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Research Article

Semigroup Method on a M^X/G/1 Queueing Model

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By using the Hille-Yosida theorem, Phillips theorem, and Fattorini theorem in functional analysis we prove that the M^X/G/1 queueing model with vacation times has a unique nonnegative time-dependent solution.

1. Introduction

The queueing system when the server become idle is not new. Miller [1] was the first to study such a model, where the server is unavailable during some random length of time for the M/G/1 queueing system. The M/G/1 queueing models of similar nature have also been reported by a number of authors, since Levy and Yechiali [2] included several types of generalizations of the classical M/G/1 queueing system. These generalizations are useful in model building in many real life situations such as digital communication, computer network, and production/inventory system [3–5].

At present, however, most studies are devoted to batch arrival queues with vacation because of its interdisciplinary character. Considerable efforts have been devoted to study these models by Baba [6], Lee and Srinivasan [7], Lee et al. [8, 9], Borthakur and Choudhury [10], and Choudhury [11, 12] among others. However, the recent progress of M^X/G/1 type queueing models of this nature has been served by Chae and Lee [13] and Medhi [14].

In 2002, Choudhury [15] studied the M^X/G/1 queueing model with vacation times. By using the supplementary variable technique [16] he established the corresponding queueing model and obtained the queue size distribution at a stationary (random) as well as a departure point of time under multiple vacation policy based on the following hypothesis. "The time-dependent solution of the model converges to a nonzero steady-state solution." By reading the paper we find that the previous hypothesis, in fact, implies the following two hypothesis.

Hypothesis 1. The model has a nonnegative time-dependent solution.

Hypothesis 2. The time-dependent solution of the model converges to a nonzero steady-state solution.

In this paper we investigate Hypothesis 1. By using the Hille-Yosida theorem, Phillips theorem, and Fattorini theorem we prove that the model has a unique nonnegative time-dependent solution, and therefore we obtain Hypothesis 1.

According to Choudhury [15], the M^X/G/1 queueing system with vacation times can be described by the following system of equations:

$$\frac{dQ(t)}{dt} = -\lambda Q(t) + \int_0^\infty v(x) P_{0,0}(x,t)\,dx + \int_0^\infty b(x) P_{1,1}(x,t)\,dx,$$

$$\frac{\partial P_{0,0}(x,t)}{\partial t} + \frac{\partial P_{0,0}(x,t)}{\partial x} = -[\lambda + v(x)] P_{0,0}(x,t),$$

$$\frac{\partial P_{0,n}(x,t)}{\partial t} + \frac{\partial P_{0,n}(x,t)}{\partial x} = -[\lambda + v(x)] P_{0,n}(x,t) + \lambda \sum_{k=1}^{n} c_k P_{0,n-k}(x,t), \quad n \geq 1,$$

$$\frac{\partial P_{1,n}(x,t)}{\partial t} + \frac{\partial P_{1,n}(x,t)}{\partial x}$$

$$
\begin{aligned}
&= -[\lambda + b(x)] P_{1,n}(x,t) \\
&\quad + \lambda \sum_{k=1}^{n} c_k P_{1,n-k+1}(x,t), \quad n \geq 1, \\
&P_{0,0}(t) = \lambda Q(t), \qquad P_{0,n}(0,t) = 0, \quad n \geq 1, \\
&P_{1,n}(0,t) = \int_0^\infty v(x) P_{0,n}(x,t)\,dx \\
&\qquad + \int_0^\infty b(x) P_{1,n+1}(x,t)\,dx, \quad n \geq 1, \\
&Q(0) = 1, \qquad P_{0,n}(x,0) = 0, \quad n \geq 0, \\
&P_{1,n}(x,0) = 0, \quad n \geq 1,
\end{aligned}
\tag{1}
$$

where $(x,t) \in [0,\infty)\times[0,\infty)$; $Q(t)$ represents the probability that there is no customer in the system and the server is idle at time t; $P_{0,n}(x,t)dx\,(n \geq 0)$ represents the probability that at time t there are n customers in the system and the server is on a vacation with elapsed vacation time of the server lying in $[x,\ x+dx)$. $P_{1,n}(x,t)dx\,(n \geq 1)$ represents the probability that at time t there are n customers in the system with elapsed service time of the customer undergoing service lying in $[x,\ x+dx)$. λ is batch arrival rate of customers. $c_k\ (k \geq 1)$ represents the probability that at every arrival epoch a batch of k external customers arrives and satisfies $\sum_{k=1}^{\infty} c_k = 1$. $v(x)$ is the vacation rate of the server, which satisfies

$$v(x) \geq 0, \qquad \int_0^\infty v(x)\,dx = \infty. \tag{2}$$

$b(x)$ is the service rate of the server satisfying

$$b(x) \geq 0, \qquad \int_0^\infty b(x)\,dx = \infty. \tag{3}$$

2. Problem Formulation

We first formulate the system (1) as an abstract Cauchy problem on a suitable state space. For convenience we take some notations as follows:

$$
\Gamma_1 = \begin{pmatrix} e^{-x} & 0 & 0 & \cdots \\ \lambda e^{-x} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad
\Gamma_2 = \begin{pmatrix} 0 & 0 & v(x) & 0 & \cdots \\ 0 & 0 & 0 & v(x) & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},
$$

$$
\Gamma_3 = \begin{pmatrix} 0 & b(x) & 0 & \cdots \\ 0 & 0 & b(x) & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4}
$$

If we take state space

$$
X = \left\{ (P_0, P_1) \;\middle|\; \begin{aligned} &P_0 = (Q, P_{0,0}, P_{0,1}, \ldots) \\ &\quad \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots \\ &P_1 = (P_{1,1}, P_{1,2}, P_{1,3}, \ldots) \\ &\quad \in L^1[0,\infty) \times L^1[0,\infty) \\ &\qquad \times L^1[0,\infty) \times \cdots \\ &\|(P_0,P_1)\| = |Q| + \sum_{i=0}^{\infty} \|P_{0,i}\|_{L^1[0,\infty)} \\ &\qquad + \sum_{i=1}^{\infty} \|P_{1,i}\|_{L^1[0,\infty)} < \infty \end{aligned} \right\}, \tag{5}
$$

then it is obvious that X is a Banach space. In the following we define operators and their domains;

$$
\begin{aligned}
&A(P_0, P_1) \\
&= \left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ P_{0,0}(x) \\ P_{0,1}(x) \\ \vdots \end{pmatrix}, \right. \\
&\qquad \left. \begin{pmatrix} -\frac{d}{dx} & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_{1,1}(x) \\ P_{1,2}(x) \\ P_{1,3}(x) \\ \vdots \end{pmatrix} \right),
\end{aligned}
$$

$$
\begin{aligned}
&D(A) \\
&= \left\{ (P_0,P_1) \in X \;\middle|\; \begin{aligned} &\frac{dP_{0,n}(x)}{dx}, \frac{dP_{1,k}(x)}{dx} \in L^1[0,\infty), \\ &P_{0,n}(x), P_{1,k}(x)\ (n \geq 0,\ k \geq 1) \\ &\text{are absolutely continuous} \\ &\text{functions and satisfy} \\ &P_0(0) = \int_0^\infty \Gamma_1 P_0(x)\,dx, \\ &P_1(0) = \int_0^\infty \Gamma_2 P_0(x)\,dx \\ &\qquad + \int_0^\infty \Gamma_3 P_1(x)\,dx \end{aligned} \right\},
\end{aligned}
$$

$$
\begin{aligned}
&U(P_0, P_1) \\
&= \left(\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \tilde{v} & 0 & 0 & \cdots \\ 0 & \lambda c_1 & \tilde{v} & 0 & \cdots \\ 0 & \lambda c_2 & \lambda c_1 & \tilde{v} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ P_{0,0}(x) \\ P_{0,1}(x) \\ P_{0,2}(x) \\ \vdots \end{pmatrix}, \right.
\end{aligned}
$$

$$\left(\begin{pmatrix} \tilde{b} & 0 & 0 & 0 & \cdots \\ \lambda c_2 & \tilde{b} & 0 & 0 & \cdots \\ \lambda c_3 & \lambda c_2 & \tilde{b} & 0 & \cdots \\ \lambda c_4 & \lambda c_3 & \lambda c_2 & \tilde{b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_{1,1}(x) \\ P_{1,2}(x) \\ P_{1,3}(x) \\ P_{1,4}(x) \\ \vdots \end{pmatrix}\right), \tag{6}$$

where $\tilde{v} = -[\lambda + v(x)]$, $\tilde{b} = -[\lambda + b(x)] + \lambda c_1$,

$$E(P_0, P_1) = \left(\begin{pmatrix} \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}\right),$$

$$D(U) = D(E) = X. \tag{7}$$

Then the previous system of equations (1) can be rewritten as an abstract Cauchy problem in the Banach space X:

$$\frac{d(P_0, P_1)(t)}{dt} = (A + U + E)(P_0, P_1)(t), \quad \forall t \in [0, \infty),$$
$$(P_0, P_1)(0) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}\right). \tag{8}$$

3. Well-Posedness of The System (8)

Theorem 1. *If $v = \sup_{x\in[0,\infty)} v(x) < \infty$, $b = \sup_{x\in[0,\infty)} b(x) < \infty$, then $A + U + E$ generates a positive contraction C_0-semigroup $T(t)$.*

Proof. We split the proof of the theorem into four steps. Firstly, we prove that $(\gamma I - A)^{-1}$ exists and is bounded for some γ. Secondly, we show that $D(A)$ is dense in X. Thirdly, we verify that U and E are bounded linear operators. Thus by using the Hille-Yosida theorem and the perturbation theorem of C_0-semigroup we deduce that $A + U + E$ generates a C_0-semigroup $T(t)$. Finally, we check that $A + U + E$ is dispersive, and therefore we obtain the desired result.

For any given $(y_0, y_1) \in X$, we consider the equation $(\gamma I - A)(P_0, P_1) = (y_0, y_1)$; that is,

$$(\gamma + \lambda) Q = y_Q, \tag{9}$$

$$\frac{dP_{0,n}(x)}{dx} = -\gamma P_{0,n}(x) + y_{0,n}(x), \quad n \geq 0, \tag{10}$$

$$\frac{dP_{1,n}(x)}{dx} = -\gamma P_{1,n}(x) + y_{1,n}(x), \quad n \geq 1, \tag{11}$$

$$P_{0,0}(0) = \lambda Q, \tag{12}$$

$$P_{0,n}(0) = 0, \quad n \geq 1, \tag{13}$$

$$P_{1,n}(0) = \int_0^\infty v(x) P_{0,n}(x)\,dx + \int_0^\infty b(x) P_{1,n+1}(x)\,dx, \quad n \geq 1. \tag{14}$$

Through solving (9)–(11), we have

$$Q = \frac{1}{\gamma + \lambda} y_Q, \tag{15}$$

$$P_{0,n}(x) = a_{0,n} e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau, \quad n \geq 0, \tag{16}$$

$$P_{1,n}(x) = a_{1,n} e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{1,n}(\tau) e^{\gamma\tau} d\tau, \quad n \geq 1. \tag{17}$$

Combining (16) with (12) and (13), we obtain

$$a_{0,0} = P_{0,0}(0) = \lambda Q = \frac{\lambda}{\gamma + \lambda} y_Q, \tag{18}$$

$$a_{0,n} = P_{0,n}(0) = 0, \quad n \geq 1. \tag{19}$$

Substituting (19) into (16), it follows that

$$P_{0,n}(x) = e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau, \quad n \geq 1. \tag{20}$$

By combining (15), (16), (17), and (20) with (14), we deduce

$$\begin{aligned} a_{1,n} &= P_{1,n}(0) \\ &= \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau\,dx \\ &\quad + a_{1,n+1} \int_0^\infty b(x) e^{-\gamma x} dx \\ &\quad + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{1,n+1}(\tau) e^{\gamma\tau} d\tau\,dx, \quad n \geq 1, \end{aligned} \tag{21}$$

$$\Longrightarrow$$

$$\begin{aligned} a_{1,n} - a_{1,n+1} \int_0^\infty b(x) e^{-\gamma x} dx &= \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau\,dx \\ &\quad + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{1,n+1}(\tau) e^{\gamma\tau} d\tau\,dx. \end{aligned} \tag{22}$$

If we set

$$\mathscr{C}=\begin{pmatrix} 1 & -\int_0^\infty b(x)e^{-\gamma x}dx & 0 & 0 & \cdots \\ 0 & 1 & -\int_0^\infty b(x)e^{-\gamma x}dx & 0 & \cdots \\ 0 & 0 & 1 & -\int_0^\infty b(x)e^{-\gamma x}dx & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\vec{a}_1=\begin{pmatrix} a_{1,1}\\ a_{1,2}\\ a_{1,3}\\ \vdots\end{pmatrix}, \tag{23}$$

then (21) can be rewritten as follows:

$$\mathscr{C}\vec{a}_1=\begin{pmatrix} \int_0^\infty v(x)e^{-\gamma x}\int_0^x y_{0,1}(\tau)e^{\gamma\tau}d\tau\,dx+\int_0^\infty b(x)e^{-\gamma x}\int_0^x y_{1,2}(\tau)e^{\gamma\tau}d\tau\,dx \\ \int_0^\infty v(x)e^{-\gamma x}\int_0^x y_{0,2}(\tau)e^{\gamma\tau}d\tau\,dx+\int_0^\infty b(x)e^{-\gamma x}\int_0^x y_{1,3}(\tau)e^{\gamma\tau}d\tau\,dx \\ \vdots \end{pmatrix}. \tag{24}$$

It is easy to calculate

$$\mathscr{C}^{-1}=\begin{pmatrix} 1 & \int_0^\infty b(x)e^{-\gamma x}dx & \left(\int_0^\infty b(x)e^{-\gamma x}dx\right)^2 & \left(\int_0^\infty b(x)e^{-\gamma x}dx\right)^3 & \cdots \\ 0 & 1 & \int_0^\infty b(x)e^{-\gamma x}dx & \left(\int_0^\infty b(x)e^{-\gamma x}dx\right)^2 & \cdots \\ 0 & 0 & 1 & \int_0^\infty b(x)e^{-\gamma x}dx & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{25}$$

From which together with (24), we derive

$$\begin{aligned} a_{1,n} &= \sum_{k=0}^{\infty}\left(\int_0^\infty b(x)e^{-\gamma x}dx\right)^k \\ &\quad\times\left\{\int_0^\infty v(x)e^{-\gamma x}\int_0^x y_{0,k+n}(\tau)e^{\gamma\tau}d\tau\,dx\right. \\ &\quad\left.+\int_0^\infty b(x)e^{-\gamma x}\int_0^x y_{1,k+n+1}(\tau)e^{\gamma\tau}d\tau\,dx\right\},\quad n\geq 1. \end{aligned} \tag{26}$$

By using Fubini theorem we estimate (16) and (17) as follows (assume that $\gamma > v + b$):

$$\begin{aligned} &\|P_{0,n}\|_{L^1[0,\infty)} \\ &\quad\leq \int_0^\infty |a_{0,n}|e^{-\gamma x}dx+\int_0^\infty e^{-\gamma x}\int_0^x |y_{0,n}(\tau)|e^{\gamma\tau}d\tau\,dx \\ &\quad=\frac{1}{\gamma}|a_{0,n}|+\int_0^\infty |y_{0,n}(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dx\,d\tau \\ &\quad=\frac{1}{\gamma}|a_{0,n}|+\frac{1}{\gamma}\|y_{0,n}\|_{L^1[0,\infty)},\quad n\geq 0, \\ &\|P_{1,n}\|_{L^1[0,\infty)}\leq\frac{1}{\gamma}|a_{1,n}|+\frac{1}{\gamma}\|y_{1,n}\|_{L^1[0,\infty)},\quad n\geq 1. \end{aligned} \tag{27}$$

From (26) and Fubini theorem, we deduce

$$\begin{aligned} \sum_{n=1}^{\infty}|a_{1,n}| &\leq \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^k \\ &\quad\times\left\{v\int_0^\infty |y_{0,k+n}(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dx\,d\tau\right. \\ &\quad\left.+b\int_0^\infty |y_{1,k+n+1}(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dx\,d\tau\right\} \end{aligned}$$

$$= \frac{v}{\gamma}\sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k}\sum_{n=1}^{\infty}\|y_{0,k+n}\|_{L^1[0,\infty)} + \sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k+1}\sum_{n=1}^{\infty}\|y_{1,k+n+1}\|_{L^1[0,\infty)}$$
$$\leq \frac{v}{\gamma}\sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k}\sum_{n=0}^{\infty}\|y_{0,n}\|_{L^1[0,\infty)} + \sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k+1}\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}$$
$$= \frac{v}{\gamma-b}\sum_{n=0}^{\infty}\|y_{0,n}\|_{L^1[0,\infty)} + \frac{b}{\gamma-b}\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}. \quad (28)$$

By inserting (18), (19), and (28) into (27) and using inequality $(\gamma+v-b)/\gamma(\gamma-b) < 1/(\gamma-v-b)$, we estimate

$$\begin{aligned}\|(P_0,P_1)\| &\leq \frac{|y_Q|}{\gamma+\lambda} + \frac{1}{\gamma}\sum_{n=0}^{\infty}|a_{0,n}| + \frac{1}{\gamma}\sum_{n=0}^{\infty}\|y_{0,n}\|_{L^1[0,\infty)} + \frac{1}{\gamma}\sum_{n=1}^{\infty}|a_{1,n}| + \frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}\\ &\leq \frac{1}{\gamma}|y_Q| + \frac{\gamma+v-b}{\gamma(\gamma-b)}\sum_{n=0}^{\infty}\|y_{0,n}\|_{L^1[0,\infty)} + \frac{1}{\gamma-b}\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}\\ &\leq \frac{1}{\gamma-v-b}|y_Q| + \frac{1}{\gamma-v-b}\sum_{n=0}^{\infty}\|y_{0,n}\|_{L^1[0,\infty)} + \frac{1}{\gamma-v-b}\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}\\ &= \frac{1}{\gamma-v-b}\|(y_0,y_1)\|.\end{aligned} \quad (29)$$

Equation (29) shows that $(\gamma I - A)^{-1}$ exists for $\gamma > v + b$ and

$$(\gamma I - A)^{-1} : X \longrightarrow D(A), \quad \left\|(\gamma I - A)^{-1}\right\| \leq \frac{1}{\gamma - v - b}. \quad (30)$$

As far as the second step is concerned, from $|Q| + \sum_{n=0}^{\infty}\|P_{0,n}\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty}\|P_{1,n}\|_{L^1[0,\infty)} < \infty$ for $(P_0,P_1) \in X$ it follows that, for any $\epsilon > 0$, there exists a positive integer N such that $\sum_{n=N}^{\infty}\|P_{0,n}\|_{L^1[0,\infty)} < \epsilon$, $\sum_{n=N}^{\infty}\|P_{1,n}\|_{L^1[0,\infty)} < \infty$. Let

$$\mathbb{L} = \left\{ (P_0,P_1) \;\middle|\; \begin{array}{l} P_0(x) = (Q, P_{0,0}(x), P_{0,1}(x), \ldots, \\ \qquad P_{0,N}(x), 0, \ldots), \\ P_1(x) = (P_{1,1}(x), P_{1,2}(x), \ldots, \\ \qquad P_{1,N}(x), 0, \ldots), \\ Q \in \mathbb{R}, P_{0,i}(x), \\ P_{1,j}(x) \in L^1[0,\infty),\ i = 0,1,\ldots,N; \\ j = 1,2,\ldots,N; \\ N \text{ is a finite positive integer} \end{array} \right\}, \quad (31)$$

then $\mathbb{L}$ is dense in X. If we set

$$\mathbb{Z} = \left\{ (P_0,P_1) \;\middle|\; \begin{array}{l} P_0(x) = (Q, P_{0,0}(x), P_{0,1}(x), \ldots, \\ \qquad P_{0,l}(x), 0, \ldots), \\ P_1(x) = (P_{1,1}(x), P_{1,2}(x), \ldots, \\ \qquad P_{1,l}(x), 0, \ldots), \\ P_{0,i}(x),\ P_{1,j}(x) \in C_0^{\infty}[0,\infty), \\ \text{and there exists positive} \\ \text{numbers } c_{0,i} > 0,\ c_{1,j} > 0, \\ \text{such that } P_{0,i}(x) = 0, x \in [0, c_{0,i}], \\ P_{1,j}(x) = 0,\ x \in [0, c_{1,j}]; \\ i = 0,1,2,\ldots,l;\ j = 1,2,\ldots,l \end{array} \right\}, \quad (32)$$

then by the relationship $C_0^{\infty}[0,\infty)$ and $L^1[0,\infty)$ in Adams [17], we know that $\mathbb{Z}$ is dense in $\mathbb{L}$. Hence in order to prove denseness of $D(A)$, it suffices to prove that $D(A)$ is dense in $\mathbb{Z}$. Take any $(P_0, P_1) \in \mathbb{Z}$, then there are a finite positive integer l and positive numbers $c_{0,i} > 0$, $c_{1,j} > 0$ ($i = 0,1,\ldots,l$; $j = 1,2,\ldots,l$) such that

$$\begin{aligned} &P_0(x) = (Q, P_{0,0}(x), P_{0,1}(x), \ldots, P_{0,l}(x), 0, \ldots), \\ &P_1(x) = (P_{1,1}(x), P_{1,2}(x), \ldots, P_{1,l}(x), 0, \ldots), \\ &P_{0,i}(x) = 0, \quad \text{for } x \in [0, c_{0,i}],\ i = 0,1,2,\ldots,l, \\ &P_{1,j}(x) = 0, \quad \text{for } x \in [0, c_{1,j}],\ j = 1,2,\ldots,l, \end{aligned} \quad (33)$$

$$\begin{aligned} &P_{0,i}(x) = 0, \quad x \in [0, 2s],\ i = 0,1,\ldots,l; \\ &P_{1,j}(x) = 0, \quad x \in [0, 2s],\ j = 1,\ldots,l, \end{aligned} \quad (34)$$

where $0 < 2s < \min\{c_{0,0}, c_{0,1}, \ldots, c_{0,l}, c_{1,1}, \ldots, c_{1,l}\}$. Define

$$\begin{aligned} f_0^s(0) &= \left(Q, f_{0,0}^s(0), f_{0,1}^s(0), \ldots, f_{0,l}^s(0), 0, \ldots\right) \\ &= (Q, \lambda Q, 0, \ldots, 0, 0, \ldots), \end{aligned} \quad (35)$$

$$f_1^s(0) = \left(f_{1,1}^s(0), f_{1,2}^s(0), \ldots, f_{1,l}^s(0), 0, \ldots\right),$$

where

$$f_{1,j}^s(0)=\int_{2s}^{\infty}v(x)P_{0,j}(x)\,dx+\int_{2s}^{\infty}b(x)P_{1,j+1}(x)\,dx,$$
$$j=1,2,\ldots,l-1,$$
$$f_{1,l}^s(0)=\int_{2s}^{\infty}v(x)P_{0,l}(x)\,dx,$$
$$f_0^s(x)=\left(Q,f_{0,0}^s(x),f_{0,1}^s(x),\ldots,f_{0,l}^s(x),0,\ldots\right),$$
$$f_1^s(x)=\left(f_{1,1}^s(x),f_{1,2}^s(x),\ldots,f_{1,l}^s(x),0,\ldots\right), \tag{36}$$

where

$$f_{0,i}^s(x)=\begin{cases}f_{0,i}^s(0)\left(1-\dfrac{x}{s}\right)^2, & x\in[0,s),\\ -\mu_{0,i}(x-s)^2(x-2s)^2, & x\in[s,2s),\\ P_{0,i}(x), & x\in[2s,\infty),\end{cases}$$
$$f_{1,j}^s(x)=\begin{cases}f_{1,j}^s(0)\left(1-\dfrac{x}{s}\right)^2, & x\in[0,s),\\ -\mu_{1,j}(x-s)^2(x-2s)^2, & x\in[s,2s),\\ P_{1,j}(x), & x\in[2s,\infty),\end{cases} \tag{37}$$
$$\mu_{0,i}=\frac{\int_0^s v(x)f_{0,i}^s(0)(1-(x/s))^2dx}{\int_s^{2s}v(x)(x-s)^2(x-2s)^2dx},$$
$$\mu_{1,j}=\frac{\int_0^s b(x)f_{1,j}^s(0)(1-(x/s))^2dx}{\int_s^{2s}b(x)(x-s)^2(x-2s)^2dx},$$
$$i=0,1,\ldots,l;\quad j=1,\ldots,l.$$

Then it is not difficult to verify that $(f_0^s,f_1^s)\in D(A)$. Moreover,

$$\begin{aligned}&\left\|(P_0,P_1)-(f_0^s,f_1^s)\right\|\\ &=\sum_{i=0}^{l}\int_0^{\infty}\left|P_{0,i}(x)-f_{0,i}^s(x)\right|dx\\ &\quad+\sum_{j=1}^{l}\int_0^{\infty}\left|P_{1,j}(x)-f_{1,j}^s(x)\right|dx\\ &=\sum_{i=0}^{l}\int_0^{s}\left|f_{0,i}^s(0)\right|\left(1-\frac{x}{s}\right)^2dx\\ &\quad+\sum_{i=0}^{l}\left|\mu_{0,i}\right|\int_s^{2s}(x-s)^2(x-2s)^2dx\\ &\quad+\sum_{j=1}^{l}\int_0^{s}\left|f_{1,j}^s(0)\right|\left(1-\frac{x}{s}\right)^2dx\\ &\quad+\sum_{j=1}^{l}\left|\mu_{1,j}\right|\int_s^{2s}(x-s)^2(x-2s)^2dx\\ &=\sum_{i=0}^{l}\left|f_{0,i}^s(0)\right|\frac{s}{3}+\sum_{i=0}^{l}\left|\mu_{0,i}\right|\frac{s^5}{30}+\sum_{j=1}^{l}\left|f_{1,j}^s(0)\right|\frac{s}{3}\\ &\quad+\sum_{j=1}^{l}\left|\mu_{1,j}\right|\frac{s^5}{30}\longrightarrow 0\quad\text{as } s\longrightarrow 0,\end{aligned} \tag{38}$$

which shows that $D(A)$ is dense in $\mathbb{Z}$. Hence, $D(A)$ is dense in X. From the first step, the second step, and the Hille-Yosida theorem [18] we know that A generates a C_0-semigroup.

Next we will verify that U and E are bounded linear operators. From the definition of U and E and $\sum_{k=1}^{\infty}c_k=1$ we have

$$\begin{aligned}\left\|U(P_0,P_1)\right\|&\le\sum_{n=0}^{\infty}\int_0^{\infty}[\lambda+v(x)]\left|P_{0,n}(x)\right|dx\\ &\quad+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=0}^{\infty}\int_0^{\infty}\left|P_{0,n}(x)\right|dx\\ &\quad+\sum_{n=1}^{\infty}\int_0^{\infty}[\lambda+b(x)]\left|P_{1,n}(x)\right|dx\\ &\quad+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=1}^{\infty}\int_0^{\infty}\left|P_{1,n}(x)\right|dx\\ &\le(2\lambda+v)\sum_{n=0}^{\infty}\int_0^{\infty}\left|P_{0,n}(x)\right|dx\\ &\quad+(2\lambda+b)\sum_{n=1}^{\infty}\int_0^{\infty}\left|P_{1,n}(x)\right|dx\\ &\le\max\{(2\lambda+v),(2\lambda+b)\}\left\|(P_0,P_1)\right\|,\\ \left\|E(P_0,P_1)\right\|&\le\int_0^{\infty}v(x)\left|P_{0,0}(x)\right|dx\\ &\quad+\int_0^{\infty}b(x)\left|P_{1,1}(x)\right|dx\\ &\le v\left\|P_{0,0}\right\|_{L^1[0,\infty)}+b\left\|P_{1,1}\right\|_{L^1[0,\infty)}\\ &\le\max\{v,b\}\left\|(P_0,P_1)\right\|.\end{aligned} \tag{39}$$

The previous two formulas show that U and E are bounded operators. It is easy to check that U and E are linear operators. Hence from the perturbation theorem of C_0-semigroup [18], we obtain that $A+U+E$ generates a C_0-semigroup $T(t)$.

Lastly, we will prove that $A+U+E$ is a dispersive operator. For $(P_0,P_1)\in D(A)$ we take (ϕ_0,ϕ_1) as

$$\phi_0(x)=\left(\frac{[Q]^+}{Q},\frac{[P_{0,0}(x)]^+}{P_{0,0}(x)},\frac{[P_{0,1}(x)]^+}{P_{0,1}(x)},\ldots\right),$$
$$\phi_1(x)=\left(\frac{[P_{1,1}(x)]^+}{P_{1,1}(x)},\frac{[P_{1,2}(x)]^+}{P_{1,2}(x)},\ldots\right), \tag{40}$$

where

$$[Q]^+ = \begin{cases} Q, & \text{if } Q > 0, \\ 0, & \text{if } Q \le 0, \end{cases}$$

$$[P_{0,n}(x)]^+ = \begin{cases} P_{0,n}(x), & \text{if } P_{0,n}(x) > 0, \\ 0, & \text{if } P_{0,n}(x) \le 0, \end{cases} \quad n \ge 0, \tag{41}$$

$$[P_{1,n}(x)]^+ = \begin{cases} P_{1,n}(x), & \text{if } P_{1,n}(x) > 0, \\ 0, & \text{if } P_{1,n}(x) \le 0, \end{cases} \quad n \ge 1.$$

If we define $V_{0,i} = \{x \in [0,\infty) \mid P_{0,i}(x) > 0\}$ and $W_{0,i} = \{x \in [0,\infty) \mid P_{0,i}(x) \le 0\}$ for $i = 0, 1, 2, \ldots$, then by a short argument we calculate

$$\begin{aligned} &\int_0^\infty \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx \\ &\quad = \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx \\ &\qquad + \int_{W_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx \\ &\quad = \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx = \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} dx \\ &\quad = \int_0^\infty \frac{d[P_{0,i}(x)]^+}{dx} dx = -[P_{0,i}(0)]^+, \quad i \ge 0. \end{aligned} \tag{42}$$

Similar to (42), we get

$$\int_0^\infty \frac{dP_{1,j}(x)}{dx} \frac{[P_{1,j}(x)]^+}{P_{1,j}(x)} dx = -[P_{1,j}(0)]^+, \quad j \ge 1. \tag{43}$$

By using boundary conditions on $(P_0, P_1) \in D(A)$, (42), (43), and $\sum_{k=1}^\infty c_k = 1$ for such (ϕ_0, ϕ_1), we derive

$$\begin{aligned} &\langle (A + U + E)(P_0, P_1), (\phi_0, \phi_1) \rangle \\ &= \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\, dx \right. \\ &\qquad \left. + \int_0^\infty b(x) P_{1,1}(x)\, dx \right\} \frac{[Q]^+}{Q} \\ &\quad + \int_0^\infty \left\{ -\frac{dP_{0,0}(x)}{dx} - [\lambda + v(x)] P_{0,0}(x) \right\} \frac{[P_{0,0}(x)]^+}{P_{0,0}(x)} dx \\ &\quad + \sum_{n=1}^\infty \int_0^\infty \left\{ -\frac{dP_{0,n}(x)}{dx} - [\lambda + v(x)] P_{0,n}(x) \right. \\ &\qquad \left. + \lambda \sum_{k=1}^n c_k P_{0,n-k}(x) \right\} \frac{[P_{0,n}(x)]^+}{P_{0,n}(x)} dx \\ &\quad + \sum_{n=1}^\infty \int_0^\infty \left\{ -\frac{dP_{1,n}(x)}{dx} - [\lambda + b(x)] P_{1,n}(x) \right. \\ &\qquad \left. + \lambda \sum_{k=1}^n c_k P_{1,n-k+1}(x) \right\} \frac{[P_{1,n}(x)]^+}{P_{1,n}(x)} dx \\ &= -\lambda [Q]^+ + \frac{[Q]^+}{Q} \int_0^\infty v(x) P_{0,0}(x)\, dx \\ &\quad + \frac{[Q]^+}{Q} \int_0^\infty b(x) P_{1,1}(x)\, dx \\ &\quad + \sum_{n=0}^\infty [P_{0,n}(0)]^+ - \sum_{n=0}^\infty \int_0^\infty [\lambda + v(x)] [P_{0,n}(x)]^+ dx \\ &\quad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty P_{0,n-k}(x) \frac{[P_{0,n}(x)]^+}{P_{0,n}(x)} dx \\ &\quad + \sum_{n=1}^\infty [P_{1,n}(0)]^+ - \sum_{n=1}^\infty \int_0^\infty [\lambda + b(x)] [P_{1,n}(x)]^+ dx \\ &\quad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty P_{1,n-k+1}(x) \frac{[P_{1,n}(x)]^+}{P_{1,n}(x)} dx \\ &\le -\lambda [Q]^+ + \frac{[Q]^+}{Q} \int_0^\infty v(x) [P_{0,0}(x)]^+ dx \\ &\quad + \frac{[Q]^+}{Q} \int_0^\infty b(x) [P_{1,1}(x)]^+ dx \\ &\quad + \lambda [Q]^+ - \sum_{n=0}^\infty \int_0^\infty [\lambda + v(x)] [P_{0,n}(x)]^+ dx \\ &\quad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty [P_{0,n-k}(x)]^+ dx \\ &\quad + \sum_{n=1}^\infty \int_0^\infty v(x) [P_{0,n}(x)]^+ dx + \sum_{n=2}^\infty \int_0^\infty b(x) [P_{1,n}(x)]^+ dx \\ &\quad - \sum_{n=1}^\infty \int_0^\infty [\lambda + b(x)] [P_{1,n}(x)]^+ dx \\ &\quad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty [P_{1,n-k+1}(x)]^+ dx \\ &\le \left(\frac{[Q]^+}{Q} - 1 \right) \\ &\quad \times \left(\int_0^\infty v(x) [P_{0,0}(x)]^+ dx + \int_0^\infty b(x) [P_{1,1}(x)]^+ dx \right) \\ &\quad - \sum_{n=0}^\infty \int_0^\infty \lambda [P_{0,n}(x)]^+ dx + \lambda \sum_{k=1}^\infty c_k \sum_{n=0}^\infty \int_0^\infty [P_{0,n}(x)]^+ dx \\ &\quad - \sum_{n=1}^\infty \int_0^\infty \lambda [P_{1,n}(x)]^+ dx + \lambda \sum_{k=1}^\infty c_k \sum_{n=1}^\infty \int_0^\infty [P_{1,n}(x)]^+ dx \\ &\le 0. \end{aligned} \tag{44}$$

Equation (44) shows that $A+U+E$ is a dispersive operator. From which together with the first step, the second step, and the Phillips theorem, we know that $A+U+E$ generates a positive contraction C_0-semigroup [18]. By the uniqueness of a C_0-semigroup we conclude that this semigroup is just $T(t)$. □

It is not difficult to see that X^*, the dual space of X, is

$$X^* = \left\{ (q_0^*, q_1^*) \;\middle|\; \begin{aligned} & q_0^* = (q^*, q_{0,0}^*, q_{0,1}^*, \ldots) \\ & \quad \in \mathbb{R} \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ & q_1^* = (q_{1,1}^*, q_{1,2}^*, \ldots) \\ & \quad \in L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ & |||(q_0^*, q_1^*)||| = \sup\left\{ |q^*|, \sup_{n\ge0} \|q_{0,n}^*\|_{L^\infty[0,\infty)}, \right. \\ & \qquad \left. \sup_{n\ge1} \|q_{1,n}^*\|_{L^\infty[0,\infty)} \right\} \end{aligned} \right\}. \quad (45)$$

It is easy to check that X^* is a Banach space. If we take a set S in X as

$$S = \{(P_0, P_1) \in X \mid Q \ge 0, P_{0,n}(x) \ge 0, n \ge 0; \quad P_{1,n}(x) \ge 0, n \ge 1, x \in [0,\infty)\}, \quad (46)$$

then S is a cone in X. For $(P_0, P_1) \in D(A) \cap S$, we take

$$(q_0^*, q_1^*) = \|(P_0, P_1)\| \left(\begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix} \right) \in X^*, \quad (47)$$

then we have

$$\begin{aligned} & \langle (P_0, P_1), (q_0^*, q_1^*) \rangle \\ & = \|(P_0, P_1)\| \left(|Q| + \sum_{n=0}^{\infty} P_{0,n}(x)\,dx + \sum_{n=1}^{\infty} P_{1,n}(x)\,dx \right) \\ & = \|(P_0, P_1)\|^2 = |||(q_0^*, q_1^*)|||^2, \end{aligned} \quad (48)$$

that is

$$\begin{aligned} & (q_0^*, q_1^*) \in \theta((P_0, P_1)) \\ & \quad = \{ (q_0^*, q_1^*) \in X^* \mid \langle (P_0, P_1), (q_0^*, q_1^*) \rangle \\ & \qquad = \|(P_0, P_1)\|^2 = |||(q_0^*, q_1^*)|||^2 \}. \end{aligned} \quad (49)$$

For such (q_0^*, q_1^*), by using boundary conditions on $(P_0, P_1) \in D(A) \cap S$ and $\sum_{k=1}^{\infty} c_k = 1$, we have

$$\begin{aligned} & \langle (A+U+E)(P_0, P_1), (q_0^*, q_1^*) \rangle \\ & = \|(P_0, P_1)\| \\ & \quad \times \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \right\} \\ & \quad + \int_0^\infty \left\{ -\frac{dP_{0,0}(x)}{dx} - [\lambda + v(x)] P_{0,0}(x) \right\} \|(P_0, P_1)\|\,dx \\ & \quad + \sum_{n=1}^{\infty} \int_0^\infty \left\{ -\frac{dP_{0,n}(x)}{dx} - [\lambda + v(x)] P_{0,n}(x) \right. \\ & \qquad \left. + \lambda \sum_{k=1}^{n} c_k P_{0,n-k}(x) \right\} \|(P_0, P_1)\|\,dx \\ & \quad + \sum_{n=1}^{\infty} \int_0^\infty \left\{ -\frac{dP_{1,n}(x)}{dx} - [\lambda + b(x)] P_{1,n}(x) \right. \\ & \qquad \left. + \lambda \sum_{k=1}^{n} c_k P_{1,n-k+1}(x) \right\} \|(P_0, P_1)\|\,dx \\ & = \|(P_0, P_1)\| \\ & \quad \times \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \right\} \\ & \quad - \|(P_0, P_1)\| \\ & \quad \times \left\{ \sum_{n=0}^{\infty} \int_0^\infty -\frac{dP_{0,n}(x)}{dx}\,dx + \sum_{n=0}^{\infty} \int_0^\infty [\lambda + v(x)] P_{0,n}(x)\,dx \right. \\ & \qquad - \lambda \sum_{k=1}^{\infty} c_k \sum_{n=0}^{\infty} \int_0^\infty P_{0,n}(x)\,dx + \sum_{n=1}^{\infty} \int_0^\infty \frac{dP_{1,n}(x)}{dx}\,dx \\ & \qquad + \sum_{n=1}^{\infty} \int_0^\infty [\lambda + b(x)] P_{1,n}(x)\,dx \\ & \qquad \left. - \lambda \sum_{k=1}^{\infty} c_k \sum_{n=1}^{\infty} \int_0^\infty P_{1,n}(x)\,dx \right\} \\ & = \|(P_0, P_1)\| \\ & \quad \times \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \right\} \\ & \quad - \|(P_0, P_1)\| \\ & \quad \times \left\{ -\lambda Q + \sum_{n=0}^{\infty} \int_0^\infty v(x) P_{0,n}(x)\,dx \right. \\ & \qquad - \sum_{n=1}^{\infty} \int_0^\infty v(x) P_{0,n}(x)\,dx \\ & \qquad \left. - \sum_{n=2}^{\infty} \int_0^\infty b(x) P_{1,n}(x)\,dx + \sum_{n=1}^{\infty} \int_0^\infty b(x) P_{1,n}(x)\,dx \right\} \\ & = 0. \end{aligned} \quad (50)$$

Which shows that $A+U+E$ is a conservative operator. So we can use the Fattorini theorem [19] and state it as follows.

Theorem 2. *$T(t)$ is isometric for the initial value of the system* (8)*; that is,*

$$\|T(t)(P_0, P_1)(0)\| = \|(P_0, P_1)(0)\|, \quad \forall t \in [0,\infty). \quad (51)$$

Proof. Since $A+U+E$ is conservative with respect to θ and $(P_0,P_1)(0)\in D(A^2)\cap S$, from the Taylor expansion of $T(t+h)$ for $t,h\geq 0$ we have

$$\begin{aligned} T(t+h)(P_0,P_1)(0) &= T(t)(P_0,P_1)(0)+h(A+U+E)T(t)(P_0,P_1)(0) \\ &\quad+\int_t^{t+h}(t+h-s)T(s)(A+U+E)^2(P_0,P_1)(0)\,ds \\ &= T(t)(P_0,P_1)(0)+h(A+U+E)T(t)(P_0,P_1)(0) \\ &\quad+h\rho(t,h), \end{aligned} \tag{52}$$

where $\|\rho(t,h)\|\to 0$ as $h\to 0$, uniformly in $t\geq 0$. Then

$$\begin{aligned} &\|T(t+h)(P_0,P_1)(0)\|\,\|T(t)(P_0,P_1)(0)\| \\ &\quad\geq\left|\left\langle T(t+h)(P_0,P_1)(0),(T(t)(P_0,P_1)(0))^*\right\rangle\right| \\ &\quad\geq\operatorname{Re}\left\langle T(t+h)(P_0,P_1)(0),(T(t)(P_0,P_1)(0))^*\right\rangle \\ &\quad=\|T(t)(P_0,P_1)(0)\|^2 \\ &\qquad+h\operatorname{Re}\left\langle\rho(t,h),(T(t)(P_0,P_1)(0))^*\right\rangle. \end{aligned} \tag{53}$$

In view of (52) and the fact that $A+U+E$ is conservative with respect to θ, consider the set

$$\Omega=\left\{t\in[0,\infty)\mid\|T(t)(P_0,P_1)(0)\|=\|(P_0,P_1)(0)\|\right\}. \tag{54}$$

Since $0\in\Omega$, Ω is nonempty, moreover Ω is obviously a closed interval because of continuity of $T(t)$. If $\Omega\neq[0,\infty)$, then let t_0 be the right end point of Ω and $\eta>0$ is so small that $\|T(t)(P_0,P_1)(0)\|$ is bounded away from zero in $t_0\leq t\leq t_0+\eta$. For any such t we divide (53) by $\|T(t)(P_0,P_1)(0)\|$ and get

$$\begin{aligned} &\|T(t+h)(P_0,P_1)(0)\| \\ &\quad\geq\|T(t)(P_0,P_1)(0)\|+\frac{h}{\|T(t)(P_0,P_1)(0)\|} \\ &\qquad\times\operatorname{Re}\left\langle\rho(t,h),(T(t)(P_0,P_1)(0))^*\right\rangle \\ &\quad\geq\|T(t)(P_0,P_1)(0)\|-h\beta(t,h), \end{aligned} \tag{55}$$

where $\beta(t,h)$ is positive and $\beta(t,h)\to 0$ when $h\to 0$, uniformly in $t_0\leq t\leq t_0+\eta$.

Let now ϵ be are small positive number and $\delta>0$ such that $\|\beta(t,h)\|\leq\epsilon$ for $0\leq h\leq\delta$ and $t_0\leq t\leq t_0+\eta$. Let $t_0<t_1<t_2<\cdots<t_m=t_0+\eta$ be partition of the interval $t_0\leq t\leq t_0+\eta$ such that $t_j-t_{j-1}\leq\delta$ $(1\leq j\leq m)$. Then by (55), one has

$$\begin{aligned} 0&\leq\|T(t_0)(P_0,P_1)(0)\|-\|T(t_0+\eta)(P_0,P_1)(0)\| \\ &\leq\sum_{j=1}^m\left(\left\|T(t_{j-1})(P_0,P_1)(0)\right\|-\left\|T(t_j)(P_0,P_1)(0)\right\|\right) \\ &\leq\sum_{j=1}^m(t_j-t_{j-1})\beta(t_j,t_j-t_{j-1})\leq\eta\epsilon. \end{aligned} \tag{56}$$

Since ϵ is arbitrary, it follows that $\|T(t_0+\eta)(P_0,P_1)(0)\|=\|T(t_0)(P_0,P_1)(0)\|$, which contradicts the fact that t_0 is the right endpoint of Ω. Hence $\Omega=[0,\infty)$. That is, $\|T(t)(P_0,P_1)(0)\|=\|(P_0,P_1)(0)\|$ for $t\in[0,\infty)$. The proof of the theorem is complete. □

From Theorems 1 and 2 we obtain the main result in this paper.

Theorem 3. *If $v=\sup_{x\in[0,\infty)}v(x)<\infty$, $b=\sup_{x\in[0,\infty)}b(x)<\infty$, then the system (8) has a unique nonnegative time-dependent solution $(P_0,P_1)(x,t)$, which satisfies*

$$\|(P_0,P_1)(\cdot,t)\|=1,\quad\forall t\in[0,\infty). \tag{57}$$

Proof. Since $(P_0,P_1)(0)\in D(A^2)\cap S$, by Theorem 1 and Theorem 11 in Gupur et al. [18], we know that the system (8) has a unique nonnegative time-dependent solution $(P_0,P_1)(x,t)$ which can be expressed as

$$(P_0,P_1)(x,t)=T(t)(P_0,P_1)(0),\quad\forall t\in[0,\infty). \tag{58}$$

From which together with Theorem 2 (i.e., (51)) we have

$$\begin{aligned} \|(P_0,P_1)(\cdot,t)\|=\|T(t)(P_0,P_1)(0)\|=\|(P_0,P_1)(0)\|=1, \\ \forall t\in[0,\infty), \end{aligned} \tag{59}$$

this just reflects the physical background of the problem. □

4. Concluding Remarks

If we know the spectrum of $A+U+E$ on the imaginary axis, then by Theorem 1 and Theorem 14 in Gupur et al. [18], we obtain the asymptotic behavior of the time-dependent solution of the system (8), which describes Hypothesis 2. It is our next research work.

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References

[1] L. W. Miller, *Alternating priorities in multi-class queue [Ph.D. thesis]*, Cornel University, Ithaca, New York, USA, 1964.

[2] Y. Levy and U. Yechiali, "Utilization of idle time in an M/G/1 queueing system," *Management Science*, vol. 22, no. 2, pp. 202–211, 1975.

[3] B. T. Doshi, "Queueing systems with vacations—a survey," *Queueing Systems*, vol. 1, no. 1, pp. 29–66, 1986.

[4] B. Doshi, "Generalizations of the stochastic decomposition results for single server queues with vacations," *Stochastic Models*, vol. 6, no. 2, pp. 307–333, 1990.

[5] H. Takagi, *Vacation and Priority Systems, Part 1*, vol. 1 of *Queueing Analysis : A Foundation of Performance Evaluation*, North-Holland, Amsterdam, The Netherland, 1991.

[6] Y. Baba, "On the $M^X/G/1$ queue with vacation time," *Operations Research Letters*, vol. 5, no. 2, pp. 93–98, 1986.

[7] H.-S. Lee and M. M. Srinivasan, "Control policies for the $M^X/G/1$ queueing system," *Management Science*, vol. 35, no. 6, pp. 708–721, 1989.

[8] H. W. Lee, S. S. Lee, J. O. Park, and K. C. Chae, "Analysis of the $M^X/G/1$ queue with N-policy and multiple vacations," *Journal of Applied Probability*, vol. 31, no. 2, pp. 476–496, 1994.

[9] S. S. Lee, H. W. Lee, S. H. Yoon, and K. C. Chae, "Batch arrival queue with N-policy and single vacation," *Computers and Operations Research*, vol. 22, no. 2, pp. 173–189, 1995.

[10] A. Borthakur and G. Choudhury, "On a batch arrival Poisson queue with generalized vacation," *Sankhyā B*, vol. 59, no. 3, pp. 369–383, 1997.

[11] G. Choudhury, "On a batch arrival Poisson queue with a random setup time and vacation period," *Computers & Operations Research*, vol. 25, no. 12, pp. 1013–1026, 1998.

[12] G. Choudhury, "An $M^X/G/1$ queueing system with a setup period and a vacation period," *Queueing Systems*, vol. 36, no. 1–3, pp. 23–38, 2000.

[13] K. C. Chae and H. W. Lee, "$M^X/G/1$ vacation models with N-policy: heuristic interpretation of the mean waiting time," *Journal of the Operational Research Society*, vol. 46, no. 2, pp. 258–264, 1995.

[14] J. Medhi, "Single server queueing system with Poisson input: a review of some recent developments," in *Advances in Combinatorical Method and Applications in Probability and Statistics*, N. Balakrishnan, Ed., pp. 317–338, Birkhäuser, Boston, Mass, USA, 1997.

[15] G. Choudhury, "Analysis of the $M^X/G/1$ queueing system with vacation times," *Sankhyā B*, vol. 64, no. 1, pp. 37–49, 2002.

[16] D. R. Cox, "The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables," vol. 51, pp. 433–441, 1955.

[17] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, NY, USA, 1975.

[18] G. Gupur, X. Z. Li, and G. T. Zhu, *Functional Analysis Method in Queueing Theory*, Research Information, Herdfortshire, UK, 2001.

[19] H. O. Fattorini, *The Cauchy Problem*, vol. 18 of *Encyclopedia of Mathematics and its Applications*, Addison-Wesley, Reading, Mass, USA, 1983.