UNIQUENESS OF POSITIVE SOLUTIONS OF A CLASS OF ODE WITH NONLINEAR BOUNDARY CONDITIONS

RUYUN MA AND YULIAN AN

Received 19 August 2004 and in revised form 27 January 2005

We study the uniqueness of positive solutions of the boundary value problem $u'' + a(t)u' + f(u) = 0$, $t \in (0,b)$, $B_1(u(0)) - u'(0) = 0$, $B_2(u(b)) + u'(b) = 0$, where $0 < b < \infty$, B_1 and $B_2 \in C^1(\mathbb{R})$, $a \in C[0,\infty)$ with $a \le 0$ on $[0,\infty)$ and $f \in C[0,\infty) \cap C^1(0,\infty)$ satisfy suitable conditions. The proof of our main result is based upon the shooting method and the Sturm comparison theorem.

1. Introduction

The existence of positive solutions of second order ordinary differential equations (ODEs) with linear boundary conditions has been extensively studied in the literature, see Coffman [1], Henderson and Wang [7], Lan and Webb [8] and the references therein. Also the existence of positive solutions of second order ODEs with nonlinear boundary conditions has been studied by several authors, see Dunninger and Wang [2], Wang [11] and Wang and Jiang [12] for some references along this line. However for the uniqueness problem of second order ODEs, even in the linear boundary conditions case, very little was known, see Ni and Nussbaum [9], Fu and Lin [6] and Peletier and Serrin [10]. To the best of our knowledge, no uniqueness results of positive solutions were established for second order ODEs subject to nonlinear boundary conditions. In this paper, we attempt to prove some uniqueness results in this direction.

More precisely, we consider the uniqueness of positive solutions of the boundary value problem

$$u'' + a(t)u' + f(u) = 0, \quad t \in (0,b) \tag{1.1}$$

$$B_1(u(0)) - u'(0) = 0, \qquad B_2(u(b)) + u'(b) = 0, \tag{1.2}$$

where $0 < b < \infty$. We make the following assumptions:

(C1) $f \in C[0,\infty) \cap C^1(0,\infty)$ with $f(0) = 0$,

$$f(u) > 0, \quad uf'(u) < f(u), \quad \text{for } u > 0; \tag{1.3}$$

Boundary Value Problems 2005:3 (2005) 289–298
DOI: 10.1155/BVP.2005.289

(C2) $a \in C[0,\infty)$ with $a(t) \le 0$ for $t \ge 0$;
(C3) $B_i \in C^1[0,\infty)$ satisfies $B_i(0) = 0$, $B_i(x) > 0$ for $x > 0$, $B_i'(x)$ is nondecreasing on $(0,\infty)$ $(i = 1,2)$.

Remark 1.1. Condition (C3) implies that $B_i'(x) \ge 0$ for $x \ge 0$ $(i = 1,2)$.

In fact, we have from $B_i(0) = 0$ and $B_i(x) > 0$ for $x > 0$ that

$$B_i'(0) \ge 0. \tag{1.4}$$

This together with the assumption $B_i'(x)$ is nondecreasing on $(0,\infty)$ implies that $B_i'(x) \ge 0$ for $x \ge 0$.

The main result of this paper is the following.

Theorem 1.2. *Let (C1)–(C3) hold. Then problem (1.1), (1.2) has at most one positive solution.*

Here we say $u(t)$ is a *positive solution* of (1.1), (1.2), if that $u(t) > 0$ on $[0,b]$ and satisfies the differential equation (1.1) as well as the boundary conditions (1.2).

Remark 1.3. As an application of Theorem 1.2, we consider the nonlinear problem

$$\begin{gathered} u'' + a(t)u' + u^p = 0, \quad t \in (0,b), \\ (u(0))^k - u'(0) = 0, \qquad (u(b))^l + u'(b) = 0, \end{gathered} \tag{1.5}$$

where $p \in (0,1)$, k, $l \in (1,\infty)$ are given, $a \in C[0,\infty)$ with $a \le 0$ on $[0,\infty)$. Clearly all of the conditions of Theorem 1.2 are satisfied. Therefore by Theorem 1.2, (1.5) has at most a positive for any $b \in (0,\infty)$.

The proof of the main result is motivated by the work of Erbe and Tang [3, 4, 5] and is based on the shooting method and the Sturm comparison theorem. The rest of the paper is organized as follows. In Section 2, we state and prove some preliminary lemmas. The proof of Theorem 1.2 will be given in Section 3.

2. The preliminary results

To apply the shooting method, we need some properties of the solutions of the initial value problem

$$u'' + \bar{a}(t)u' + \bar{f}(u) = 0, \tag{2.1}$$

$$u(0) = \delta, \qquad u'(0) = \lambda. \tag{2.2}$$

Lemma 2.1. *Let $\bar{a} \in C[0,\infty)$, $\bar{f} \in C[0,\infty) \cap C^1(0,\infty)$ with $\bar{f}(0) = 0$ and $\bar{f}(s) > 0$ for $s > 0$. Let $\delta \in (0,\infty)$ and $\lambda \in \mathbb{R}$ be two given constants. Then (2.1), (2.2) has a unique solution u satisfying either*

(I) *$u(t) > 0$ for $t \in [0,\infty)$; or*
(II) *there exists $\rho \in (0,\infty)$ such that*

$$u(t) > 0 \quad \text{on } t \in [0,\rho), \qquad u(\rho) = 0, \qquad u'(\rho) < 0. \tag{2.3}$$

Proof. For any $r \in (0,\infty)$, let

$$\Omega_r := \{(t,u,p) \mid t \in [0,r],\ u > 0\}. \tag{2.4}$$

Then the function

$$F(t,u,p) := \bar{a}(t)p + \bar{f}(u) \tag{2.5}$$

satisfies locally Lipschitz condition in Ω_r, and consequently (2.1), (2.2) has a unique solution $u(t)$ such that one of the following cases must occur

(i) $u > 0$ on $[0,\infty)$;

(ii) there exists $\rho \in (0,\infty)$ such that $u > 0$ on $[0,\rho)$, and $\lim_{t\to\rho^-} u(t) = 0$;

(iii) there exists $T \in (0,\infty)$ such that $u > 0$ on $[0,T)$ and $\limsup_{t\to T^-} u(t) = \infty$.

We claim that (iii) can not occur.

Assume on the contrary that (iii) occurs, then

$$\limsup_{t\to T^-} u'(t) = \infty. \tag{2.6}$$

On the other hand, we have from (2.1) that

$$\left(u'(t)\exp\left(\int_0^t \bar{a}(s)ds\right)\right)' + \exp\left(\int_0^t \bar{a}(s)ds\right)\bar{f}(u) = 0, \quad t \in [0,T) \tag{2.7}$$

which together with the condition $\bar{f}(s) > 0$ for $s > 0$ implies that

$$u'(t)\exp\left(\int_0^t \bar{a}(s)ds\right) \text{ is strictly decreasing on } [0,T). \tag{2.8}$$

However this contradicts the fact (2.6).

Therefore either (i) or (ii) must occur.

Suppose on the contrary that (ii) occurs and $u'(\rho) = 0$. Using the similar argument of proving (2.8), we conclude that $u'(t)\exp(\int_0^t \bar{a}(s)ds)$ is strictly decreasing on $[0,\rho)$. Thus $u'(t)\exp(\int_0^t \bar{a}(s)ds) > 0$ on $[0,\rho)$, and accordingly $u'(t) > 0$ on $[0,\rho)$. However this contradicts the fact $\delta = u(0) > u(\rho) = 0$. Therefore $u'(\rho) < 0$ if (ii) occurs. This completes the proof. □

In order to prove Theorem 1.2, we introduce an initial value problem

$$u'' + a(t)u' + f(u) = 0, \tag{2.9}$$

$$u(0) = \alpha > 0, \qquad u'(0) = B_1(\alpha). \tag{2.10}$$

For any $\alpha > 0$, we know from Lemma 2.1 that (2.9), (2.10) has a unique solution u such that one of the cases occurs:

(i) $u > 0$ in $[0,\infty)$;

(ii) there exists a unique $\rho = \rho(\alpha) \in (0,\infty)$ such that $u(t) > 0$ on $[0,\rho)$, $u(\rho) = 0$ and $u'(\rho) < 0$.

Let

$$T_\alpha = \begin{cases} \infty, & \text{if (i) occurs} \\ \rho(\alpha), & \text{if (ii) occurs.} \end{cases} \tag{2.11}$$

From $\alpha > 0$, we have that $u(0,\alpha) = \alpha > 0$ and $u'(0,\alpha) = B_1(\alpha) > 0$, and consequently

$$B_2(u(0,\alpha)) + u'(0,\alpha) = B_2(\alpha) + B_1(\alpha) > 0. \tag{2.12}$$

Therefore, there exists $\epsilon \in (0, T_\alpha)$ such that

$$B_2(u(t,\alpha)) + u'(t,\alpha) > 0, \quad t \in [0,\epsilon). \tag{2.13}$$

Denote

$$B(t,\alpha) := B_2(u(t,\alpha)) + u'(t,\alpha). \tag{2.14}$$

When $B(t,\alpha)$ vanishes at some $t_0 \in (0,T_\alpha)$, we define $b(\alpha)$ to be the first zero of $B(t,\alpha)$ in $(0,T_\alpha)$. More precisely, $b(\alpha)$ is a function of α which has the properties

$$B(b(\alpha),\alpha) = 0, \qquad B(t,\alpha) > 0, \quad t \in [0,\, b(\alpha)). \tag{2.15}$$

If $B(t,\alpha)$ is positive in $[0,T_\alpha)$, then we define $b(\alpha) = T_\alpha$. Let

$$N := \{\alpha \mid \alpha > 0,\ b(\alpha) < T_\alpha\}. \tag{2.16}$$

It is obvious that (1.1), (1.2) has no positive solution if N is an empty set. (We recall that u is a positive solution means $u(t) > 0$ in $[0,b]$. So in the case $B(T_\alpha,\alpha) = 0$, $u(t,\alpha)$ is not a positive solution of (1.1), (1.2) since $u(T_\alpha,\alpha) = 0$). Hence we suppose $N \neq \emptyset$.

Remark 2.2. It is worth remarking here that if (ii) occurs, and accordingly $u(\rho(\alpha),\alpha) = 0$, then $b(\alpha) \in (0,\rho(\alpha))$,

$$B(b(\alpha),\alpha) = 0, \qquad B(t,\alpha) > 0 \quad \text{on } [0,b(\alpha)). \tag{2.17}$$

In fact, we have from Lemma 2.1 that

$$B(\rho(\alpha),\alpha) = B_2(u(\rho,\alpha)) + u'(\rho(\alpha),\alpha) < 0, \tag{2.18}$$

which together with the fact $B(0,\alpha) > 0$ yields the existence of zero of $B(t,\alpha)$ in $(0,\rho(\alpha))$.

LEMMA 2.3. *Let (C1)–(C3) hold and let $\alpha \in N$. Let $u(t,\alpha)$ be the unique solution of (2.9), (2.10) on $[0,T_\alpha)$. Then*

$$\begin{gathered} u(t,\alpha) > 0, \quad t \in [0,b(\alpha)], \\ u'(b(\alpha),\alpha) < 0. \end{gathered} \tag{2.19}$$

Proof. By Remark 2.2, $b(\alpha) \in (0,\rho(\alpha))$. Applying Lemma 2.1, we get that

$$u(t,\alpha) > 0, \quad t \in [0,b(\alpha)]. \tag{2.20}$$

The second inequality in (2.19) can be easily deduced from (2.20) and (C3) and the fact

$$B(b(\alpha),\alpha) = B_2(u(b(\alpha),\alpha)) + u'(b(\alpha),\alpha) = 0. \tag{2.21}$$

□

LEMMA 2.4. *Let (C1)–(C3) hold. Let $u(t,\alpha)$ be the unique solution of (2.9), (2.10) on $[0,T_\alpha)$. If $\eta \in (0,T_\alpha)$ is such that*

$$B(\eta,\alpha) = 0, \tag{2.22}$$

then

$$B(t,\alpha) > 0, \quad t \in [0,\eta). \tag{2.23}$$

Proof. From (2.9), we conclude that

$$\left(u' \exp\left(\int_0^t a(s)ds\right)\right)' + \exp\left(\int_0^t a(s)ds\right) f(u) = 0. \tag{2.24}$$

Since $u(t,\alpha) > 0$ for all $t \in [0,\eta]$, we have

$$\left(u'(t,\alpha)\exp\left(\int_0^t a(s)ds\right)\right)' = -\exp\left(\int_0^t a(s)ds\right) f(u(t,\alpha)) < 0, \quad \forall t \in [0,\eta]. \tag{2.25}$$

Suppose on the contrary that there exists $\tau_2 \in [0,\eta)$ such that

$$B(\tau_2,\alpha) = B_2(u(\tau_2,\alpha)) + u'(\tau_2,\alpha) = 0. \tag{2.26}$$

Then we have from condition (C3) and the fact $u(\tau_2,\alpha) > 0$ that

$$u'(\tau_2,\alpha) = -B_2(u(\tau_2,\alpha)) < 0 \tag{2.27}$$

and accordingly

$$u'(\tau_2,\alpha)\exp\left(\int_0^{\tau_2} a(s)ds\right) < 0. \tag{2.28}$$

This together with (2.25) implies that

$$u'(t,\alpha)\exp\left(\int_0^t a(s)ds\right) < 0, \quad t \in [\tau_2,\eta], \tag{2.29}$$

and consequently

$$u'(t,\alpha) < 0, \quad t \in [\tau_2,\eta]. \tag{2.30}$$

This implies

$$u(\tau_2,\alpha) > u(\eta,\alpha). \tag{2.31}$$

By Remark 1.1 and (2.31), we get

$$B_2(u(\tau_2,\alpha)) \geq B_2(u(\eta,\alpha)). \tag{2.32}$$

From (2.30) and (C1)–(C2) and the fact $u''(t,\alpha) = -a(t)u'(t,\alpha) - f(u(t,\alpha))$, it follows that

$$u''(t,\alpha) < 0, \quad t \in [\tau_2,\eta] \tag{2.33}$$

and consequently

$$u'(\tau_2,\alpha) > u'(\eta,\alpha), \tag{2.34}$$

which together with (2.32) implies that

$$B(\tau_2,\alpha) = B_2(u(\tau_2,\alpha)) + u'(\tau_2,\alpha) > B_2(u(\eta,\alpha)) + u'(\eta,\alpha) = 0. \tag{2.35}$$

However this contradicts (2.26). □

Remark 2.5. From Lemmas 2.3 and 2.4, we have that if $\eta \in (0,T_\alpha)$ satisfies

$$B(\eta,\alpha) = 0. \tag{2.36}$$

Then

$$\eta = b(\alpha). \tag{2.37}$$

In other words, if $\alpha \in N$, then $b(\alpha)$ is the unique zero of $B(t,\alpha) = 0$ in $[0,\rho(\alpha))$. Therefore to prove that (1.1), (1.2) has at most one positive solution, it is sufficient to show that for any $l > 0$, there exists at most one $\alpha \in N$ such that $b(\alpha) = l$.

Now we denote the *variation* of $u(t,\alpha)$ by $\phi(t,\alpha) = \partial u(t,\alpha)/\partial\alpha$. Then, $\phi(t,\alpha)$ satisfies

$$\phi'' + a(t)\phi' + f'(u)\phi = 0, \tag{2.38}$$

$$\phi(0,\alpha) = 1, \qquad \phi'(0,\alpha) = B_1'(\alpha). \tag{2.39}$$

Lemma 2.6. *Suppose that*

$$B_2'(u(b(\alpha),\alpha))\phi(b(\alpha),\alpha) + \phi'(b(\alpha),\alpha) \neq 0, \quad \alpha \in N. \tag{2.40}$$

Then one of the following cases must occur

(i) *N is an open interval;*

(ii) *$N = (0,j_1) \cup (j_2,\infty)$ with $0 < j_1 < j_2 < +\infty$. Moreover, $b'(\alpha) > 0$ for all $(0,j_1)$; $b'(\alpha) < 0$ for all (j_2,∞).*

Proof. We firstly show that $b(\alpha) \in C^1(N)$ and $b'(\alpha) \neq 0$.

From Lemma 2.3, (C1)–(C2), we conclude that

$$u''(b(\alpha),\alpha) = -a(b(\alpha))u'(b(\alpha),\alpha) - f(u(b(\alpha),\alpha)) < 0. \tag{2.41}$$

This together with

$$B(b(\alpha),\alpha) = 0 \tag{2.42}$$

and (C3) and (2.19) implies that

$$\frac{\partial}{\partial t}B(t,\alpha)\Big|_{t=b(\alpha)} = B_2'(u(b(\alpha),\alpha))u'(b(\alpha),\alpha) + u''(b(\alpha),\alpha) < 0. \tag{2.43}$$

So by Implicit Function theorem, $b(\alpha)$ is well-defined as a function of α in N and $b(\alpha) \in C^1(N)$. Furthermore, it follows from (2.43) that N is an open set.

Differentiating both sides of (2.42) with respect to α, we obtain

$$B_2'(u(b(\alpha),\alpha))[u'(b(\alpha),\alpha)b'(\alpha)+\phi(b(\alpha),\alpha)] + u''(b(\alpha),\alpha)b'(\alpha)+\phi'(b(\alpha),\alpha) = 0, \tag{2.44}$$

that is,

$$\begin{aligned}&[B_2'(u(b(\alpha),\alpha))u'(b(\alpha),\alpha)+u''(b(\alpha),\alpha)]b'(\alpha)\\ &\quad + B_2'(u(b(\alpha),\alpha))\phi(b(\alpha),\alpha)+\phi'(b(\alpha),\alpha) = 0.\end{aligned} \tag{2.45}$$

which together with (2.40) implies that

$$b'(\alpha) \neq 0. \tag{2.46}$$

Next we show that if $\bar{\alpha} \in (0,\infty)\setminus N$ is such that there is a sequence $\{\alpha_n\} \subset N$ and $\alpha_n \to \bar{\alpha}$ as $n \to \infty$, then $b(\alpha_n) \to +\infty$.

Suppose on the contrary that $b(\alpha_n) \nrightarrow +\infty$, then there exists a subsequence of $\{b(\alpha_n)\}$ which converges to a limit number t^*. Without loss of generality, we may suppose that $b(\alpha_n) \to t^*$ as $n \to \infty$, and consequently

$$B(t^*,\bar{\alpha}) = \lim_{n\to\infty} B(b(\alpha_n),\alpha_n) = 0. \tag{2.47}$$

However this contradicts $\bar{\alpha} \notin N$.

Finally we show that if N is not an open interval, then (ii) must occur.

Suppose $J_1 = (j_0, j_1)$ and $J_2 = (j_2, j_3)$ are two distinct components of N with $0 < j_1 < j_2 < \infty$. Then

$$\lim_{\alpha\to j_1^-} b(\alpha) = \lim_{\alpha\to j_2^+} b(\alpha) = +\infty. \tag{2.48}$$

Since $b(\alpha)$ is strictly monotonic in each component of N, we have that $b'(\alpha) > 0$ in J_1, and $b'(\alpha) < 0$ in J_2. Meanwhile

$$\lim_{\alpha\to j_0^+} b(\alpha) < +\infty, \qquad \lim_{\alpha\to j_3^-} b(\alpha) < +\infty. \tag{2.49}$$

It follows that $j_0 = 0$ and $j_3 = +\infty$, and accordingly $N = (0,j_1)\cup(j_2,\infty)$ with $b'(\alpha) > 0$ in $(0,j_1)$, and $b'(\alpha) < 0$ in (j_2,∞). $\square$

3. Proof of Theorem 1.2

By Remark 2.5, we only need to show that for any $l > 0$, there exists at most one $\alpha \in N$ such that $b(\alpha) = l$.

Recall that for any given $\alpha \in N$, (2.43), (2.45) hold. If we can show that

$$B_2'(u(b(\alpha),\alpha))\phi(b(\alpha),\alpha) + \phi'(b(\alpha),\alpha) > 0, \quad \alpha \in N \tag{3.1}$$

then it follows from (2.43) and (2.45) that

$$b'(\alpha) > 0, \quad \alpha \in N. \tag{3.2}$$

Thus by Lemma 2.6, N must be an open interval. Moreover we know from (3.2) that $b(\alpha)$ is a strictly increasing function on N. Thus, for any given $l > 0$, there is at most one $\alpha \in N$ such that $b(\alpha) = l$, and consequently, (1.1), (1.2) has at most one positive solution.

Proof of Theorem 1.2. Now we prove (3.1).

First we claim that

$$\phi(t,\alpha) > 0, \quad t \in [0, b(\alpha)]. \tag{3.3}$$

Suppose on the contrary that $\phi(t,\alpha)$ has a zero in $(0, b(\alpha)]$. We denote the first zero of $\phi(t,\alpha)$ in $(0,b(\alpha)]$ by t_3, then $0 < t_3 \le b(\alpha)$ and

$$u'\phi - u\phi'\big|_{t=t_3} = -u(t_3,\alpha)\phi'(t_3,\alpha) \ge 0 \tag{3.4}$$

since $\phi(t_3,\alpha) = 0$ and $\phi(t,\alpha) > 0$ on $(0,t_3)$ implies $\phi'(t_3,\alpha) \le 0$.

Notice that

$$\phi'' + a(t)\phi' + f'(u)\phi = 0 \tag{3.5}$$

so that using (C1) and (1.1) we can compute

$$\left[\exp\left(\int_0^t a(s)ds\right)(u'\phi - u\phi')\right]' = \exp\left(\int_0^t a(s)ds\right)[f'(u)u - f(u)]\phi < 0 \tag{3.6}$$

for $t \in (0,t_3)$. Next we compute from (C3) and (2.39) and (2.10)

$$\exp\left(\int_0^t a(s)ds\right)(u'\phi - u\phi')\big|_{t=0} = B_1(\alpha) - \alpha B_1'(\alpha) = (B_1'(\xi_1(\alpha)) - B_1'(\alpha))\alpha \le 0, \tag{3.7}$$

where $\xi_1(\alpha) \in (0,\alpha)$. This means that

$$\exp\left(\int_0^t a(s)ds\right)(u'\phi - u\phi')\big|_{t=t_3} < 0 \tag{3.8}$$

and accordingly

$$u'\phi - u\phi'\big|_{t=t_3} < 0. \tag{3.9}$$

However this contradicts (3.4). Therefore (3.3) is true.

Using (3.5), (1.1), (C1) and (3.3), we can conclude

$$\left[\exp\left(\int_0^t a(s)ds\right)(u'\phi - u\phi')\right]' = \exp\left(\int_0^t a(s)ds\right)[f'(u)u - f(u)]\phi < 0, \quad t \in (0, b(\alpha)] \tag{3.10}$$

which together with (3.7) implies that

$$(u'\phi - u\phi')\big|_{t=b(\alpha)} < 0. \tag{3.11}$$

Since

$$\begin{aligned} 0 &= B(b(\alpha),\alpha) \\ &= B_2(u(b(\alpha),\alpha)) + u'(b(\alpha),\alpha) \\ &= B_2'(\xi_2(\alpha))u(b(\alpha),\alpha) + u'(b(\alpha),\alpha) \end{aligned} \tag{3.12}$$

for some $\xi_2(\alpha) \in (0, u(b(\alpha),\alpha))$, we have that

$$u'(b(\alpha),\alpha) = -B_2'(\xi_2(\alpha))u(b(\alpha),\alpha). \tag{3.13}$$

This together with (3.11) implies

$$\begin{aligned} &-u(b(\alpha),\alpha)[B_2'(\xi_2(\alpha))\phi(b(\alpha),\alpha) + \phi'(b(\alpha),\alpha)] \\ &\quad = -B_2'(\xi_2(\alpha))u(b(\alpha),\alpha)\phi(b(\alpha),\alpha) - u(b(\alpha),\alpha)\phi'(b(\alpha),\alpha) \\ &\quad = u'(b(\alpha),\alpha)\phi(b(\alpha),\alpha) - u(b(\alpha),\alpha)\phi'(b(\alpha),\alpha) \\ &\quad = u'\phi - u\phi'\big|_{t=b(\alpha)} < 0 \end{aligned} \tag{3.14}$$

and consequently

$$B_2'(\xi_2(\alpha))\phi(b(\alpha),\alpha) + \phi'(b(\alpha),\alpha) > 0. \tag{3.15}$$

Now we have from (C3) and the facts $\xi_2(\alpha) \le u(b(\alpha),\alpha)$ and $\phi(b(\alpha),\alpha) > 0$ that

$$B_2'(u(b(\alpha),\alpha))\phi(b(\alpha),\alpha) + \phi'(b(\alpha),\alpha) \ge B_2'(\xi_2(\alpha))\phi(b(\alpha),\alpha) + \phi'(b(\alpha),\alpha) > 0. \tag{3.16}$$

Therefore (3.1) holds. □

Acknowledgments

The authors are very grateful to the anonymous referee for his/her valuable suggestions. Supported by the NSFC (no. 10271095), GG-110-10736-1003, NSF of Gansu province (no. 3ZS051-A25-016), the Foundation of Excellent Young Teacher of the Chinese Education Ministry.

References

[1] C. V. Coffman, *Uniqueness of the positive radial solution on an annulus of the Dirichlet problem for* $\Delta u - u + u^3 = 0$, J. Differential Equations **128** (1996), no. 2, 379–386.

[2] D. R. Dunninger and H. Y. Wang, *Multiplicity of positive solutions for a nonlinear differential equation with nonlinear boundary conditions*, Ann. Polon. Math. **69** (1998), no. 2, 155–165.

[3] L. Erbe and M. Tang, *Structure of positive radial solutions of semilinear elliptic equations*, J. Differential Equations **133** (1997), no. 2, 179–202.

[4] ———, *Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball*, J. Differential Equations **138** (1997), no. 2, 351–379.

[5] ———, *Uniqueness of positive radial solutions of* $\Delta u + f(|x|, u) = 0$, Differential Integral Equations **11** (1998), no. 5, 725–743.

[6] C.-C. Fu and S.-S. Lin, *Uniqueness of positive radial solutions for semilinear elliptic equations on annular domains*, Nonlinear Anal. Ser. A: Theory Methods **44** (2001), no. 6, 749–758.

[7] J. Henderson and H. Y. Wang, *Positive solutions for nonlinear eigenvalue problems*, J. Math. Anal. Appl. **208** (1997), no. 1, 252–259.

[8] K. Q. Lan and J. R. L. Webb, *Positive solutions of semilinear differential equations with singularities*, J. Differential Equations **148** (1998), no. 2, 407–421.

[9] W.-M. Ni and R. D. Nussbaum, *Uniqueness and nonuniqueness for positive radial solutions of* $\Delta u + f(u, r) = 0$, Comm. Pure Appl. Math. **38** (1985), no. 1, 67–108.

[10] L. A. Peletier and J. Serrin, *Uniqueness of positive solutions of semilinear equations in* R^n, Arch. Ration. Mech. Anal. **81** (1983), no. 2, 181–197.

[11] J. Y. Wang, *The existence of positive solutions for the one-dimensional* p*-Laplacian*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 2275–2283.

[12] J. Y. Wang and J. Jiang, *The existence of positive solutions to a singular nonlinear boundary value problem*, J. Math. Anal. Appl. **176** (1993), no. 2, 322–329.

Ruyun Ma: Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
E-mail address: mary@maths.uq.edu.au

Yulian An: Physical Software & Engineering, Lanzhou Jiaotong University, Lanzhou 730070, Gansu, China
E-mail address: an_yulian@tom.com