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Research Article

Entire Bounded Solutions for a Class of Quasilinear Elliptic Equations

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We consider the problem $-\text{div}(|\nabla u|^{p-2}\nabla u) = a(x)(u^m+\lambda u^n), x\in\mathbb{R}^N, N\geq 3$, where $0 < m < p-1 < n, a(x)\geq 0, a(x)$ is not identically zero. Under the condition that $a(x)$ satisfies (H), we show that there exists $\lambda_0 > 0$ such that the above-mentioned equation admits at least one solution for all $\lambda\in(0,\lambda_0)$. This extends the results of Laplace equation to the case of p-Laplace equation.

In this work, we are interested in studying the existence of solutions to the following quasilinear equation:

$$-\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = a(x)\left(u^m+\lambda u^n\right),\quad x\in\mathbb{R}^N,\ N\geq 3, \tag{1}$$

where $0 < m < p-1 < n$, $a(x)\geq 0$, $a(x)$ is not identically zero. We will assume throughout the paper that $a(x)\in C(\mathbb{R}^N)$. Equations of the above form are mathematical models occuring in studies of the p-Laplace equation, generalized reaction-diffusion theory [1], non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium [2]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

Problem (1) for bounded domains with zero Dirichlet condition has been extensively studied (even for more general sublinear functions). We refer in particular to [3–10] (see also the references therein). When $p = 2$, the related results have been obtained by [11–16] (including bounded domains with zero Dirichlet condition or $\mathbb{R}^N$). Our existence results extend that of Brezis and Kamin (see [11, Theorem 1]) for semilinear problem, and complement results in [3–10].

$u \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ is called a entire weak solution to (1) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla \psi \, dx = \int_{\mathbb{R}^N} a(x)(u^m + \lambda u^n)\psi \, dx \quad \forall \psi \in C_0^\infty(\mathbb{R}^N) \tag{2}$$

and $u > 0$ in $\mathbb{R}^N$.

Definition 1. $\overline{u} \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ is called a supersolution to problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x,u) = 0 \tag{3}$$

if

$$\int_{\mathbb{R}^N} |\nabla \overline{u}|^{p-2}\nabla \overline{u} \cdot \nabla \psi \, dx \geq \int_{\mathbb{R}^N} f(x,\overline{u})\psi \, dx \quad \forall \psi \in C_0^\infty(\mathbb{R}^N) \tag{4}$$

and $\overline{u} > 0$ in $\mathbb{R}^N$. As always, a subsolution $\underline{u}$ is defined by reversing the inequalities.

From [3], we have the following lemma.

Lemma 1. *Suppose that $f(x,u)$ is defined on $\mathbb{R}^{N+1}$ and is locally Hölder continuous (with exponent $\lambda \in (0,1)$) in x. $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution to (3) with $\underline{u} \leq \overline{u}$ on $\mathbb{R}^N$, and suppose that $f(x,u)$ is locally Lipschitz continuous in u on the set*

$$\{(x,u) : x \in \mathbb{R}^N,\ w(x) \leq u \leq v(x)\}. \tag{5}$$

Then, (3) possesses an entire solution $u(x)$ satisfying

$$w(x) \leq u(x) \leq v(x), \quad x \in \mathbb{R}^N. \tag{6}$$

Definition 2. Say that a function $a(x) \in C(\mathbb{R}^N)$, $a(x) \geq 0$, has the property (H) if the linear problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(x), \quad \text{in } \mathbb{R}^N, \tag{7}$$

has a bounded solution.

Remark 1. If $a(x)$ satisfies

$$H_\infty = \int_0^\infty \left(s^{1-N}\int_0^s t^{N-1}\psi(t)dt\right)^{1/(p-1)} ds < \infty, \tag{8}$$

where $\psi(r) = \max_{|x|=r} a(x)$, then $a(x)$ has the property (H).

In fact, because

$$V(x) = \int_{|x|}^\infty \left(\frac{1}{s^{N-1}}\int_0^s \sigma^{N-1}\psi(\sigma)d\sigma\right)^{1/p-1} ds \tag{9}$$

which is a solution for the $-\operatorname{div}(|\nabla V|^{p-2}\nabla V) = \psi(r)$ in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} V(x) = 0$, so V is a supersolution for (7). On the other hand, 0 is a subsolution for (7), then (7) exists bounded entire solution.

Remark 2. If $N \geq 3$, $N > p$, then condition (8) of Remark 1 is replaced by

$$0 < \int_1^{\infty} r^{1/(p-1)} \psi(r)^{1/(p-1)} dr < \infty \quad \text{if } 1 < p \leq 2, \tag{A}$$

$$0 < \int_1^{\infty} r^{((p-2)N+1)/(p-1)} \psi(r) dr < \infty \quad \text{if } p \geq 2. \tag{B}$$

Let

$$J(r) = \int_0^r \left(t^{1-N} \int_0^t s^{N-1} \psi(s) ds \right)^{1/(p-1)} dt. \tag{10}$$

In fact, if $1 < p \leq 2$, by estimating the above integral,

$$J(r) \leq C_1 + \int_1^r t^{(1-N)/(p-1)} \left[\int_0^t s^{N-1} \psi(s) ds \right]^{1/(p-1)} dt. \tag{11}$$

Using the assumption $N \geq 3$ in the computation of the first integral above and Jensen's inequality to estimate the last one,

$$J(r) \leq C_2 + C_3 \int_1^r t^{(3-N-p)/(p-1)} \int_1^t s^{(N-1)/(p-1)} \psi(s)^{1/(p-1)} ds\, dt. \tag{12}$$

Computing the above integral, we obtain

$$J(r) \leq C_2 + C_4 \int_1^r t^{1/(p-1)} \psi(t)^{1/(p-1)} dt. \tag{13}$$

Applying (A) in the above integral, we infer that $H_\infty = \lim_{r \to \infty} J(r) < \infty$. On the other hand, if $p \geq 2$, set

$$H(t) = \int_0^t s^{N-1} \psi(s) ds \tag{14}$$

and note that either $H(t) \leq 1$ for $t > 0$ or $H(t_0) = 1$ for some $t_0 > 0$. In the first case, $H^{1/(p-1)} \leq 1$, and hence,

$$J(r) = \int_0^r t^{(1-N)/(p-1)} H(t)^{1/(p-1)} dt \leq C_5 + \int_1^r t^{(1-N)/(p-1)} dt \tag{15}$$

so that $J(r)$ has a finite limit because $p < N$. In the second case, $H(s)^{1/(p-1)} \leq H(s)$ for $s \geq s_0$ and hence,

$$J(r) \leq C_6 + \int_1^r t^{(1-N)/(p-1)} \int_0^t s^{N-1} \psi(s) ds\, dt. \tag{16}$$

Estimating and integrating by parts, we obtain

$$\begin{aligned} J(r) &\le C_6 + \frac{p-1}{N-p}\int_0^1 t^{N-1}\psi(t)dt \\ &\quad + \frac{p-1}{N-p}\left[\int_1^r t^{((p-2)N+1)/(p-1)}\psi(t)dt - r^{(p-N)/(p-1)}\int_0^r t^{N-1}\psi(t)dt\right] \\ &\le C_7 + C_8\int_1^r t^{((p-2)N+1)/(p-1)}\psi(t)dt. \end{aligned} \tag{17}$$

By (B), $H_\infty = \lim_{r\to\infty} J(r) < \infty$.

Lemma 2. *Problem*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla v) = a(x)u^m, \quad in\ \mathbb{R}^N,\ N \ge 3, \tag{18}$$

has a bounded solution if and only if $a(x)$ *satisfies (H). Moreover, there is a minimal positive solution of (18).*

Proof

Sufficient condition. Let

$$B_R = \{x \in \mathbb{R}^N : |x| < R\} \tag{19}$$

and let u_R be the solution of

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(x)u^m \quad \text{in } B_R, \\ u = 0 \quad \text{on } \partial B_R. \end{gathered} \tag{20}$$

It is well known that u_R exists and is unique (see [5]). The sequence u_R is increasing with R. Indeed, let $R' > R$. Then $u_{R'}$ is a supersolution for (20). We now construct a subsolution $\underline{u}$ for (20) and $\underline{u} \le u_{R'}$. From Lemma 1, we will imply that there is a solution u for (20) between $\underline{u}$ and $u_{R'}$. Since the unique solution is u_R, it follows that $u_R \le u_{R'}$ in B_R. For $\underline{u}$, we may take $\varepsilon\psi_1$ where ψ_1 satisfies

$$\begin{gathered} -\operatorname{div}(|\nabla\psi_1|^{p-2}\nabla\psi_1) = \lambda_1 a(x)|\psi_1|^{p-2}\psi_1 \quad \text{in } B_R, \\ \psi_1 = 0 \quad \text{on } \partial B_R. \end{gathered} \tag{21}$$

We now prove that the sequence u_R remains bounded as $R \to \infty$. In fact,

$$u_R \le CU \tag{22}$$

for some appropriate constant C. Indeed, CU is a supersolution for the (20) since

$$-\operatorname{div}(|\nabla(CU)|^{p-2}\nabla(CU)) = C^{p-1}a(x) \ge a(x)(CU)^m, \tag{23}$$

provided that

$$C^{p-1-m} \geq \|U\|_{\infty}^{m}. \tag{24}$$

Therefore $u = \lim_{R\to\infty} u_R$ exists and u is a solution of (18) satisfying

$$u \leq CU. \tag{25}$$

Clearly, u is the minimal solution. In fact, if $\overline{u}$ is another solution of (18) then $u_R \leq \overline{u}$ on B_R by the above argument and thus $u \leq \overline{u}$.

Necessary condition. Suppose u is bounded positive solution of (18) and set

$$v = \frac{p-1}{p-1-m} u^{(p-1-m)/(p-1)}. \tag{26}$$

Then

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) = mu^{-m-1}|\nabla u|^{p} + a(x) \geq a(x). \tag{27}$$

The solution w_R of the problem

$$\begin{gathered} -\operatorname{div}(|\nabla w_R|^{p-2}\nabla w_R) = a(x), \quad x \in B_R, \\ w_R = 0, \quad x \in \partial B_R \end{gathered} \tag{28}$$

satisfies $w_R \leq v$. Thus w_R increases as $R \to \infty$ to a bounded solution of (7). □

Theorem 1. *Suppose that $a(x)$ satisfies (H), then there exists*

$$\lambda_0 = \frac{p-1-m}{n-p+1} E^{(p-1-n)/(p-1-m)-n} \left(\frac{n-p+1}{n-m}\right)^{(n-m)/(p-1-m)}, \tag{29}$$

here $E = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} e(x)$, $e(x)$ is a bounded solution of (18), such that for $\lambda \in (0,\lambda_0)$, (1) has an entire bounded solution. If (1) has an entire bounded solution, then (7) has an entire bounded solution.

Proof. Firstly, we prove that there exists $\lambda_0 > 0$ such that for all $\lambda \in (0,\lambda_0)$, (1) has a bounded solution. Since $a(x)$ satisfies (H), we have that

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(x) \tag{30}$$

has a bounded solution $e(x)$, let $E = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} e(x)$, we consider the following function:

$$\lambda(t) = \frac{t^{p-1} - E^m t^m}{t^n E^n} = \frac{1}{E^n}\left(t^{p-1-n} - E^m t^{m-n}\right), \quad t > 0, \tag{31}$$

for $\lambda(t)$ first derivation, we have

$$\lambda'(t) = \frac{1}{E^n}\left((p-1-n)t^{p-2-n} - (m-n)E^m t^{m-n-1}\right) \tag{32}$$

let $\lambda'(t) = 0$, it follows that

$$t_0 = \left(\frac{E^m(n-m)}{n-p+1}\right)^{1/(p-1-m)}. \tag{33}$$

By simple calculation, we obtain that t_0 is maximal value point of $\lambda(t)$, it is clear that $\lambda(t_0) = \lambda_0$. Then for all $\lambda \in [0,\lambda_0]$, $\exists T = T(\lambda) > 0$ satisfies $(T^{p-1} - E^m T^m)/T^n E^n \geq \lambda$, it follows that for all $\lambda \in [0,\lambda_0]$, such that $T^{p-1} \geq T^m E^m + \lambda T^n E^n, Te$ is a supersolution of (1), in fact

$$\begin{aligned} -\operatorname{div}\left(\left|\nabla(Te)\right|^{p-2}\nabla(Te)\right) &= -T^{p-1}\operatorname{div}\left(|\nabla e|^{p-2}\nabla e\right) = T^{p-1}a(x) \\ &\geq a(x)\left(T^m E^m + \lambda T^n E^n\right) \geq a(x)\left[(Te)^m + \lambda(Te)^n\right]. \end{aligned} \tag{34}$$

From Lemma 2, problem (18) has a positive solution u_0, then εu_0 is a subsolution of (1), in fact , for all λ and sufficiently small, we have ε $(0 < \varepsilon < 1)$,

$$\begin{aligned} -\operatorname{div}\left(\left|\nabla\left(\varepsilon^{1/(p-1)}u_0\right)\right|^{p-2}\nabla\left(\varepsilon^{1/(p-1)}u_0\right)\right) &= -\varepsilon\operatorname{div}\left(\left|\nabla u_0\right|^{p-2}\nabla u_0\right) \\ &= \varepsilon a(x)u_0^m \leq a(x)\left[\left(\varepsilon u_0\right)^m + \lambda\left(\varepsilon u_0\right)^n\right]. \end{aligned} \tag{35}$$

□

Set ε sufficiently small, such that $\varepsilon^{1/(p-1)}u_0 < Te$, then for $0 < \lambda < \lambda_0$, $\varepsilon^{1/(p-1)}u_0 < u < Te$, therefore (1) has a bounded solution.

Secondly, if (1) has a positive solution, then (3) has a positive solution. Let us define

$$\lambda^* = \sup\{\lambda > 0 \mid (1) \text{ has at least one bounded positive solution}\}. \tag{36}$$

Apparently, $0 < \lambda < \lambda^*$. Suppose u is a bounded positive solution of (1) and for all $\lambda \in (0,\lambda^*)$, set $v = ((p-1)/(p-1-m))u^{(p-1-m)/(p-1)}$. Then

$$\begin{aligned} -\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) &= \left(\frac{p-1}{p-1-m}\right)^{p-1}\left[-\operatorname{div}\left(\left|\nabla\left(u^{(p-1-m)/(p-1)}\right)\right|^{p-2}\nabla\left(u^{(p-1-m)/(p-1)}\right)\right)\right] \\ &= -\left(\frac{p-1}{p-1-m}\right)^{p-1}\operatorname{div}\left(\left(\frac{p-1-m}{p-1}\right)^{p-1}u^{-m}|\nabla u|^{p-2}\nabla u\right) \\ &= -\operatorname{div}\left(u^{-m}|\nabla u|^{p-2}\nabla u\right) = mu^{-m-1}|\nabla u|^p - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)u^{-m} \\ &= mu^{-m-1}|\nabla u|^p + a(x)\left(1 + \lambda u^{n-m}\right) \geq a(x). \end{aligned} \tag{37}$$

The solution w_R of the problem

$$-\operatorname{div}(|\nabla w_R|^{p-2}\nabla w_R) = a(x), \qquad x \in B_R,$$
$$w_R = 0, \quad x \in \partial B_R \tag{38}$$

satisfies $w_R \leq v$. Thus w_R increases as $R \to \infty$ to a bounded solution of (3).

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