonlinear, since the supplementary systems (3.14) with $n \geq 1$ are nonlinear with respect to the four dependent variables u_3, u_4, u_5, u_6. So, in term of the definition of integrable couplings, we conclude that the system (3.14) is nonlinear bi-integrable couplings of the multicomponent Guo hierarchy.

In order to deduce to the Hamiltonian structure of (3.14), we rewrite (3.5) as follows:

$$[a, b] = a^T R(b), \tag{3.15}$$

where

$R(b)$

$$
=\begin{pmatrix}
0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 & 0 & 2b_8 & -2b_9 \\
b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 & b_9 & -2b_7 & 0 \\
-b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 & -b_8 & 0 & 2b_7 \\
0 & 0 & 0 & 0 & 2(b_2+b_5) & -2(b_3+2b_6) & 0 & 2(b_5+b_8) & -2(b_6+b_9) \\
0 & 0 & 0 & b_3+b_6 & -2(b_1+b_4) & 0 & b_6+b_9 & -2(b_4+b_7) & 0 \\
0 & 0 & 0 & -b_2-b_5 & 0 & 2(b_1+b_4) & -b_5-b_8 & 0 & 2(b_4+b_7) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2(b_2+b_5) & -2(b_3+b_6) \\
0 & 0 & 0 & 0 & 0 & 0 & b_3+b_6 & -2(b_1+b_4) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(b_2+b_5) & 0 & 2(b_1+b_4)
\end{pmatrix}. \tag{3.16}
$$

Solving the matrix equation for the constant matrix F: $R(b) * F = -(R(b) * F)^T$, $F^T = F$, presents

$$
F = \begin{pmatrix}
2I_M & 0 & 0 & 2I_M & 0 & 0 & 2I_M & 0 & 0 \\
0 & 0 & I_M & 0 & 0 & I_M & 0 & 0 & I_M \\
0 & I_M & 0 & 0 & I_M & 0 & 0 & I_M & 0 \\
2I_M & 0 & 0 & 4I_M & 0 & 0 & 2I_M & 0 & 0 \\
0 & 0 & I_M & 0 & 0 & 2I_M & 0 & 0 & I_M \\
0 & I_M & 0 & 0 & 2I_M & 0 & 0 & I_M & 0 \\
2I_M & 0 & 0 & 2I_M & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_M & 0 & 0 & I_M & 0 & 0 & 0 \\
0 & I_M & 0 & 0 & I_M & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{3.17}
$$

Then in terms of F, define a linear functional

$$
\begin{aligned}
\{a,b\} &= a^T * F * b \\
&= 2(a_1+a_4+a_7)*b_1 + (a_3+a_6+a_9)*b_2 \\
&\quad + (a_2+a_5+a_8)*b_3 + 2(a_1+2a_4+a_7)*b_4 + (a_3+2a_6+a_9)*b_5 \\
&\quad + (a_2+2a_5+a_8)*b_6 + 2(a_1+a_4)*b_7 + (a_3+a_6)*b_8 + (a_2+a_5)\ *b_9.
\end{aligned} \tag{3.18}
$$

It is easy to find that $\{a,b\}$ satisfies the variational identity (2.18). By using the above linear functional, we have

$$
\begin{aligned}
\{V,U_\lambda\} &= \lambda(v_1+v_4+v_7) + \frac{1}{2}(v_2+v_3+v_5+v_6+v_8+v_9)*(u_1-u_2) \\
&\quad + \frac{1}{2}(v_2+v_3+2v_5+2v_6+v_8+v_9)*(u_3-u_4) + \frac{1}{2}(v_2+v_3+v_5+v_6)*(u_5-u_6),
\end{aligned}
$$

$$
\{V,U_{u_1}\} = \lambda(v_2+v_5+v_8), \qquad \{V,U_{u_2}\} = \lambda(-v_3-v_6-v_9), \qquad \{V,U_{u_3}\} = \lambda(v_2+2v_5+v_8),
$$

$$
\{V,U_{u_4}\} = \lambda(-v_3-v_6-v_9), \qquad \{V,U_{u_5}\} = \lambda(v_2+v_5), \qquad \{V,U_{u_6}\} = \lambda(-v_3-v_6).
$$

$$\tag{3.19}$$

Substituting (3.19) into the variational identity (2.18), we have

$$
\frac{\delta}{\delta u}\left(\lambda(v_1 + v_4 + v_7) + \frac{1}{2}(v_2 + v_3 + v_5 + v_6 + v_8 + v_9) * (u_1 - u_2)\right.
$$

$$
+ \frac{1}{2}(v_2 + v_3 + 2v_5 + 2v_6 + v_8 + v_9) * (u_3 - u_4) + \left.\frac{1}{2}(v_2 + v_3 + v_5 + v_6) * (u_5 - u_6)\right) \tag{3.20}
$$

$$
= \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(\lambda(v_2 + v_5 + v_8), \lambda(-v_3 - v_6 - v_9), \lambda(v_2 + 2v_5 + v_8), \lambda(-v_3 - v_6 - v_9),
$$

$$
\lambda(v_2 + v_5), \lambda(-v_3 - v_6))^T.
$$

Comparing the coefficients of λ^{-n-1} yields

$$
\frac{\delta}{\delta u}\left(v_{1,n+1} + v_{4,n+1} + v_{7,n+1} + \frac{1}{2}(v_{2,n} + v_{3,n} + v_{5,n} + v_{6,n} + v_{8,n} + v_{9,n}) * (u_1 - u_2)\right.
$$

$$
+ \frac{1}{2}(v_{2,n} + v_{3,n} + 2v_{5,n} + 2v_{6,n} + v_{8,n} + v_{9,n})(u_3 - u_4)
$$

$$
+ \left.\frac{1}{2}(v_{2,n} + v_{3,n} + v_{5,n} + v_{6,n}) * (u_5 - u_6)\right) \tag{3.21}
$$

$$
= (\gamma - n)((v_{2,n} + v_{5,n} + v_{8,n}), (-v_{3,n} - v_{6,n} - v_{9,n}), (v_{2,n} + 2v_{5,n} + v_{8,n}),
$$

$$
(-v_{3,n} - 2v_{6,n} - v_{9,n}), (v_{2,n} + v_{5,n}), (-v_{3,n} - v_{6,n}))^T.
$$

Taking $n = 0$ in the above equation gives $\gamma = 0$.

Hence, the nonlinear bi-integrable couplings of the multicomponent Guo hierarchy (3.14) can be written as a Hamiltonian form

$$
U_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_t = J_2 \begin{pmatrix} \dfrac{\delta}{\delta u_1} \\[4pt] \dfrac{\delta}{\delta u_2} \\[4pt] \dfrac{\delta}{\delta u_3} \\[4pt] \dfrac{\delta}{\delta u_4} \\[4pt] \dfrac{\delta}{\delta u_5} \\[4pt] \dfrac{\delta}{\delta u_6} \end{pmatrix} \qquad \overline{H}_n = J_2 \frac{\delta \overline{H}_n}{\delta u}, \tag{3.22}
$$

where $\overline{H}_n = -(1/n)[v_{1,n+1} + v_{4,n+1} + v_{7,n+1} + (1/2)(v_{2,n} + v_{3,n} + v_{5,n} + v_{6,n} + v_{8,n} + v_{9,n}) * (u_1 - u_2) + (1/2)(v_{2,n} + v_{3,n} + 2v_{5,n} + 2v_{6,n} + v_{8,n} + v_{9,n})(u_3 - u_4) + (1/2)(v_{2,n} + v_{3,n} + v_{5,n} + v_{6,n}) * (u_5 - u_6)]$.

Next, we will derive the nonlinear bi-integrable couplings of the multicomponent Guo hierarchy with self-consistent sources. Consider the linear problem

$$
\begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \psi_{3,j} \\ \psi_{4,j} \\ \psi_{5,j} \\ \psi_{6,j} \end{pmatrix}_x = U(u,\lambda_j) \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \psi_{3,j} \\ \psi_{4,j} \\ \psi_{2,j} \\ \psi_{6,j} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \psi_{3,j} \\ \psi_{4,j} \\ \psi_{5,j} \\ \psi_{6,j} \end{pmatrix}_t = V(u,\lambda_j) \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \psi_{3,j} \\ \psi_{4,j} \\ \psi_{5,j} \\ \psi_{6,j} \end{pmatrix}. \tag{3.23}
$$

According to (2.24) and the definition of $\{a,b\}$, we get

$$
\Psi_j = \big(\phi_{1,j} * \phi_{2,j}, -\phi_{1,j} * \phi_{1,j}, \phi_{2,j} * \phi_{2,j}, \phi_{3,j} * \phi_{4,j}, -\phi_{3,j} * \phi_{3,j}, \phi_{4,j} * \phi_{4,j}, \phi_{5,j} * \phi_{6,j},
$$
$$
-\phi_{5,j} * \phi_{5,j}, \phi_{6,j} * \phi_{6,j} \big),
$$

$$
\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \frac{1}{4} W
$$

$$
= \frac{1}{4} \begin{pmatrix} \langle \Phi_2, \Phi_2 \rangle + \langle \Phi_4, \Phi_4 \rangle + \langle \Phi_6, \Phi_6 \rangle - \wedge(\langle \Phi_1, \Phi_1 \rangle + \langle \Phi_3, \Phi_3 \rangle + \langle \Phi_5, \Phi_5 \rangle) \\ \langle \Phi_2, \Phi_2 \rangle + \langle \Phi_4, \Phi_4 \rangle + \langle \Phi_6, \Phi_6 \rangle + \wedge(\langle \Phi_1, \Phi_1 \rangle + \langle \Phi_3, \Phi_3 \rangle + \langle \Phi_5, \Phi_5 \rangle) \\ \langle \Phi_2, \Phi_2 \rangle + 2\langle \Phi_4, \Phi_4 \rangle + \langle \Phi_6, \Phi_6 \rangle - \wedge(\langle \Phi_1, \Phi_1 \rangle + 2\langle \Phi_3, \Phi_3 \rangle + \langle \Phi_5, \Phi_5 \rangle) \\ \langle \Phi_2, \Phi_2 \rangle + 2\langle \Phi_4, \Phi_4 \rangle + \langle \Phi_6, \Phi_6 \rangle + \wedge(\langle \Phi_1, \Phi_1 \rangle + 2\langle \Phi_3, \Phi_3 \rangle + \langle \Phi_5, \Phi_5 \rangle) \\ \langle \Phi_2, \Phi_2 \rangle + \langle \Phi_4, \Phi_4 \rangle - \wedge(\langle \Phi_1, \Phi_1 \rangle + \langle \Phi_3, \Phi_3 \rangle) \\ \langle \Phi_2, \Phi_2 \rangle + \langle \Phi_4, \Phi_4 \rangle - \wedge(\langle \Phi_1, \Phi_1 \rangle + \langle \Phi_3, \Phi_3 \rangle) \end{pmatrix},
$$

$$
\tag{3.24}
$$

where $\phi_{i,j} = (\phi_{i_1,j}, \phi_{i_2,j}, \ldots, \phi_{i_M,j})^T$, $\wedge = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $\Phi_i = (\phi_{i,1}, \ldots, \phi_{i,N})^T$, $i = 1, \ldots, 6$; $j = 1, \ldots, N$; $\langle \cdot, \cdot \rangle$ stands for the inner product in R^N.

Then, according to (2.23) and taking $\alpha_i = -4$, the nonlinear bi-integrable couplings of the multicomponent Guo hierarchy with self-consistent sources can be rewritten as follows:

$$
(u_1, u_2, u_3, u_4, u_5, u_6)_t^T = J_2 \frac{\delta \overline{H}_n}{\delta u} + J_2 W. \tag{3.25}
$$

4. Conclusions

In this paper, we derive the multicomponent Guo hierarchy with self-consistent sources and its nonlinear bi-integrable couplings by using the Lie algebra Q_M. As far as we know, the existing papers are not designed for constructing the nonlinear bi-integrable couplings of multicomponent hierarchy with self-consistent sources. This method can be used to derive the other multicomponent soliton hierarchies with self-consistent sources and their nonlinear bi-integrable couplings.
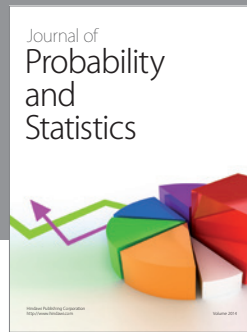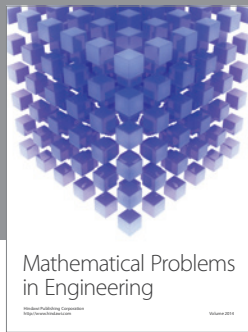
Acknowledgments

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