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Research Article

Deriving Weights of Criteria from Inconsistent Fuzzy Comparison Matrices by Using the Nearest Weighted Interval Approximation

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Deriving the weights of criteria from the pairwise comparison matrix with fuzzy elements is investigated. In the proposed method we first convert each element of the fuzzy comparison matrix into the nearest weighted interval approximation one. Then by using the goal programming method we derive the weights of criteria. The presented method is able to find weights of fuzzy pairwise comparison matrices in any form. We compare the results of the presented method with some of the existing methods. The approach is illustrated by some numerical examples.

1. Introduction

Weight estimation technique in multiple criteria decision making (MADM) problem has been extensively applied in many areas such as selection, evaluation, planning and development, decision making, and forecasting [1]. The conventional MADM requires exact judgments.

In the process of multiple criteria decision making, a decision maker sometimes uses a fuzzy preference relation to express his/her uncertain preference information due to the complexity and uncertainty of real-life decision making problem and the time pressure, lack of knowledge, and the decision maker's limited expertise about problem domain. The priority weights derived from a fuzzy preference relation can also be used as the weights of criteria or used to rank the given alternatives.

Xu and Da [2] utilized the fuzzy preference relation to rank a collection of interval numbers. Fan et al. [3] studied the multiple attribute decision-making problem in which the decision maker provides his/her preference information over alternatives with fuzzy preference relation. They first established an optimization model to derive the attribute weights and then to select the most desirable alternative(s). Xu and Da [4] developed an approach to improving consistency of fuzzy preference relation and gave a practical iterative algorithm to derive a modified fuzzy preference relation with acceptable consistency.

Xu and Da [5] proposed a least deviation method to obtain a priority vector from a fuzzy preference relation.

Determining criteria weights is a central problem in MCDM. Weights are used to express the relative importance of criteria in MCDM. When the decision maker is unable to rank the alternatives holistically and directly with respect to a criterion, pairwise comparisons are often used as intermediate decision support. In the other words, in evaluating n competing alternatives $A_1, \ldots, A_n$ under a given criterion, it is natural to use the framework of pairwise comparisons represented by a $n \times n$ square matrix from which a set of preference values for the alternatives is derived.

Because of ease of understanding and application, pairwise comparisons play an important role in assessing the priority weights of decision criteria. Geoffrion's gradient search method [6], Haimes' surrogate worth tradeoff method [7], Zionts-Wallenius' method [8], Saaty's analytic hierarchy process [9], Cogger and Yu's eigenvector method [10], Takeda et al.'s GEM [11], and the logarithmic least square method are just some methods which are primarily based on pairwise comparisons.

The classical pairwise comparison matrix requires the decision maker (DM) to express his/her preferences in the form of a precise ratio matrix encoding a valued preference relation. However it can often be difficult for the DM to express exact estimates of the ratios of importance and therefore express his/her estimates as fuzzy numbers. The theory of fuzzy numbers is based on the theory of fuzzy sets which Zadeh [12] introduced in 1965. First, Bellman and Zadeh [13] incorporate the concept of fuzzy numbers into decision analysis.

The methodology presented in this paper is useful in assisting decision makers to determine criteria fuzzy weights from criteria, and it is helpful in alternative selection when these fuzzy weights are used with one of the techniques of MCDM. To derive the weights of criteria from this fuzzy pairwise comparison matrix is an important problem. Islam et al. [14] and Wang [15] developed a lexicographic goal programming to generate weights from inconsistent pairwise interval comparison matrices. Wang and Chin [16] proposed an eigenvector method (EM) to generate interval or fuzzy weight estimate from an interval or fuzzy comparison matrix. Also in Xu and Chen [17], the concepts of additive and multiplicative consistent interval fuzzy preference relations were defined, and some simple and practical linear programming models for deriving the priority weights of various interval fuzzy preference relations established. Wang and Chin [18] proposed a sound yet simple priority method for fuzzy AHP which utilized a linear goal programming model to derive normalized fuzzy weights for fuzzy pairwise comparison matrices. Taha and Rostam [19] proposed a decision support system for machine tool selection in flexible manufacturing cell using fuzzy analytic hierarchy process (fuzzy AHP) and artificial neural network. A program is developed in that model to find the priority weights of the evaluation criteria and alternative's ranking called PECAR for fuzzy AHP model. Ayağ and Özdemir [20] proposed a fuzzy ANP-based approach to evaluate a set of conceptual design alternatives developed in an NPD environment in order to reach to the best one satisfying both the needs and expectations of customers and the engineering specifications of company.

Many methods for estimating the preference values from the pairwise comparison matrix have been proposed and their effectiveness comparatively evaluated. Some of the proposed estimating methods presume interval-scaled preference values (Barzilai, [21] and Salo and Haimalainen,[22]).

In this paper we first introduce the nearest weighting interval approximation of a fuzzy number (see Saeidifar [23]), and then by using some weighting functions we convert each fuzzy element of the pairwise comparison matrix to its nearest weighting interval approximation. Then we apply the goal programming method to derive weights of criteria. Goal programming was originally proposed by Charnes and Cooper [24] and is an important technique for DMs to consider simultaneously several objectives in finding a set of acceptable solution.

The structure of the rest of this paper is as follows. Section 2 provides some required preliminaries. Section 3 of the paper gives a goal programming approach for deriving weights of criteria. Some examples are presented in Section 4. The paper ends with conclusion.

2. Preliminaries

In this section we review some basic definitions about fuzzy numbers, fuzzy pairwise comparison matrix, and goal programming method.

2.1. Fuzzy Numbers

Fuzzy numbers are one way to describe the vagueness and lack of precision of data. The theory of fuzzy numbers is based on the theory of fuzzy sets which Zadeh [12] introduced in 1965.

Definition 2.1. A fuzzy number is a fuzzy set like $A : R \to I = [0,1]$ which satisfies

(i) A is continuous,

(ii) $A(x) = 0$ outside some interval $[a,d]$,

(iii) there are real numbers b, c such that $a \leq b \leq c \leq d$,

and

(1) $A(x)$ is increasing on $[a,b]$,

(2) $A(x)$ is decreasing on $[c,d]$,

(3) $A(x) = 1, b \leq x \leq c$.

We denote the set of all fuzzy numbers by $F(R)$.

Definition 2.2. A triangular fuzzy number is denoted as $\widetilde{A} = (a,b,c)$; see Figure 1.

The membership function of a triangular fuzzy number is expressed as formula (2.1):

$$A(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x \leq b, \\ \dfrac{c-x}{c-b}, & b \leq x \leq c, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

2.1.1. Comparison between Two Fuzzy Numbers

In this subsection, in order to compare two fuzzy numbers, we use the concept of ranking function.

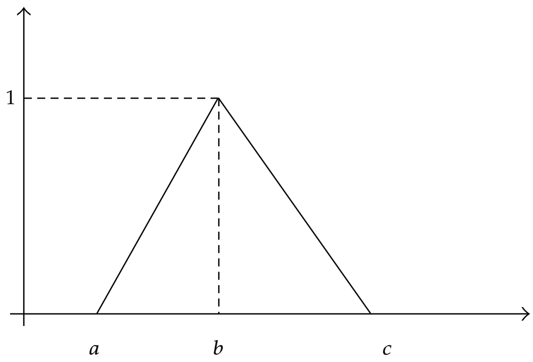

Figure 1: Triangular fuzzy number.

A ranking function is a function $g : F(R) \rightarrow R$, which maps each fuzzy number into the real line, where a natural order exists. Asady and Zendehnam [25] proposed a defuzzification using minimizer of the distance between two the fuzzy number.

If $\tilde{A} = (a, b, c)$ be a triangular fuzzy number, then they introduced distance minimization of a fuzzy number $\tilde{A}$ that denoted by $M(\tilde{A})$ which was defined as follows:

$$M\left(\tilde{A}\right) = \frac{1}{4}\{a + 2b + c\}. \tag{2.2}$$

This ranking function has the following properties.

Property 1. If $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers, then,

$$\begin{aligned} &\text{(i)}\ M\left(\tilde{A}\right) > M\left(\tilde{B}\right) \quad \text{iff } \tilde{A} \succ \tilde{B}, \\ &\text{(ii)}\ M\left(\tilde{A}\right) < M\left(\tilde{B}\right) \quad \text{iff } \tilde{A} \prec \tilde{B}, \\ &\text{(iii)}\ M\left(\tilde{A}\right) = M\left(\tilde{B}\right) \quad \text{iff } \tilde{A} \approx \tilde{B}. \end{aligned} \tag{2.3}$$

Property 2. If $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers, then,

$$M\left(\tilde{A} \oplus \tilde{B}\right) = M\left(\tilde{A}\right) + M\left(\tilde{B}\right). \tag{2.4}$$

2.2. Fuzzy Pairwise Comparison Matrix

Suppose the decision maker provides fuzzy judgments instead of precise judgments for a pairwise comparison. Without loss of generality we assume that we deal with pairwise comparison matrix with triangular fuzzy numbers being the elements of the matrix. We consider a pairwise comparison matrix where all its elements are triangular fuzzy numbers as follows:

$$\tilde{A} = \begin{bmatrix} (a_{11}^L, a_{11}^M, a_{11}^U) & \cdots & (a_{1n}^L, a_{1n}^M, a_{1n}^U) \\ \vdots & \ddots & \vdots \\ (a_{n1}^L, a_{n1}^M, a_{nn}^U) & \cdots & (a_{nn}^L, a_{nn}^M, a_{nn}^U) \end{bmatrix}, \tag{2.5}$$

where $\tilde{a}_{ij} = (a_{ij}^L, a_{ij}^M, a_{ij}^U)$ is a triangular fuzzy number; see Chen et al. [26]. We say that $\tilde{A}$ is reciprocal, if the following condition is satisfied (Ramík and Korviny, [27]):

$$\tilde{a}_{ij} = \left(a_{ij}^L, a_{ij}^M, a_{ij}^U\right) \quad \text{implies } \tilde{a}_{ji} = \left(\frac{1}{a_{ij}^U}, \frac{1}{a_{ij}^M}, \frac{1}{a_{ij}^L}\right) \quad \forall\, i, j = 1, \ldots, n. \tag{2.6}$$

2.3. Goal Programming

Consider the following problem:

$$\begin{aligned} \max \quad & \{f_1(x), \ldots, f_k(x)\} \\ \text{s.t.} \quad & x \in \mathrm{X}, \end{aligned} \tag{2.7}$$

where $f_1, \ldots, f_k$ are objective functions and X is nonempty feasible region. Model (2.7) is called multiple objective programming. Goal programming is now an important area of multiple criteria optimization. The idea of goal programming is to establish a goal level of achievement for each criterion. Goal programming method requires the decision maker to set goals for each objective that he/she wishes to obtain. A preferred solution is then defined as the one which minimizes the deviations from the set goals. Then GP can be formulated as the following achievement function:

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} \left(d_i^+ + d_i^-\right) \\ \text{s.t.} \quad & f_i(x) + d_i^+ - d_i^- = b_i, \quad i = 1, \ldots, k, \\ & x \in \mathrm{X}, \\ & d_i^+ d_i^- = 0, \quad i = 1, \ldots, k, \\ & d_i^+, d_i^- \geq 0, \quad i = 1, \ldots, k. \end{aligned} \tag{2.8}$$

The DMs for their goals set some acceptable aspiration levels, b_i $(i = 1, \ldots, k)$, for these goals, and try to achieve a set of goals as closely as possible. The purpose of GP is to minimize the deviations between the achievement of goals, $f_i(x)$, and these acceptable aspiration levels, b_i $(i = 1, \ldots, k)$. Also, d_i^+ and d_i^- are, respectively, over- and underachievement of the ith goal.

2.4. The Nearest Interval Approximation

In this section, we introduce an interval operator of a fuzzy number, which is called the nearest weighted possibilistic interval approximation. First we introduce an f-weighted distance quantity on the fuzzy numbers, and then we obtain the interval approximations for a fuzzy number.

Definition 2.3. A weighting function is a function as $f = (\underline{f}, \overline{f}) : ([0,1],[0,1]) \to (R,R)$ such that the functions $\underline{f}, \overline{f}$ are nonnegative, monotone increasing and satisfies the following normalization condition: $\int_0^1 \underline{f}(\alpha)d\alpha = \int_0^1 \overline{f}(\alpha)d\alpha = 1$.

Definition 2.4 (see [23]). Let $\widetilde{A}$ be a fuzzy number with $A_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and $f(\alpha) = (\underline{f}(\alpha), \overline{f}(\alpha))$ being a weighted function. Then the interval

$$\mathrm{NWIA}_f(A) = \left[\int_0^1 \underline{f}(\alpha)\underline{a}(\alpha)d\alpha, \int_0^1 \overline{f}(\alpha)\overline{a}(\alpha)d\alpha\right] \tag{2.9}$$

is the nearest weighted interval approximation to fuzzy number $\widetilde{A}$.

Remark 2.5. The function $f(\alpha)$ can be understood as the weight of our interval approximation; the property of monotone increasing of function $f(\alpha)$ means that the higher the cut level is, the more important its weight is in determining the interval approximation of fuzzy numbers. In applications, the function $f(\alpha)$ can be chosen according to the actual situation.

Corollary 2.6. *Let $\widetilde{A} = (a,b,c)$ be a triangular fuzzy number and let $f(\alpha) = (n\alpha^{n-1}, n\alpha^{n-1})$ be a weighting function. Then*

$$\mathrm{NWIA}_f(A) = \left[\frac{a+nb}{n+1}, \frac{nb+c}{n+1}\right]. \tag{2.10}$$

Example 2.7. Let $\widetilde{A} = (3,4,7)$ be a triangular fuzzy number and also let $f_1(\alpha) = (2\alpha, 2\alpha)$ and $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$ be two weighting functions. Then the nearest weighted intervals to $\widetilde{A}$ is as follows (see Figure 2):

$$\mathrm{NWIA}_{f_1}(A) = \left[\frac{11}{3}, 5\right], \qquad \mathrm{NWIA}_{f_2}(A) = \left[\frac{19}{5}, \frac{23}{5}\right]. \tag{2.11}$$

Example 2.8. Let $\widetilde{A} = (3,7,8,13)$ be a trapezoidal fuzzy number and also let $f_1(\alpha) = (2\alpha, 2\alpha)$ and $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$ be two weighting functions. Then the nearest weighted interval to $\widetilde{A}$ is as follows (see Figure 3):

$$\begin{aligned} \mathrm{NWIA}_{f_1}(A) &= \left[\frac{17}{3}, \frac{29}{3}\right], \\ \mathrm{NWIA}_{f_2}(A) &= \left[\frac{31}{5}, 9\right]. \end{aligned} \tag{2.12}$$

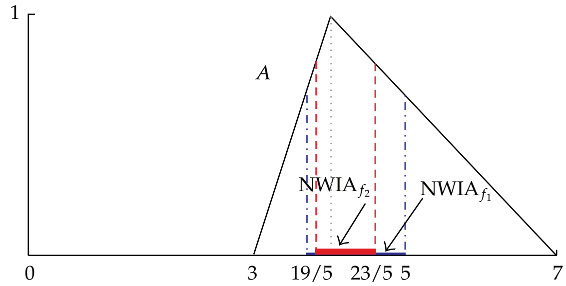

Figure 2: Triangular fuzzy number and its interval approximation.

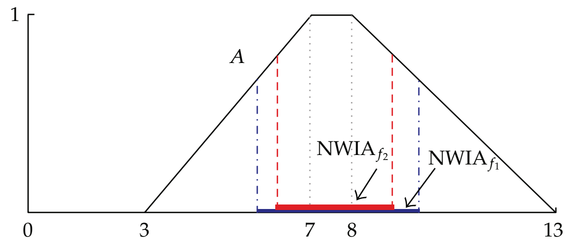

Figure 3: Trapezoidal fuzzy number and its interval approximation.

3. Deriving the Weights of Criteria

In the conventional case, if a pairwise comparison matrix A be reciprocal and consistent then the weights of each criterion are simply calculated as $w_i = a_{ij}/\sum_{k=1}^{n} a_{kj},\ i = 1,\dots,n$. In the case of inconsistent matrix, we must obtain the importance weights $w_i,\ i = 1,\dots,n$ such that $a_{ij} = w_i/w_j$ or equivalently $a_{ij}w_j - w_i = 0$. Therefore in the case of uncertainty, for deriving the weights of criteria from inconsistent fuzzy comparison matrix we follow the following procedure.

Step 1. First by formula (2.10) we convert each fuzzy element $\tilde{a}_{ij} = (a_{ij}^L, a_{ij}^M, a_{ij}^U)$ of the pairwise comparison matrix to the nearest weighted interval approximation $\overline{a}_{ij} = [\overline{a}_{ij}^L, \overline{a}_{ij}^U]$. Hence the fuzzy pairwise comparison matrix $\widetilde{A}$ is converted to an interval pairwise comparison matrix $\overline{A}$.

Step 2. Now we must calculate the weight vector $w_i,\ i = 1,\dots,n$ such that $\overline{a}_{ij}^L \le w_i/w_j \le \overline{a}_{ij}^U$; therefore we must have $\overline{a}_{ij}^L w_j \le w_i \le \overline{a}_{ij}^U w_j$. Hence we introduce deviation variables p_{ij}^-, p_{ij}^+ and q_{ij}^-, q_{ij}^+ which lead to

$$\begin{aligned}
&\overline{a}_{ij}^L w_j - w_i + p_{ij}^- - p_{ij}^+ = 0,\\
&w_i - \overline{a}_{ij}^U w_j + q_{ij}^- - q_{ij}^+ = 0,
\end{aligned} \tag{3.1}$$

where deviation variables p_{ij}^-, p_{ij}^+ and q_{ij}^-, q_{ij}^+ are nonnegative real numbers but cannot be positive at the same time, that is, $p_{ij}^- p_{ij}^+ = 0$ and $q_{ij}^- q_{ij}^+ = 0$. Now we apply the goal programming method. It is desirable that the deviation variables p_{ij}^{+} and q_{ij}^{+} are kept to be as small as possible, which leads to the following goal programming model:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_{ij}^{+}+q_{ij}^{+}\right) \\
\text{s.t.} \quad & \overline{a}_{ij}^{L} w_j - w_i + p_{ij}^{-} - p_{ij}^{+} = 0, \quad i, j = 1,\ldots,n, \\
& w_i - \overline{a}_{ij}^{U} w_j + q_{ij}^{-} - q_{ij}^{+} = 0, \quad i, j = 1,\ldots,n, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i, p_{ij}^{-}, p_{ij}^{+}, q_{ij}^{-}, q_{ij}^{+} \geq 0.
\end{aligned} \tag{3.2}$$

By solving model (3.2) the optimal weight vector $W = (w_1,\ldots,w_n)$ which shows the importance of each criterion will be obtained. We can use these weights in the process of solving a multiple criteria decision-making problem. Also, these weights show which criterion is more important than others.

Theorem 3.1. *The model* (3.2) *is always feasible.*

Proof. Consider $\widehat{W} = (\widehat{w}_1,\ldots,\widehat{w}_n)$ which has the condition $\sum_{i=1}^{n}\widehat{w}_i = 1,\ \widehat{w}_i \geq 0,\ i = 1,\ldots,n$. Then we define

$$\begin{aligned}
\widehat{p}_{ij}^{-} &= \max\left\{-\left(\overline{a}_{ij}^{L} w_j - w_i\right), 0\right\}, \qquad \widehat{q}_{ij}^{-} = \max\left\{-\left(w_i - \overline{a}_{ij}^{U} w_j\right), 0\right\}, \\
\widehat{p}_{ij}^{+} &= \max\left\{\left(\overline{a}_{ij}^{L} w_j - w_i\right), 0\right\}, \qquad \widehat{q}_{ij}^{+} = \max\left\{\left(w_i - \overline{a}_{ij}^{U} w_j\right), 0\right\},
\end{aligned} \tag{3.3}$$

It is clear that $(\widehat{W}, \widehat{p}_{ij}^{-}, \widehat{p}_{ij}^{+}, \widehat{q}_{ij}^{-}, \widehat{q}_{ij}^{+})$ is a feasible solution for model (3.2). □

Remark 3.2. For ranking of these criteria, we assign rank 1 to the criterion with the maximal value of w_i, and so forth, in a decreasing order of w_i.

Remark 3.3. The proposed method is able to derive the weights of criteria when the elements of the pairwise comparison matrix are fuzzy in any form (see Example 3 in Section 4.3).

Special Case: The Case of Matrix with Crisp Elements

In the case of matrix with crisp data, in order to derive the weights of criteria from the inconsistent pairwise comparison matrix, the goal programming model (3.2) can be converted to the following model:

$$
\begin{aligned}
d^* = \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}(p_{ij}+q_{ij}) \\
\text{s.t.} \quad & a_{ij}w_j - w_i + p_{ij} - q_{ij} = 0, \quad i,j=1,\ldots,n \\
& \sum_{j=1}^{n} w_j = 1, \\
& w_j, p_{ij}, q_{ij} \geq 0, \quad i,j=1,\ldots,n,
\end{aligned}
\tag{3.4}
$$

where p_{ij} and q_{ij} are deviation variables. By solving model (3.4) the optimal weight vector w_j, $j = 1,\ldots,n$, which shows the importance of each criterion will be obtained.

Theorem 3.4. *In the case of crisp data, the pairwise comparison matrix A is consistent if and only if $d^* = 0$.*

Proof. Let us first prove that if $d^* = 0$, then matrix A is consistent.

Since $d^* = 0$, we have $p_{ij} = q_{ij} = 0$. Therefore $a_{ij}w_j - w_i = 0$ and hence $a_{ij} = w_i/w_j$. This gives $a_{ij}a_{jk} = a_{ik}$, and we conclude that matrix A is consistent.

Conversely, suppose that matrix A is consistent. That is

$$a_{ij}a_{jk} = a_{ik}, \quad i,j,k = 1,\ldots,n. \tag{3.5}$$

Now, if we define

$$
\begin{aligned}
\overline{w}_j &= \frac{a_{jk}}{\sum_{t=1}^{n} a_{tk}}, \quad j = 1,\ldots,n, \\
\overline{p}_{ij} &= \overline{q}_{ij} = 0,
\end{aligned}
\tag{3.6}
$$

then it is easy to check that $(\overline{W}, \overline{p}_{ij}, \overline{q}_{ij})$ is feasible for model (3.4). Since model (3.4) has minimization form, we conclude that $d^* = 0$. □

Theorem 3.5. *Model* (3.4) *is always feasible.*

Proof. By Theorem 3.1, proof is evident. □

4. Illustrating Example

In this section we present an illustrating example showing that the proposed approach is a convenient tool not only for calculating the weights of criteria of a pairwise comparison

Table 1: The result of proposed method for Example 2 (see Section 4.1).

Criteria	The obtained weights	Rank of criteria
1	$w_1 = 0.16667$	2
2	$w_2 = 0.16667$	2
3	$w_3 = 0.66666$	1

matrices with fuzzy elements but also for calculating the weights of criteria of crisp pairwise comparison matrices.

4.1. Example 1: Matrix with Crisp Elements

Consider 3×3 reciprocal matrix A with crisp elements:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ 2 & 1 & \frac{1}{4} \\ 4 & 4 & 1 \end{bmatrix}. \tag{4.1}$$

We can easily check that the pairwise comparison matrix A is reciprocal but it is inconsistent. Now, for deriving the weights of criteria we apply a goal programming model (3.4) to matrix A. Therefore we must solve the following goal programming model:

$$\begin{aligned} d^* = \min \quad & p_{12} + q_{12} + p_{13} + q_{13} + p_{21} + q_{21} \\ & + p_{23} + q_{23} + p_{31} + q_{31} + p_{32} + q_{32} \\ \text{s.t.} \quad & 0.50w_2 - w_1 + p_{12} - q_{12} = 0, \\ & 0.25w_3 - w_1 + p_{13} - q_{13} = 0, \\ & 2.00w_1 - w_2 + p_{21} - q_{21} = 0, \\ & 0.25w_3 - w_2 + p_{23} - q_{23} = 0, \\ & 4.00w_1 - w_3 + p_{31} - q_{31} = 0, \\ & 4.00w_2 - w_3 + p_{32} - q_{32} = 0, \\ & w_1 + w_2 + w_3 = 1, \\ & w_i, p_{ij}, q_{ij} \geq 0, \quad 1 \leq i, j \leq 3. \end{aligned} \tag{4.2}$$

By solving model (4.2), we obtain the optimal vector $W = (w_1, w_2, w_3)$. We assign rank 1 to the criteria with the maximal value of w_j, and so forth, in a decreasing order of w_j. The result is shown in Table 1. The optimal objective of model (4.2) is $d^* = 0.249$, which shows that the pairwise comparison matrix A is inconsistent by Theorem 3.4.

In this example the rank order of these criteria is as

$$w_3 > w_1 \sim w_2. \tag{4.3}$$

The results of ranking these criteria are shown in last column of Table 1.

4.2. Example 2: Matrix with Fuzzy Elements in Triangular Form

Consider 3×3 reciprocal matrix $\widetilde{A}$ with triangular fuzzy elements:

$$\widetilde{A} = \begin{bmatrix} (1,1,1) & (2,3,4) & (4,5,6) \\ \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right) & (1,1,1) & (3,4,5) \\ \left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) & \left(\frac{1}{5}, \frac{1}{4}, \frac{1}{3}\right) & (1,1,1) \end{bmatrix}. \tag{4.4}$$

Now we convert the above fuzzy matrix to the equivalent interval approximation pairwise comparison matrix. We consider two cases.

Case 1. We use the weighting function $f_1(\alpha) = (2\alpha, 2\alpha)$.

Then by using (2.10) the interval approximation pairwise comparison matrix is obtained as follows:

$$\overline{A} = \begin{bmatrix} [1.000, 1.000] & [2.667, 3.333] & [4.667, 5.333] \\ [0.303, 0.387] & [1.000, 1.000] & [3.667, 4.333] \\ [0.192, 0.217] & [0.233, 0.278] & [1.000, 1.000] \end{bmatrix}. \tag{4.5}$$

We construct the goal programming model for the above interval approximation pairwise comparison matrix as model (4.6):

$$\begin{aligned} \text{Min} \quad & \left(p_{12}^{+} + q_{12}^{+} + p_{13}^{+} + q_{13}^{+} + p_{21}^{+} + q_{21}^{+} + p_{23}^{+} + q_{23}^{+} + p_{31}^{+} + q_{31}^{+} + p_{32}^{+} + q_{32}^{+}\right) \\ \text{s.t.} \quad & 2.667w_2 - w_1 + p_{12}^{-} - p_{12}^{+} = 0, \\ & w_1 - 3.333w_2 + q_{12}^{-} - q_{12}^{+} = 0, \\ & 4.667w_3 - w_1 + p_{13}^{-} - p_{13}^{+} = 0, \\ & w_1 - 5.333w_2 + q_{13}^{-} - q_{13}^{+} = 0, \\ & 0.303w_1 - w_2 + p_{21}^{-} - p_{21}^{+} = 0, \\ & w_2 - 0.387w_1 + q_{21}^{-} - q_{21}^{+} = 0, \\ & 3.667w_3 - w_2 + p_{23}^{-} - p_{23}^{+} = 0, \\ & w_2 - 4.333w_3 + q_{23}^{-} - q_{23}^{+} = 0, \\ & 0.192w_1 - w_3 + p_{31}^{-} - p_{31}^{+} = 0, \end{aligned}$$

$$\begin{aligned}
&w_3 - 0.217w_1 + q_{31}^- - q_{31}^+ = 0,\\
&0.233w_2 - w_3 + p_{32}^- - p_{32}^+ = 0,\\
&w_3 - 0.278w_2 + q_{32}^- - q_{32}^+ = 0,\\
&w_1 + w_2 + w_3 = 1,\\
&w_i, p_{ij}^-, p_{ij}^+, q_{ij}^-, q_{ij}^+ \geq 0.
\end{aligned} \tag{4.6}$$

By solving the goal programming model (4.6), we obtain the weight vector $W = (0.64, 0.24, 0.12)$. We can use these weights in the process of solving a multiple criteria decision-making problem. Also, these weights show that criterion 1 is important than others (see Table 2).

Case 2. We use the weighting function $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$.

Then the interval approximation pairwise comparison matrix is obtained as follows:

$$\overline{A} = \begin{bmatrix} [1.000, 1.000] & [2.800, 3.200] & [4.800, 5.200] \\ [0.316, 0.366] & [1.000, 1.000] & [3.800, 4.200] \\ [0.193, 0.210] & [0.240, 0.267] & [1.000, 1.000] \end{bmatrix}. \tag{4.7}$$

Similar to model (4.6), by constructing the corresponding goal programming model and solving it, we obtain the weight vector as shown in Table 3.

It can be seen that in two above cases we derive the weights of criteria when the elements of their pairwise comparison matrix are in the form of triangular fuzzy numbers.

4.3. Example 3: Matrix with Fuzzy Elements in any Form

Consider 3×3 reciprocal matrix $\widetilde{A}$ with fuzzy elements in any form:

$$\widetilde{A} = \begin{bmatrix} (1,1,1) & (2,3,4) & (4,7,8,9) \\ \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right) & (1,1,1) & \widetilde{x}_{23} \\ \left(\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{4}\right) & \dfrac{1}{\widetilde{x}_{23}} & (1,1,1) \end{bmatrix}, \tag{4.8}$$

where

$$\widetilde{x}_{23} = \begin{cases} \dfrac{1 - (x-5)^2}{4}, & x \in [3, 7], \\ 0, & \text{otherwise}. \end{cases} \tag{4.9}$$

We see that there is a trapezoidal fuzzy number and there is a fuzzy number in general form. In order to obtain the interval approximation of $1/\widetilde{x}_{23}$, first we obtain the interval approximation of $\widetilde{x}_{23}$ by formula (2.9). Therefore we obtain $\widetilde{x}_{23} \approx [4.086, 5.914]$. Then we can obtain

Table 2: The result of proposed method for Example 2 (see Section 4.2).

Criteria	The obtained weights	Rank of criteria
1	$w_1 = 0.64$	1
2	$w_2 = 0.24$	2
3	$w_3 = 0.12$	3

Table 3: The result of proposed method for Example 2 (see Section 4.2).

Criteria	The obtained weights	Rank of criteria
1	$w_1 = 0.645$	1
2	$w_2 = 0.231$	2
3	$w_3 = 0.124$	3

the interval approximation of $1/\tilde{x}_{23}$, as $1/\tilde{x}_{23} \approx [0.169, 0.245]$. Now we can convert the above fuzzy matrix to the equivalent interval approximation pairwise comparison matrix. We consider the case that we use the weighting function $f_1(\alpha) = (2\alpha, 2\alpha)$.

Then by using (2.10) the interval approximation pairwise comparison matrix is obtained as follows:

$$\overline{A} = \begin{bmatrix} [1.000, 1.000] & [2.667, 3.333] & [6.000, 8.333] \\ [0.303, 0.387] & [1.000, 1.000] & [4.086, 5.914] \\ [0.120, 0.179] & [0.169, 0.245] & [1.000, 1.000] \end{bmatrix}. \quad (4.10)$$

Similar to model (4.6), by constructing the corresponding goal programming model and solving it, we obtain the weight vector as Table 4.

We can use these weights in the process of solving a multiple criteria decision-making problem. Also, these weights show that criterion 1 is more important than others.

Note 1. We claim that none of the existing methods can find the weights for such pairwise comparison matrices as Example 3 (see Section 4.3).

5. Comparing with the Existing Methods

In this section, we provide four numerical examples to illustrate the potential applications of the proposed method. And also we use them for comparing the proposed method with some of the existing methods. These methods propose some methods to derive weights for fuzzy pairwise comparison matrices. Among the existing methods, we consider the following methods.

(i) Wang and Chin [16] proposed an eigenvector method (EM) to generate interval or fuzzy weight estimate from an interval or fuzzy comparison matrix.

(ii) Wang and Chin [18] proposed a sound yet simple priority method for fuzzy AHP which utilized a linear goal programming model to derive normalized fuzzy weights for fuzzy pairwise comparison matrices.

(iii) Taha and Rostam [19] proposed a decision support system for machine tool selection in flexible manufacturing cell using fuzzy analytic hierarchy process (fuzzy AHP) and artificial neural network. A program is developed in that model to

Table 4: The result of proposed method for Example 3 (see Section 4.3).

Criteria	The obtained weights	Rank of criteria
1	$w_1 = 0.6689$	1
2	$w_2 = 0.2508$	2
3	$w_3 = 0.0803$	3

Table 5: The obtained weights of proposed method and Wang and Chin [16] method for $\widetilde{A}_1$.

Criteria	Proposed method	Wang and Chin [16] method	Rank of criteria
1	$w_1 = 0.135$	$w_1 = (0.1265, 0.1428, 0.1812)$ $M(w_1) = 0.1481$	2
2	$w_2 = 0.4325$	$w_2 = (0.4094, 0.4286, 0.4641)$ $M(w_2) = 0.432675$	1
3	$w_3 = 0.4325$	$w_3 = (0.4094, 0.4286, 0.4641)$ $M(w_3) = 0.432675$	1

find the priority weights of the evaluation criteria and alternative's ranking called PECAR for fuzzy AHP model.

(iv) Ayağ and Özdemir [20] proposed a fuzzy ANP-based approach to evaluate a set of conceptual design alternatives developed in an NPD environment in order to reach to the best one satisfying both the needs and expectations of customers, and the engineering specifications of company.

Consider the following fuzzy comparison matrix which is derived from Wang and Chin [16]:

$$\widetilde{A}_1 = \begin{bmatrix} (1,1,1) & (2,3,4)^{-1} & (2,3,4)^{-1} \\ (2,3,4) & (1,1,1) & (1,1,1) \\ (2,3,4) & (1,1,1) & (1,1,1) \end{bmatrix}. \tag{5.1}$$

By constructing the corresponding goal programming model and solving it, we obtain the weight vector as shown in Table 5. We consider the case that we use the weighting function $f(\alpha) = (3\alpha^2, 3\alpha^2)$.

Consider the following fuzzy comparison matrix which is derived from Wang and Chin [18]:

$$\widetilde{A}_2 = \begin{bmatrix} (1,1,1) & (1,2,3) & (2,3,4) \\ \left(\frac{1}{3}, \frac{1}{2}, 1\right) & (1,1,1) & (1,2,3) \\ \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right) & \left(\frac{1}{3}, \frac{1}{2}, 1\right) & (1,1,1) \end{bmatrix}. \tag{5.2}$$

By constructing the corresponding goal programming model and solving it, we obtain the weight vector as shown in Table 6.

Table 6: The obtained weights of proposed method and Wang and Chin [18] method for $\tilde{A}_2$.

Criteria	Proposed method	Wang and Chin [18] method	Rank of criteria
1	$w_1 = 0.532$	$w_1 = (0.4194, 0.5405, 0.5927)$ $M(w_1) = 0.523275$	1
2	$w_2 = 0.304$	$w_2 = (0.2016, 0.2973, 0.4274)$ $M(w_2) = 0.3059$	2
3	$w_3 = 0.164$	$w_3 = (0.1452, 0.1622, 0.2056)$ $M(w_3) = 0.1688$	3

In two previous examples we see that both of the Wang and Chin methods produce the fuzzy weights, and when we defuzzificate them by ranking function $M(\cdot)$, we can see that the results of proposed method and their methods are very close.

Now, consider the following fuzzy comparison matrix which is derived from Ayağ and Özdemir [20]:

$$\tilde{A}_3 = \begin{bmatrix} (1,1,1) & (1,3,5) & (5,7,9) \\ \left(\frac{1}{5},\frac{1}{3},1\right) & (1,1,1) & (1,3,5) \\ \left(\frac{1}{9},\frac{1}{7},\frac{1}{5}\right) & \left(\frac{1}{5},\frac{1}{3},1\right) & (1,1,1) \end{bmatrix}. \tag{5.3}$$

By constructing the corresponding goal programming model and solving it, we obtain the weight vector as shown in Table 7.

Consider the following fuzzy comparison matrix which is derived from Taha and Rostam [19]:

$$\tilde{A}_4 = \begin{bmatrix} (1,1,1) & \left(\frac{1}{8},\frac{1}{7},\frac{1}{6}\right) & \left(\frac{1}{6},\frac{1}{5},\frac{1}{4}\right) & \left(\frac{1}{6},\frac{1}{5},\frac{1}{4}\right) \\ (6,7,8) & (1,1,1) & (4,5,6) & (2,3,4) \\ (4,5,6) & \left(\frac{1}{6},\frac{1}{5},\frac{1}{4}\right) & (1,1,1) & \left(\frac{1}{2},1,1\right) \\ (4,5,6) & \left(\frac{1}{4},\frac{1}{3},\frac{1}{2}\right) & (1,1,2) & (1,1,1) \end{bmatrix}. \tag{5.4}$$

By constructing the corresponding goal programming model and solving it, we obtain the weight vector as shown in Table 8.

In two previous examples we see that both of the Ayağ and Özdemir method and the Taha and Rostam method produce the exact (nonfuzzy) weights, and again we can see that the results of proposed method and their methods are very close.

Note 2. The above-mentioned methods are not able to derive weights of fuzzy pairwise comparison matrices as Example 3 (see Section 4.3). But the presented method is able to find weights of fuzzy pairwise comparison matrices in any form.

Table 7: The obtained weights of proposed method and Ayağ and Özdemir method for $\widetilde{A}_3$.

Criteria	Proposed method	Ayağ and Özdemir method	Rank of criteria
1	$w_1 = 0.682$	$w_1 = 0.660$	1
2	$w_2 = 0.227$	$w_2 = 0.249$	2
3	$w_3 = 0.091$	$w_3 = 0.091$	3

Table 8: The obtained weights of proposed method and Taha and Rostam method for $\widetilde{A}_4$.

Criteria	Proposed method	Taha and Rostam method	Rank of criteria
1	$w_1 = 0.0474$	$w_1 = 0.0522$	4
2	$w_2 = 0.6009$	$w_2 = 0.5552$	1
3	$w_3 = 0.1256$	$w_3 = 0.1698$	3
4	$w_4 = 0.2252$	$w_4 = 0.2227$	2

6. Conclusion

Finding the weights of criteria has been one of the most important issues in the field of decision making. In this paper, we have investigated the problem of deriving the weights of criteria from the pairwise comparison matrix with fuzzy elements. In the presented method we first convert the elements of the fuzzy comparison matrix into the nearest weighted interval approximation ones. Then by using the goal programming method we derive the weights of criteria. The presented method is able to find weights of fuzzy pairwise comparison matrices in any form. Also it is shown that the results of proposed method and the existing methods are very close. We saw that the existing methods are not able to derive weights of fuzzy pairwise comparison matrices in any form such as Example 3 (see Section 4.3), but the presented method is able to find weights of such fuzzy pairwise comparison matrices. The approach is illustrated by using some examples.

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