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Research Article

Dead Core Problems for Singular Equations with ϕ-Laplacian

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The paper discusses the existence of positive solutions, dead core solutions, and pseudo dead core solutions of the singular problem $(\phi(u'))' + f(t,u') = \lambda g(t,u,u')$, $u'(0) = 0$, $\beta u'(T) + \alpha u(T) = A$. Here λ is a positive parameter, $\beta \geq 0$, α, $A > 0$, f may be singular at $t = 0$ and g is singular at $u = 0$.

1. Introduction

Throughout the paper $L^1[a,b]$ denotes the set of integrable functions on $[a,b]$, $L^1_{\mathrm{loc}}(a,b]$ the set of functions $x:(a,b] \to \mathbb{R}$ which are integrable on $[a-\varepsilon,b]$ for arbitrary small $\varepsilon > 0$, $AC[a,b]$ is the set of absolutely continuous function on $[a,b]$, and $AC_{\mathrm{loc}}(a,b]$ is the set of functions $x:(a,b] \to \mathbb{R}$ which are absolutely continuous on $[a-\varepsilon,b]$ for arbitrary small $\varepsilon > 0$.

Let T be a positive number. If $G \subset \mathbb{R}^j$ $(j = 1,2)$ then $\mathrm{Car}([0,T] \times G)$ stands for the set of functions $h:[0,T] \times G \to \mathbb{R}$ satisfying the local Carathéodory conditions on $[0,T] \times G$, that is, (j) for each $z \in G$, the function $h(\cdot,z):[0,T] \to \mathbb{R}$ is measurable; (jj) for a.e. $t \in [0,T]$, the function $h(t,\cdot):G \to \mathbb{R}$ is continuous; (jjj) for each compact set $M \subset G$, there exists $\delta_M \in L^1[0,T]$ such that $|h(t,z)| \leq \delta_M(t)$ for a.e. $t \in [0,T]$ and all $z \in M$. We will write $h \in \mathrm{Car}((0,T] \times G)$ if $h \in \mathrm{Car}([a,T] \times G)$ for each $a \in (0,T]$.

We consider the singular boundary value problem

$$(\phi(u'(t)))' + f(t,u'(t)) = \lambda g(t,u(t),u'(t)), \quad \lambda > 0, \tag{1.1}$$

$$u'(0) = 0, \qquad \beta u'(T) + \alpha u(T) = A, \quad \beta \geq 0,\ \alpha, A > 0, \tag{1.2}$$

depending on the positive parameter λ. Here $\phi \in C^0[0,\infty)$, $f \in \mathrm{Car}((0,T] \times [0,\infty))$ is nonnegative, $f(t,0) = 0$ for a.e. $t \in [0,T]$, $g \in \mathrm{Car}([0,T] \times D)$ is positive, where $D = (0,A/\alpha] \times [0,\infty)$ and g is singular at the value 0 of its first space variable. We say that g is singular at the value 0 of its first space variable provided

$$\lim_{x\to 0^+} g(t,x,y) = \infty \quad \text{for a.e. } t \in [0,T] \text{ and each } y \in [0,\infty). \tag{1.3}$$

A function $u \in C^1[0,T]$ is called *a positive solution of problem* (1.1), (1.2) if $u > 0$ on $[0,T]$, $\phi(u') \in AC_{\mathrm{loc}}(0,T]$, u satisfies (1.2), and (1.1) holds for a.e. $t \in [0,T]$. We say that $u \in C^1[0,T]$ satisfying (1.2) is *a dead core solution of problem* (1.1), (1.2) if there exists $t_0 \in (0,T)$ such that $u = 0$ on $[0,t_0]$, $u > 0$ on $(t_0,T]$, $\phi(u') \in AC[t_0,T]$ and (1.1) holds for a.e. $[t_0,T]$. The interval $[0,t_0]$ is called the *dead core of* u. If $u(0) = 0$, $u > 0$ on $(0,T]$, $\phi(u') \in AC_{\mathrm{loc}}(0,T]$, u satisfies (1.2) and (1.1) a.e. on $[0,T]$, then u is called *a pseudo dead core solution of problem* (1.1), (1.2).

The aim of this paper is to discuss the existence of positive solutions, dead core solutions, and pseudo dead core solutions to problem (1.1), (1.2). Although problem (1.1), (1.2) is singular, all types of solutions are considered in the space $C^1[0,T]$.

The study of problem (1.1), (1.2) was motivated from the paper by Baxley and Gersdorff [2]. Here the singular reaction-diffusion boundary value problem

$$\begin{gathered} u'' + f_1(t,u') = \lambda g_1(t,u), \\ u'(a) = 0, \qquad \beta u'(b) + \alpha u(b) = A, \quad \beta \ge 0,\ \alpha, A > 0 \end{gathered} \tag{1.4}$$

is considered with $f_1 \in C^0((a,b] \times [0,\infty))$ nonnegative, $f_1(t,0) = 0$ for $t \in (a,b]$, and $g_1 \in C^0([a,b] \times (0,A/\alpha])$ positive. The authors presented conditions guaranteeing that for sufficiently small positive λ problem, (1.4) has a positive solution and for sufficiently large λ, it has a dead core solution (see [2, Theorem 17]). We notice that the inspiration for paper [2] were the results by Bobisud [3] dealing with the Robin problem

$$\begin{gathered} u'' = \lambda g_2(u), \\ -u'(-1) + \alpha u(-1) = A, \quad u'(1) + \alpha u(1) = A, \quad \alpha, A > 0, \end{gathered} \tag{1.5}$$

where $g_2 \in C^1(0,A/\alpha]$ is positive. Bobisud proved that if $g_2 \in L^1[0,A/\alpha]$, then for λ sufficiently large problem (1.5) has a dead core solution. In [1] the authors considered positive and dead core solutions of the Dirichlet problem

$$\begin{gathered} (\phi(u'))' = \lambda f_2(t,u,u'), \\ u(0) = A, \quad u(T) = A, \quad A > 0. \end{gathered} \tag{1.6}$$

Here $f_2 \in \mathrm{Car}([0,T] \times (0,A) \times (\mathbb{R} \setminus \{0\}))$ and f_2 is singular at the value 0 of its first space variable and admits singularity at the value A of its first one and at the value 0 of its second one.

The results presented in this paper improve and extend the corresponding results in [2].

In this paper, we work with the following conditions on the functions ϕ, f, and g in the differential equation (1.1).

(H_1) $\phi \in C^0[0,\infty)$ is increasing, $\lim_{x\to\infty}\phi(x) = \infty$, and $\phi(0) = 0$.

(H_2) $f \in \mathrm{Car}((0,T]\times[0,\infty))$ is nonnegative and $f(t,0) = 0$ for a.e. $t \in [0,T]$.

(H_3) $g \in \mathrm{Car}([0,T]\times D)$, $D = (0,A/\alpha]\times[0,\infty)$, $g(t,x,y)$ is positive on $[0,T]\times D$ and singular at $x = 0$,

$$g(t,x,y) \le p(x)\omega(y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in D \tag{1.7}$$

with $p : (0,A/\alpha] \to (0,\infty)$ nonincreasing, $p \in L^1[0,A/\alpha]$, $\omega : [0,\infty) \to (0,\infty)$ nondecreasing and

$$\int_0^\infty \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,ds = \infty. \tag{1.8}$$

(H_4) For each $B > 0$, there exists a positive constant m_B such that $m_B \le g(t,x,y)$ for a.e. $t \in [0,T]$ and all $(x,y) \in (0,A/\alpha]\times[0,B]$.

Define $\phi^* \in C^0(\mathbb{R})$, $f^* \in \mathrm{Car}((0,T]\times\mathbb{R})$, and $g_n \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$, $n \in \mathbb{N}$, by the formulas

$$\begin{gathered}
\phi^*(x) = \begin{cases} \phi(x) & \text{for } x \in [0,\infty), \\ -\phi(-x) & \text{for } x \in (-\infty,0), \end{cases} \\
f^*(t,y) = \begin{cases} f(t,y) & \text{for } t \in (0,T],\ y \in [0,\infty), \\ y & \text{for } t \in (0,T],\ y \in (-\infty,0), \end{cases} \\
g_n(t,x,y) = \begin{cases} g_n^*(t,x,y) & \text{for } t \in [0,T],\ (x,y) \in \mathbb{R}\times[0,\infty), \\ g_n^*(t,x,0) & \text{for } t \in [0,T],\ (x,y) \in \mathbb{R}\times(-\infty,0), \end{cases}
\end{gathered} \tag{1.9}$$

where

$$g_n^*(t,x,y) = \begin{cases} g\left(t,\dfrac{A}{\alpha},y\right) & \text{for } t \in [0,T],\ (x,y) \in \left(\dfrac{A}{\alpha},\infty\right)\times[0,\infty), \\ g(t,x,y) & \text{for } t \in [0,T],\ (x,y) \in \left[\dfrac{A}{2n\alpha},\dfrac{A}{\alpha}\right]\times[0,\infty), \\ \left[\phi\left(\dfrac{A}{2n\alpha}\right)\right]^{-1}\phi(x)g\left(t,\dfrac{A}{2n\alpha},y\right) & \text{for } t \in [0,T],\ (x,y) \in \left[0,\dfrac{A}{2n\alpha}\right)\times[0,\infty), \\ 0 & \text{for } t \in [0,T],\ (x,y) \in (-\infty,0)\times[0,\infty). \end{cases} \tag{1.10}$$

We have due to (H_3),

$$0 < g_n(t,x,y) \le p(x)\omega(y) \quad \text{for a.e. } t \in [0,T] \text{ and each } (x,y) \in \left(0, \frac{A}{\alpha}\right] \times [0,\infty), \tag{1.11}$$

and due to (H_4),

for each $B > 0$, there exists a positive constant m_B such that

$$m_B \le g_n(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \left[\frac{A}{2n\alpha}, \frac{A}{\alpha}\right] \times [0,B],\ n \in \mathbb{N}. \tag{1.12}$$

Since $f^*(t,0) = 0$ for a.e. $t \in [0,T]$, $g_n(t,x,y) = 0$ for a.e. $t \in [0,T]$ and each $(x,y) \in (-\infty,0] \times \mathbb{R}$ and $\lim_{n\to\infty} g_n(t,x,y) = g(t,x,y)$ for a.e. $t \in [0,T]$ and each $(x,y) \in (0,A/\alpha] \times [0,\infty)$, we consider the existence of positive solutions, pseudo dead core solutions and dead core solutions of problem (1.1), (1.2) by considering solutions of the sequence of auxiliary regular problems

$$(\phi^*(u'(t)))' + f^*(t,u'(t)) = \lambda g_n(t,u(t),u'(t)), \quad \lambda > 0, \tag{1.13}$$

$$u'\left(\frac{1}{n}\right) = 0, \qquad \beta u'(T) + \alpha u(T) = A, \quad \beta \ge 0,\ \alpha, A > 0. \tag{1.14}$$

We may assume without loss of generality that $1/n < T$ for all $n \in \mathbb{N}$, otherwise we consider $n \in \mathbb{N}'$ where $\mathbb{N}' = \{n \in \mathbb{N} : 1/n < T\}$. A function $u \in C^1[1/n,T]$ is called a solution of problem (1.13), (1.14) if $\phi(u') \in AC[1/n,T]$, u satisfies (1.14) and (1.13) holds for a.e. $t \in [1/n,T]$.

We introduce also the notion of a sequential solution of problem (1.1), (1.2). We say that $u \in C^0[0,T]$ is *a sequential solution of problem* (1.1), (1.2) if there exists a subsequence $\{k_n\}$ such that $\lim_{n\to\infty} u_{k_n}^{(j)}(t) = u^{(j)}(t)$ locally uniformly on $(0,T]$ for $j = 0,1$, where u_{k_n} is a solution of problem (1.13), (1.14) with k_n instead of n. In Section 3 (see Theorem 3.1), we show that any sequential solution of problem (1.1), (1.2) is either a positive solution or a pseudo dead core solution or a dead core solution of this problem. Our results are proved by a combination of the method of lower and upper functions with regularization and sequential techniques.

The next part of our paper is divided into two sections. In Section 2, we discuss existence and properties of solutions to the auxiliary regular problem (1.13), (1.14). The main results are given in Section 3. Under assumptions (H_1)–(H_3), for each $\lambda > 0$, problem (1.1), (1.2) has a sequential solution and any sequential solution is either a positive solution or a pseudo dead core solution or a dead core solution (Theorem 3.1). Corollary 3.2 shows that for sufficiently small λ, all sequential solutions of problem (1.1), (1.2) are positive solutions and under the additional assumption (H_4) all sequential solutions are dead core solutions if λ is sufficiently large by Corollary 3.3. Finally, Corollary 3.4 states a relation between sequential solutions of problem (1.1), (1.2) with distinct values of parameter λ. An example demonstrates the application of our results.

2. Auxiliary regular problems

The properties of solutions of problem (1.13), (1.14) are presented in the following lemma.

Lemma 2.1. *Let (H_1)–(H_3) hold and let u_n be a solution of problem (1.13), (1.14). Then*

$$0 < u_n(t) \le \frac{A}{\alpha} \quad \textit{for } t \in \left[\frac{1}{n}, T\right], \tag{2.1}$$

and there exists a positive constant S independent of n (and depending on λ) such that

$$0 \le u_n'(t) < S \quad \textit{for } t \in \left[\frac{1}{n}, T\right]. \tag{2.2}$$

Proof. We start by showing that $u_n' \ge 0$ on $[1/n,T]$. Suppose that $\min\{u_n'(t) : 1/n \le t \le T\} = u_n'(t_1) < 0$. Since $u_n'(1/n) = 0$ by (1.14), there exists $\xi \in [1/n,t_1)$ such that $u_n'(\xi) = 0$ and $u_n' < 0$ on $(\xi,t_1]$. Consequently,

$$\left(\phi^*\left(u_n'(t)\right)\right)' = \lambda g_n\left(t,u_n(t),u_n'(t)\right) - u_n'(t) > 0 \quad \text{for a.e. } t \in [\xi,t_1]. \tag{2.3}$$

Integrating $(\phi^*(u_n'(t)))' > 0$ over $[\xi,t] \subset [\xi,t_1]$ yields $\phi^*(u_n'(t)) > 0$ for $t \in (\xi,t_1]$. Hence $u_n' > 0$ on $(\xi,t_1]$, which is impossible. We have shown that

$$u_n'(t) \ge 0 \quad \text{for } t \in \left[\frac{1}{n}, T\right], \tag{2.4}$$

and consequently, $u_n(1/n) = \min\{u_n(t) : 1/n \le t \le T\}$. We claim that $u_n(1/n) > 0$ and so $u_n > 0$ on $[1/n,T]$. Suppose $u_n(1/n) \le 0$. Put $\tau = \max\{t \in [1/n,T] : u_n(s) \le 0 \text{ for } s \in [1/n,t]\}$. We can see that $\tau \ge 1/n$. If $\tau = 1/n$, then $u_n(\tau) = 0$ and, by (1.14), $u_n'(\tau) = 0$. Let $\tau > 1/n$. Then $(\phi^*(u_n'(t)))' = -f(t,u_n'(t)) \le 0$ for a.e. $t \in [1/n,\tau]$ and integrating $(\phi^*(u_n'(t)))' \le 0$ over $[1/n,t] \subset [1/n,\tau]$ gives $\phi^*(u_n'(t)) \le 0$ on $[1/n,\tau]$. Hence $u_n' \le 0$ on $[1/n,\tau]$, which combining with (2.4) yields $u_n'(t) = 0$ for $t \in [1/n,\tau]$. If $\tau = T$, we have $u_n(T) = u_n(1/n)$ and therefore, $A = \alpha u_n(1/n)$ by (1.14), contrary to $u_n(1/n) \le 0$. It follows that $\tau < T$. Then $u_n(\tau) = 0$ and $u_n'(\tau) = 0$. We have proved that $\tau \in [1/n,T)$, $u_n(\tau) = 0$, $u_n'(\tau) = 0$, and, by the definition of τ, $u_n > 0$ on $(\tau,T]$. Put $v(t) = \max\{u_n'(s) : \tau \le s \le t\}$ for $t \in [\tau,T]$. Then v is continuous and nondecreasing on $[\tau,T]$, $v(\tau) = 0$, and $v > 0$ on $(\tau,T]$. Let $t_* \in (\tau,\tau+1] \cap (\tau,T]$ be such that $0 \le u_n(t) \le A/2n\alpha$ for $t \in [\tau,t_*]$. Then

$$\left(\phi\left(u_n'(t)\right)\right)' = -f\left(t,u_n'(t)\right) + \lambda g_n\left(t,u_n(t),u_n'(t)\right) \le B\phi\left(u_n(t)\right)r(t) \tag{2.5}$$

for a.e. $t \in [\tau,t_*]$, where $B = \lambda[\phi(A/2n\alpha)]^{-1}$ and $r(t) = g(t,A/2n\alpha,u_n'(t))$. Clearly, $r \in L^1[\tau,t_*]$ and $r > 0$ a.e. on $[\tau,t_*]$. Integrating $(\phi(u_n'(t)))' \le B\phi(u_n(t))r(t)$ over $[\tau,t]$ yields

$$\phi\left(u_n'(t)\right) \le B\int_\tau^t \phi\left(u_n(s)\right)r(s)ds \le B\phi\left(u_n(t)\right)\int_\tau^t r(s)ds, \tag{2.6}$$

and using $u_n(t) = \int_\tau^t u_n'(s)ds \le v(t)(t-\tau) \le v(t)$, we get

$$\phi\left(u_n'(t)\right) \le B\phi\left(v(t)\right)\int_\tau^t r(s)ds, \quad t \in [\tau,t_*]. \tag{2.7}$$

Hence $u_n'(t) \le \phi^{-1}(B\phi(v(t))\int_\tau^t r(s)ds)$ and therefore

$$\begin{aligned} v(t) &= \max\{u_n'(s) : \tau \le s \le t\} \le \max\left\{\phi^{-1}(B\phi(v(s))\int_\tau^s r(z)dz) : \tau \le s \le t\right\} \\ &= \phi^{-1}\left(B\phi(v(t))\int_\tau^t r(s)ds\right), \end{aligned} \tag{2.8}$$

which gives

$$\phi(v(t)) \le B\phi(v(t))\int_\tau^t r(s)ds, \quad t \in [\tau, t_*]. \tag{2.9}$$

Since we know that $v > 0$ on $(\tau, t_*]$, it follows that $1 \le B\int_\tau^t r(s)ds$ for $t \in (\tau, t_*]$, which is impossible. Hence $u_n > 0$ on $[1/n, T]$. We conclude from the last inequality, from (2.4) and from $u_n(T) = 1/\alpha(A - \beta u_n'(T)) \le A/\alpha$ that u_n fulfils inequality (2.1).

It remains to verify (2.2) with a positive constant S. By (1.11),

$$\begin{aligned} (\phi(u_n'(t)))' &= -f(t, u_n'(t)) + \lambda g_n(t, u_n(t), u_n'(t)) \le \lambda g_n(t, u_n(t), u_n'(t)) \\ &\le \lambda p(u_n(t))\omega(u_n'(t)) \end{aligned} \tag{2.10}$$

and therefore

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega(u_n'(t))} \le \lambda p(u_n(t))u_n'(t) \tag{2.11}$$

for a.e. $t \in [1/n, T]$. Integrating (2.11) from $1/n$ to T yields (see (2.1))

$$\int_0^{\phi(u_n'(t))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}ds \le \lambda\int_{u_n(1/n)}^{u_n(t)} p(s)ds \le \lambda\int_0^{A/\alpha} p(s)ds \tag{2.12}$$

for $t \in [1/n, T]$. By (H_3), there exists a positive constant S_1 independent of n such that

$$\int_0^v \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}ds > \lambda\int_0^{A/\alpha} p(s)ds, \tag{2.13}$$

whenever $v \ge S_1$. Hence (2.12) shows that $u_n' < S$ on $[1/n, T]$, where $S = \phi^{-1}(S_1)$. □

In order to prove the existence of a solution of problem (1.13), (1.14), we use the method of lower and upper functions. Let $a \in (0, T)$, $h \in \mathrm{Car}([a, T] \times \mathbb{R}^2)$, and let ϕ^* be given in (1.9) where ϕ satisfies (H_1). Consider the boundary value problem

$$\begin{gathered} (\phi^*(u'(t)))' = h(t, u(t), u'(t)), \\ u'(a) = 0, \qquad \beta u'(T) + \alpha u(T) = A, \quad \beta \ge 0,\ \alpha, A > 0. \end{gathered} \tag{2.14}$$

We say that $v \in C^1[a, T]$ is a *lower function of problem* (2.14) if $\phi^*(v') \in AC[a, T]$, $(\phi^*(v'(t)))' \ge h(t, v(t), v'(t))$ for a.e. $t \in [a, T]$, $v'(a) \ge 0$, $\beta v'(T) + \alpha v(T) \le A$. If the reverse inequalities hold, we say that v is an *upper function of problem* (2.14).

For the solvability of problem (2.14), the following result (which is a special case of the general existence principle by Cabada and Pouso [4, page 230]) holds.

Proposition 2.2. *If there exists a lower function v and an upper function z of problem (2.14), $v(t) \le z(t)$ for $t \in [a,T]$ and there exists $q \in L^1[a,T]$ such that*

$$|h(t,x,y)| \le q(t) \quad \text{for a.e. } t \in [a,T] \text{ and all } v(t) \le x \le z(t),\ y \in \mathbb{R}, \tag{2.15}$$

then problem (2.14) has a solution u and $v(t) \le u(t) \le z(t)$ for $t \in [a,T]$.

We are now in a position to give the existence result for problem (1.13), (1.14).

Lemma 2.3. *Let (H_1)–(H_3) hold. Then problem (1.13), (1.14) has a solution and any solution u_n satisfies inequalities (2.1) and (2.2), where S is a positive constant independent of n.*

Proof. Let S be the positive constant in Lemma 2.1. Put

$$h(t,x,y) = \chi(y)\left[-f^*(t,y) + \lambda g_n(t,x,y)\right] \quad \text{for } t \in \left[\frac{1}{n},T\right],\ (x,y) \in \mathbb{R}^2, \tag{2.16}$$

where

$$\chi(y) = \begin{cases} 1 & \text{for } |y| \le S, \\ 2 - \dfrac{|y|}{S} & \text{for } S < |y| \le 2S, \\ 0 & \text{for } |y| > 2S. \end{cases} \tag{2.17}$$

Since $h(t,0,0) = 0$ and $h(t,A/\alpha,0) = \lambda g_n(t,A/\alpha,0) \ge 0$ for a.e. $t \in [1/n,T]$, we see that $v = 0$ and $z = A/\alpha$ is a lower and an upper function of problem (2.14) with $a = 1/n$. It follows from $f^* \in \mathrm{Car}([1/n,T] \times \mathbb{R})$ and $g_n \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$ that

$$|h(t,x,y)| \le q(t) \quad \text{for a.e. } t \in \left[\frac{1}{n},T\right] \text{ and all } 0 \le x \le \frac{A}{\alpha},\ y \in \mathbb{R}, \tag{2.18}$$

where $q \in L^1[1/n,T]$. Hence Proposition 2.2 guarantees the existence of a solution u_n of problem (2.14) satisfying $0 \le u_n(t) \le A/\alpha$ for $t \in [1/n,T]$. If $\min\{u_n'(t) : 1/n \le t \le T\} = u_n'(t_1) < 0$, we can prove as in the first part of the proof of Lemma 2.1 that there exists $\xi \in [1/n,t_1)$ such that $u_n'(\xi) = 0$ and $u_n' < 0$ on $(\xi,t_1]$. We now deduce from the equality

$$\left(\phi^*(u_n'(t))\right)' + \chi(u_n'(t))f^*(t,u_n'(t)) = \lambda\chi(u_n'(t))g_n(t,u_n(t),u_n'(t)) \tag{2.19}$$

for a.e. $t \in [1/n,T]$ that $(\phi^*(u_n'(t)))' \ge -u_n'(t)\chi(u_n'(t)) \ge 0$ for a.e. $t \in [\xi,t_1]$. Integrating $(\phi^*(u_n'(t)))' \ge 0$ over $[\xi,t]$ gives $\phi^*(u_n'(t)) \ge 0$ for $t \in [\xi,t_1]$, contrary to $u_n' < 0$ on $(\xi,t_1]$. Consequently, $u_n' \ge 0$ on $[1/n,T]$ and

$$\left(\phi(u_n'(t))\right)' = \left(\phi^*(u_n'(t))\right)' \le \lambda\chi(u_n'(t))g_n(t,u_n(t),u_n'(t)) \le \lambda p(u_n(t))\omega(u_n'(t)) \tag{2.20}$$

for a.e. $t \in [1/n,T]$. Now the second part of the proof of Lemma 2.1 (see (2.11)) shows that $u_n' < S$ on $[1/n,T]$. Hence $h(t,u_n(t),u_n'(t)) = -f^*(t,u_n'(t)) + \lambda g_n(t,u_n(t),u_n'(t))$ for $t \in [1/n,T]$ and u_n is a solution of problem (1.13), (1.14). The fact that any solution u_n of problem (1.13), (1.14) satisfies inequalities (2.1) and (2.2) follows immediately from Lemma 2.1. $\square$

The next two results will be used for the proof of the existence of positive solutions of problem (1.1), (1.2).

Lemma 2.4. *Let* (H_1)*–*(H_3) *hold. Then there exists a nonincreasing function* $\Lambda : (0,\infty) \to (0,\infty)$ *such that for all* $\lambda > 0$, $n \in \mathbb{N}$, *and each solution* u_n *of problem (1.13), (1.14), the estimate*

$$u_n(T) \geq \Lambda(\lambda) \tag{2.21}$$

is true.

Proof. If $\beta = 0$, then $u_n(T) = A/\alpha$ for all solution u_n of problem (1.13), (1.14). Let $\beta > 0$. By (H_3), $p \in L^1[0, A/\alpha]$, and therefore there exists a nonincreasing function $\varrho : (0,\infty) \to (0,\infty)$ such that for any interval $[c,d] \subset [0, A/\alpha]$, we have

$$\int_c^d p(s)ds < \frac{1}{\lambda}\int_0^{\phi(A/(2\beta))} \frac{\phi^{-1}}{\omega(\phi^{-1}(s))}ds \tag{2.22}$$

provided $d - c < \varrho(\lambda)$. We claim that

$$u_n(T) \geq \min\left\{\frac{A}{2\alpha}, \varrho(\lambda)\right\} \quad \text{for } n \in \mathbb{N}, \tag{2.23}$$

where u_n is a solution of problem (1.13), (1.14). If $u_n(T) \geq A/2\alpha$ for $n \in \mathbb{N}$, then (2.23) is true. If not, $u_{n_0}(T) < A/2\alpha$ for some $n_0 \in \mathbb{N}$. Then $\beta u'_{n_0}(T) = A - \alpha u_{n_0}(T) > A/2$ and so $u'_{n_0}(T) > A/2\beta$. Hence (see (2.12) with $n = n_0$ and $t = T$)

$$\int_0^{\phi(A/(2\beta))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}ds < \int_0^{\phi(u'_{n_0}(T))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}ds \leq \lambda \int_{u_{n_0}(1/n_0)}^{u_{n_0}(T)} p(s)ds \tag{2.24}$$

and on account of (2.22), we get $u_{n_0}(T) - u_{n_0}(1/n_0) \geq \varrho(\lambda)$. Therefore, $u_{n_0}(T) \geq \varrho(\lambda)$ and $u_{n_0}(T) \geq \min\{A/2\alpha, \varrho(\lambda)\}$. We have proved that (2.23) holds. Consequently, inequality (2.21) is satisfied with the nonincreasing function $\Lambda(\lambda) = \min\{A/2\alpha, \varrho(\lambda)\}$ for $\lambda \in (0,\infty)$. □

Lemma 2.5. *Let* (H_1)*–*(H_3) *hold. Then there exist* $\lambda_0 > 0$ *and* $d > 0$ *such that for all* $\lambda \in (0, \lambda_0]$, $t \in [1/n, T]$, $n \in \mathbb{N}$, *and each solution* u_n *of problem (1.13), (1.14), the estimate*

$$u_n(t) > d \tag{2.25}$$

is true.

Proof. Let Λ be the function in Lemma 2.4. Notice that Λ is positive and nonincreasing on $(0,\infty)$. Put

$$\lambda_0 = \min\left\{1, \left[\int_0^{A/\alpha} p(s)ds\right]^{-1} \int_0^{\phi[\Lambda(1)/(2T)]} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}ds\right\}. \tag{2.26}$$

Let $\lambda \in (0,\lambda_0]$ be arbitrary but fixed and let u_n be a solution of problem (1.13), (1.14). Then (see (2.12))

$$\begin{aligned}\int_0^{\phi(u_n'(t))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}ds &\le \lambda\int_0^{A/\alpha} p(s)ds \le \lambda_0\int_0^{A/\alpha} p(s)ds \\ &\le \int_0^{\phi[\Lambda(1)/(2T)]} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}ds\end{aligned} \tag{2.27}$$

for $t \in [1/n, T]$ and therefore $u_n' \le \Lambda(1)/2T$ on this interval. Whence $u_n(T) - u_n(1/n) = u_n'(\xi)(T - 1/n) \le \Lambda(1)/2T(T - 1/n)$, where $\xi \in (1/n, T)$. Since, by Lemma 2.4, $u_n(T) \ge \Lambda(\lambda)$ and $\Lambda(\lambda) \ge \Lambda(\lambda_0) \ge \Lambda(1)$, we have $u_n(1/n) \ge u_n(T) - \Lambda(1)/2T(T - 1/n) > \Lambda(1)/2$. Since $u_n' \ge 0$ on $[1/n, T]$, inequality (2.25) holds with $d = \Lambda(1)/2$. □

The following results will be needed for the existence of dead core solutions of problem (1.1), (1.2).

Lemma 2.6. *Let* (H_1)*–*(H_4) *hold. Then for every* $c \in (0,T)$*, there exists* $\lambda_c > 0$ *such that for all* $\lambda > \lambda_c$ *and each solution* u_n *of problem (1.13), (1.14), the equality*

$$\lim_{n\to\infty} u_n(c) = 0 \tag{2.28}$$

is true.

Proof. Fix $c \in (0,T)$, choose $\varepsilon \in (0, A/\alpha)$ and set $B = 3A/\alpha(T-c)$. Due to (H_2) and (H_4) (see (1.12)), there exist $\varphi \in L^1[c,T]$ and $m_B > 0$ such that

$$\begin{aligned} 0 \le f(t,y) \le \varphi(t) \quad &\text{for a.e. } t \in [c,T] \text{ and all } y \in [0,B], \\ m_B \le g_n(t,x,y) \quad &\text{for a.e. } t \in [c,T] \text{ and all } (x,y) \in \left[\frac{A}{2n\alpha}, \frac{A}{\alpha}\right] \times [0,B]. \end{aligned} \tag{2.29}$$

Put

$$\lambda_c = \frac{3(\phi(B) + \|\varphi\|_*)}{m_B(T-c)}, \tag{2.30}$$

where $\|\varphi\|_* = \int_c^T \varphi(s)ds$. We claim that if $\lambda > \lambda_c$ in (1.13), then

$$u_n(c) < \varepsilon \quad \text{for } n \ge \max\left\{\frac{1}{c}, \frac{A}{2\alpha\varepsilon}\right\}, \tag{2.31}$$

where u_n is a solution of (1.13), (1.14). If not, there exists $\lambda_0 > \lambda_c$ and $n_0 \ge \max\{1/c, A/2\alpha\varepsilon\}$ such that $u_{n_0}(c) \ge \varepsilon$, where u_{n_0} is a solution of problem (1.13), (1.14) with $\lambda = \lambda_0$ and $n = n_0$. Since $u_{n_0}' \ge 0$ on $[1/n_0, T]$, we have $u_{n_0} \ge \varepsilon(\ge A/2\alpha n_0)$ on $[c,T]$. We now show that $u_{n_0}'(c_1) > B$ for some $c_1 \in [c, (2c+T)/3]$. Assuming the contrary, then $u_{n_0}' \le B$ on $[c, (2c+T)/3]$ and consequently, $g_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) \ge m_B$ and $f(t, u_{n_0}'(t)) \le \varphi(t)$ for a.e.

$t \in [c, (2c+T)/3]$. Hence

$$\begin{aligned}
\phi\left(u'_{n_0}\left(\frac{2c+T}{3}\right)\right) - \phi(u'_{n_0}(c)) &= \int_c^{(2c+T)/3} (\phi(u'_{n_0}(t)))' dt \\
&= \int_c^{(2c+T)/3} [-f(t, u'_{n_0}(t)) + \lambda_0 g_{n_0}(t, u_{n_0}(t), u'_{n_0}(t))] dt \\
&\geq \int_c^{(2c+T)/3} [-\varphi(t) + \lambda_0 m_B] dt \\
&> -\|\varphi\|_* + \lambda_c m_B \frac{T-c}{3} \\
&= \phi(B).
\end{aligned} \tag{2.32}$$

Therefore $u'_{n_0}((2c+T)/3) > B$, which is impossible. It follows that $u'_{n_0}(c_1) > B$ for some $c_1 \in [c, (2c+T)/3]$. If $u'_{n_0} \geq B$ on $[c_1, T]$, then

$$u_{n_0}(T) \geq u_{n_0}(T) - u_{n_0}(c_1) = \int_{c_1}^T u'_{n_0}(t) dt \geq B(T - c_1) = \frac{3A(T-c_1)}{\alpha(T-c)} \geq \frac{2A}{\alpha}, \tag{2.33}$$

contrary to $u_{n_0}(T) \leq A/\alpha$. Therefore $u'_{n_0} \geq B$ on $[c_1, T]$ is false. Set $\mathcal{M} = \{t \in [c_1, T] : u'_{n_0}(t) < B\}$. Then $\mathcal{M}$ is an open and nonempty set, and as a consequence, $\mathcal{M}$ is the union of at most countable set $\mathbb{J}$ of mutually disjoint intervals (a_k, b_k), $\mathcal{M} = \sum_{k \in \mathbb{J}} (a_k, b_k)$. Since $\phi(u'_{n_0}) \in AC[c, T]$ and $\phi(u'_{n_0}(b_k)) - \phi(u'_{n_0}(a_k)) = 0$ for all $k \in \mathbb{J}$ with at most one exception when the difference is negative, we have

$$\int_{\mathcal{M}} (\phi(u'_{n_0}(t)))' dt \leq 0. \tag{2.34}$$

Denoting the characteristic function of the set $\mathcal{M}$ by $\chi_{\mathcal{M}}$ and the Lebesgue measure of $\mathcal{M}$ by meas$(\mathcal{M})$, we have

$$\begin{aligned}
0 &\geq \int_{c_1}^T (\phi(u'_{n_0}(t)))' \chi_{\mathcal{M}}(t) dt \geq \int_{\mathcal{M}} (-\varphi(t) + \lambda_0 m_B) dt \\
&> -\|\varphi\|_* + \lambda_c m_B \text{ meas } (\mathcal{M}) \\
&= -\|\varphi\|_* + \text{ meas } (\mathcal{M})(\phi(B) + \|\varphi\|_*) \frac{3}{T-c}.
\end{aligned} \tag{2.35}$$

In particular, meas$(\mathcal{M}) < (T-c)/3$. Hence $u'_{n_0} \geq B$ on the measurable set $[c_1, T] \setminus \mathcal{M}$ and meas$([c_1, T] \setminus \mathcal{M}) > (T-c)/3$. Then

$$u_{n_0}(T) - u_{n_0}(c_1) = \int_{c_1}^T u'_{n_0}(t) dt \geq \int_{[c_1, T] \setminus \mathcal{M}} u'_{n_0}(t) dt > \frac{B(T-c)}{3} = \frac{A}{\alpha} \tag{2.36}$$

and $u_{n_0}(T) > A/\alpha$, which is impossible. It follows that (2.31) is true.

We have proved that for each $\lambda > \lambda_c$ in (1.13) and for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $(0 \leq) u_n(c) < \varepsilon$ for all $n \geq n_\varepsilon$ which proves that (2.28) is true. □

Our next result concerns sequences of solutions of problem (1.13), (1.14).

Lemma 2.7. *Let* (H_1)–(H_3) *hold. Let* u_n *be a solution of problem, (1.13), (1.14) and let* $m \in \mathbb{N}$. *Then the sequence* $\{u_n\}_{n\geq m} \subset C^1[1/m,T]$ *is equicontinuous on the interval* $[1/m,T]$.

Proof. First we define functions $H \in C^0[0,\infty)$ and $P \in AC[0,A/\alpha]$ by the formulas

$$\begin{aligned} H(v) &= \int_0^{\phi(v)} \phi^{-1}(s)ds \quad \text{for } v \in [0,\infty), \\ P(v) &= \int_0^{v} p(s)ds \quad \text{for } v \in \left[0,\frac{A}{\alpha}\right]. \end{aligned} \tag{2.37}$$

By Lemma 2.1, u_n satisfies inequalities (2.1) and (2.2), where S is a positive constant. Hence $\{u_n\}_{n\geq m}$ is equicontinuous on $[1/m,T]$ and so is $\{P(u_n)\}_{n\geq m}$. It follows from (H_2) that $0 \leq f(t,y) \leq \varrho(t)$ for a.e. $t \in [1/m,T]$ and each $y \in [0,S]$, where $\varrho \in L^1[1/m,T]$ and consequently,

$$0 \leq f(t,u_n'(t)) \leq \varrho(t) \quad \text{for a.e. } t \in \left[\frac{1}{m},T\right] \text{ and all } n \geq m. \tag{2.38}$$

Let us choose an arbitrary $\varepsilon > 0$. Then there exists $\nu > 0$ such that

$$0 \leq P(u_n(t_2)) - P(u_n(t_1)) < \varepsilon, \qquad 0 \leq \int_{t_1}^{t_2} \varrho(t)dt < \varepsilon, \tag{2.39}$$

whenever $1/m \leq t_1 < t_2 \leq T$ and $t_2 - t_1 < \nu$. From the inequalities (for $n \geq m$)

$$\begin{aligned} (\phi(u_n'(t)))' u_n'(t) &\geq -f(t,u_n'(t))u_n'(t) \geq -S\varrho(t), \\ (\phi(u_n'(t)))' u_n'(t) &\leq \lambda g_n(t,u_n(t),u_n'(t)) \leq \lambda\omega(S)p(u_n(t))u_n'(t) \end{aligned} \tag{2.40}$$

for a.e. $t \in [1m,T]$, we obtain

$$(\phi(u_n'(t)))' u_n'(t) \geq -S\varrho(t) \quad \text{for a.e. } t \in \left[\frac{1}{m},T\right], \tag{2.41}$$

$$(\phi(u_n'(t)))' u_n'(t) \leq \lambda\omega(S)p(u_n(t))u_n'(t) \quad \text{for a.e. } t \in \left[\frac{1}{m},T\right]. \tag{2.42}$$

Let $1/m \leq t_1 < t_2 \leq T$, $t_2 - t_1 < \nu$ and $n \geq m$. Integrating (2.41) and (2.42) over $[t_1,t_2]$ yields

$$\begin{aligned} H(u_n'(t_2)) - H(u_n'(t_1)) &\geq -S\int_{t_1}^{t_2} \varrho(t)dt > -S\varepsilon, \\ H(u_n'(t_2)) - H(u_n'(t_1)) &\leq \lambda\omega(S)\int_{u_n(t_1)}^{u_n(t_2)} p(s)ds \\ &= \lambda\omega(S)[P(u_n(t_2)) - P(u_n(t_1))] \\ &< \lambda\omega(S)\varepsilon. \end{aligned} \tag{2.43}$$

Summarizing, we have $|H(u_n'(t_2)) - H(u_n'(t_1))| < V\varepsilon$ for $1/m \leq t_1 < t_2 \leq T$, $t_2 - t_1 < \nu$, and $n \geq m$, where $V = \max\{S,\lambda\omega(S)\}$. Hence $\{H(u_n')\}_{n\geq m}$ is equicontinuous on $[1/m,T]$

and since H is continuous and increasing on $[0,\infty)$ and $\{u_n'\}_{n\geq m}$ is bounded on $[1/m,T]$, we see that $\{u_n'\}_{n\geq m}$ is equicontinuous on $[1/m,T]$. □

We now state a relation between solutions of problem (1.13), (1.14) with distinct values of parameter λ in (1.13).

Lemma 2.8. *Let (H_1)–(H_3) hold and let $0<\lambda_1<\lambda_2$. If u_n is a solution of problem (1.13), (1.14) with $\lambda=\lambda_1$, then there exists a solution v_n of problem (1.13), (1.14) with $\lambda=\lambda_2$ such that*

$$0\leq v_n(t)\leq u_n(t) \quad \text{for } t\in\left[\frac{1}{n},T\right]. \tag{2.44}$$

Proof. Let $j=1,2$ and let S_j be a positive constant in Lemma 2.1 which gives a priori bound for the derivative of solutions to problem (1.13), (1.14) with $\lambda=\lambda_j$. Put

$$h_j(t,x,y)=\chi(y)\left[-f^*(t,y)+\lambda_j g_n(t,x,y)\right] \quad \text{for } t\in\left[\frac{1}{n},T\right],\ (x,y)\in\mathbb{R}^2,\ j=1,2, \tag{2.45}$$

where the function χ is given in (2.17) with $S=\max\{S_1,S_2\}$. Consider the differential equations

$$\left(\phi^*\left(u'(t)\right)\right)'=h_j\left(t,u(t),u'(t)\right), \quad j=1,2. \tag{2.46}$$

Let u_n be a solution of problem (1.13), (1.14) with $\lambda=\lambda_1$. Since $0\leq u_n'(t)<S_1$ for $t\in[1/n,T]$, u_n is also a solution of problem (2.46), (1.14) with $j=1$. The function $v=0$ is a lower function of problem (2.46), (1.14) with $j=2$, and the relations

$$\begin{aligned}\left(\phi\left(u_n'(t)\right)\right)'&=-f\left(t,u_n'(t)\right)+\lambda_1 g_n\left(t,u_n(t),u_n'(t)\right)\\&<-f\left(t,u_n'(t)\right)+\lambda_2 g_n\left(t,u_n(t),u_n'(t)\right)\\&=h_2\left(t,u_n(t),u_n'(t)\right)\end{aligned} \tag{2.47}$$

show that u_n is an upper function of this problem. Now Proposition 2.2 guarantees that problem (2.46), (1.14) with $j=2$ has a solution v_n satisfying (2.44). Arguing as in the proof of Lemma 2.3, v_n is a solution of problem (1.13), (1.14) with $\lambda=\lambda_2$. □

3. Main results and an example

Theorem 3.1. *Let (H_1)–(H_3) hold. Then problem (1.1), (1.2) has a sequential solution for each $\lambda>0$. Moreover, any sequential solution of problem (1.1), (1.2) is either a positive solution or a pseudo dead core solution or a dead core solution.*

Proof. Fix $\lambda>0$. Let u_n be a solution of problem (1.13), (1.14) where its existence is guaranteed by Lemma 2.3. Then the inequalities (2.1) and (2.2) are satisfied for $n\in\mathbb{N}$, where S is a positive constant and, by Lemma 2.7, $\{u_n'\}_{n\geq m}$ is equicontinuous on $[1/m,T]$ for each $m\in\mathbb{N}$. Applying the Arzelà-Ascoli theorem and the diagonal method, we obtain a subsequence $\{k_n\}$ such that $\{u_{k_n}^{(j)}\}$ is locally uniformly convergent on $(0,T]$ for $j=0,1$

and let $\lim_{n\to\infty} u_{k_n}(t) = \widetilde{u}(t)$ for $t \in (0,T]$. Since $u_{k_n} > 0$ and $u'_{k_n} \geq 0$ on $[1/k_n, T]$, we have $\widetilde{u} \geq 0$ and $\widetilde{u}' \geq 0$ on $(0,T]$. Hence there exists $\lim_{t\to 0^+} \widetilde{u}(t) = a \geq 0$. Put

$$u(t) = \begin{cases} \widetilde{u}(t) & \text{for } t \in (0,T], \\ a & \text{for } t = 0. \end{cases} \tag{3.1}$$

Then $u \in C^0[0,T]$ and, by the definition, u is a sequential solution of problem (1.1), (1.2).

Let u be a sequential solution of problem (1.1), (1.2). Then $u \in C^0[0,T]$ and there exists a subsequence of $\{n\}$, for simplicity denoting again by $\{n\}$, such that $\lim_{n\to\infty} u_n^{(j)}(t) = u^{(j)}(t)$ locally uniformly on $(0,T]$ for $j = 0,1$. Here u_n is a solution of problem (1.13), (1.14). Hence $\beta u'(T) + \alpha u(T) = A$, $u \in C^0[0,T] \cap C^1(0,T]$, and (see (2.2)) $0 \leq u'(t) \leq S$ for $t \in (0,T]$. We now show that $u \in C^1[0,T]$ and $u'(0) = 0$. First we prove that

$$\lim_{n\to\infty} u_n\left(\frac{1}{n}\right) = u(0). \tag{3.2}$$

For this, choose an arbitrary $\varepsilon \in (0, ST)$ and put $\delta = \varepsilon/S$. Then there exists $n_0 \in \mathbb{N}$, $n_0 > 1/\delta$, such that $|u_n(\delta) - u(\delta)| < \varepsilon$ for $n \geq n_0$. Since

$$\begin{gathered} 0 \leq u(\delta) - u(0) = u'(\xi)\delta \leq S\delta = \varepsilon, \\ 0 \leq u_n(\delta) - u_n\left(\frac{1}{n}\right) = u'_n(\tau_n)\left(\delta - \frac{1}{n}\right) < S\left(\delta - \frac{1}{n}\right) < \varepsilon \end{gathered} \tag{3.3}$$

for $n \geq n_0$, where $\xi \in (0,\delta)$ and $\tau_n \in (1/n, \delta)$, we have

$$\left| u_n\left(\frac{1}{n}\right) - u(0) \right| \leq \left| u_n\left(\frac{1}{n}\right) - u_n(\delta) \right| + \left| u_n(\delta) - u(\delta) \right| + \left| u(\delta) - u(0) \right| < 3\varepsilon \tag{3.4}$$

for $n \geq n_0$ and therefore (3.2) is true. Passing to the limit as $n \to \infty$ in (see (2.12))

$$\int_0^{\phi(u'_n(t))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))} ds \leq \lambda \int_{u_n(1/n)}^{u_n(t)} p(s) ds, \quad t \in \left[\frac{1}{n}, T\right], \tag{3.5}$$

we obtain (see (3.2))

$$\int_0^{\phi(u'(t))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))} ds \leq \lambda \int_{u(0)}^{u(t)} p(s) ds, \quad t \in (0,T]. \tag{3.6}$$

Hence $\lim_{t\to 0^+} \phi(u'(t)) = 0$, and therefore $\lim_{t\to 0^+} u'(t) = 0$. Since for $t \in (0,T)$, $(u(t) - u(0))/t = u'(\eta)$, where $\eta \in (0,T)$ by the mean value theorem, letting $t \to 0^+$ gives 0 on the right side and $u'(0)$ on the left side so that $u'(0) = 0$ as desired. Hence $u \in C^1[0,T]$ and $u'(0) = 0$. The next part of the proof is broken into three cases.

Case 1. Let $u > 0$ on $[0,T]$. Put $\varepsilon = u(0)$ and let $t_1 \in (0,T)$ be arbitrary but fixed. Then $u \geq \varepsilon$ on $[0,T]$ and there exists $n_1 \in \mathbb{N}$ such that $u_n(t) \geq \varepsilon/2$ for $t \in [t_1, T]$ and $n \geq n_1$. Therefore

$$\lim_{n\to\infty} f(t, u'_n(t)) = f(t, u'(t)), \qquad \lim_{n\to\infty} g_n(t, u_n(t), u'_n(t)) = g(t, u(t), u'(t)) \tag{3.7}$$

for a.e. $t \in [t_1, T]$ and (see (H_2), (H_3), and (1.11))

$$0 \le g_n(t, u_n(t), u_n'(t)) \le p\left(\frac{\varepsilon}{2}\right)\omega(S), \qquad 0 \le f(t, u_n'(t)) \le \varphi(t) \tag{3.8}$$

for a.e. $t \in [t_1, T]$ and $n \ge n_1$, where $\varphi \in L^1[t_1, T]$. Letting $n \to \infty$ in

$$\phi(u_n'(t)) = \phi(u_n'(T)) + \int_t^T [f(s, u_n'(s)) - \lambda g_n(s, u_n(s), u_n'(s))]ds \tag{3.9}$$

yields

$$\phi(u'(t)) = \phi(u'(T)) + \int_t^T [f(s, u'(s)) - \lambda g(s, u(s), u'(s))]ds \tag{3.10}$$

for $t \in [t_1, T]$ by the Lebesgue dominated convergence theorem. Since $t_1 \in (0, T)$ is arbitrary, (3.10) holds for $t \in (0, T]$ and, moreover, the functions $f(t, u(t))$, $g(t, u(t), u'(t))$ belong to the class $L^1_{\text{loc}}(0, T]$. Hence $\phi(u') \in AC_{\text{loc}}(0, T]$ and (1.1) is satisfied a.e. on $[0, T]$. We have proved that u is a positive solution of problem (1.1), (1.2).

Case 2. Let $u(0) = 0$ and $u > 0$ on $(0, T]$. Choose $t_1 \in (0, T)$ and put $\varepsilon = u(t_1) > 0$. Then there exists $n_1 \in \mathbb{N}$ such that $u_n(t) \ge \varepsilon/2$ for $t \in [t_1, T]$ and $n \ge n_1$. By a similar argument as in Case 1, we can verify that $\phi(u') \in AC_{\text{loc}}(0, T]$ and (1.1) is satisfied a.e. on $[0, T]$. Hence u is a pseudo dead core solution of problem (1.1), (1.2).

Case 3. Let $u = 0$ on $[0, t_0]$ for some $t_0 \in (0, T)$ and $u > 0$ on $(t_0, T]$. Then there exists $\psi \in L^1[t_0, T]$ such that $0 \le f(t, u_n'(t)) \le \psi(t)$ for a.e. $t \in [t_0, T]$. Essentially the same reasoning as in Cases 1 and 2 (now on the interval $[t_0, T]$) shows that

$$\lim_{n \to \infty} [\lambda g_n(t, u_n(t), u_n'(t)) - f(t, u_n'(t))] = \lambda g(t, u(t), u'(t)) - f(t, u'(t)) \tag{3.11}$$

for a.e. $t \in [t_0, T]$ and equality (3.10) holds for $t \in (t_0, T]$. In addition,

$$\int_{t_0}^T [\lambda g_n(t, u_n(t), u_n'(t)) - f(t, u_n'(t))]dt = \phi(u_n'(T)) - \phi(u_n'(t_0)) \le \phi(S) \tag{3.12}$$

and $\lambda g_n(t, u_n(t), u_n'(t)) - f(t, u_n'(t)) \ge -\psi(t)$ a.e. on $[t_0, T]$ and all $n \in \mathbb{N}$. Hence, by the Fatou lemma, the function $\lambda g(t, u(t), u'(t)) - f(t, u'(t))$ is Lebesgue integrable on $[t_0, T]$ and consequently, equality (3.10) holds on $[t_0, T]$. Therefore $\phi(u') \in AC[t_0, T]$ and u satisfies (1.1) a.e. on $[t_0, T]$. We have proved that u is a dead core solution of problem (1.1), (1.2). □

Theorem 3.1 guarantees that problem (1.1), (1.2) has a sequential solution for every $\lambda > 0$ and that any sequential solution is either a positive solution or a pseudo dead core solution or a dead core solution. The next corollaries show that all sequential solutions of problem (1.1), (1.2) are positive solutions for sufficiently small values of λ and dead core solutions for sufficiently large values of λ, and also that for "larger" values of λ problem (1.1), (1.2) has "smaller" sequential solutions.

Corollary 3.2. *Let* (H_1)*–*(H_3) *hold. Then there exists* $\lambda_0 > 0$ *such that for each* $\lambda \in (0,\lambda_0]$*, all sequential solutions of problem (1.1), (1.2) are positive solutions.*

Proof. Let $\lambda_0 > 0$ and $d > 0$ be given in Lemma 2.5. Choose an arbitrary $\lambda \in (0,\lambda_0]$. Then inequalities (2.25) are satisfied where u_n is a solution of problem (1.13), (1.14). Let u be a sequential solution of problem (1.1), (1.2). Then $u \in C^0[0,T]$ and $u(t) = \lim_{n\to\infty} u_{k_n}(t)$ locally uniformly on $(0,T]$ for a subsequence $\{k_n\}$. Hence $u \geq d$ on $[0,T]$, which shows that u is a positive solution. □

Corollary 3.3. *Let* (H_1)*–*(H_4) *hold. Then for each* $c \in (0,T)$*, there exists* $\lambda_c > 0$ *such that any sequential solution* u *of problem (1.1), (1.2) with* $\lambda > \lambda_c$ *satisfies*

$$u(t) = 0 \quad \textit{for } t \in [0,c]. \tag{3.13}$$

Consequently, all sequential solutions of problem (1.1), (1.2) are dead core solutions for sufficiently large values of λ*.*

Proof. Fix $c \in (0,T)$. Let $\lambda_c > 0$ be given in Lemma 2.6. Choose $\lambda > \lambda_c$. Then (2.28) holds where u_n is a solution of problem (1.13), (1.14). Let u be a sequential solution of problem (1.1), (1.2). Then $u(t) = \lim u_{k_n}(t)$ locally uniformly on $(0,T]$ for a subsequence $\{k_n\}$. Since $u \geq 0$, (2.28) shows that $u(c) = 0$, and therefore $u = 0$ on $[0,c]$ because we know that $u' \geq 0$ on $[0,T]$. □

Corollary 3.4. *Let* (H_1)*–*(H_3) *hold. Let* $0 < \lambda_1 < \lambda_2$ *and let* u *be a sequential solution of problem (1.1), (1.2) with* $\lambda = \lambda_1$*. Then there exists a sequential solution* v *of problem (1.1), (1.2) with* $\lambda = \lambda_2$ *such that*

$$0 \leq v(t) \leq u(t) \quad \textit{for } t \in [0,T]. \tag{3.14}$$

Proof. Let $u(t) = \lim_{n\to\infty} u_{k_n}(t)$ locally uniformly on $(0,T]$, where $\{k_n\}$ is a subsequence of $\{n\}$ and u_{k_n} is a solution of problem (1.13), (1.14) with k_n instead of n and with $\lambda = \lambda_1$. By Lemma 2.8, for each $n \in \mathbb{N}$, there exists a solution v_{k_n} of problem (1.13), (1.14) with $\lambda = \lambda_2$ and with k_n instead of n such that

$$0 \leq v_{k_n}(t) \leq u_{k_n}(t) \quad \text{for } t \in \left[\frac{1}{k_n}, T\right],\ n \in \mathbb{N}. \tag{3.15}$$

Since, by Lemma 2.1, $0 < v_{k_n}(t) \leq A/\alpha$, $0 \leq v'_{k_n}(t) < S_1$ for $t \in [1/k_n, T]$ and $n \in \mathbb{N}$, and $\{v'_{k_n}\}_{n\geq m}$ is equicontinuous on $[1/k_m, T]$ by Lemma 2.7, the Arzelà-Ascoli theorem and the diagonal method guarantee that $\{v^{(j)}_{k_{n'}}\}$ is locally uniformly convergent on $(0,T]$ for $j = 0,1$, where $\{k_{n'}\}$ is a subsequence of $\{k_n\}$. Set $\tilde{v}(t) = \lim_{n'\to\infty} v_{k_{n'}}(t)$ for $t \in (0,T]$ and

$$v(t) = \begin{cases} \tilde{v}(t) & \text{for } t \in (0,T], \\ \lim_{t\to 0^+} \tilde{v}(t) & \text{for } t = 0. \end{cases} \tag{3.16}$$

Then (see the first part of the proof of Theorem 3.1) $v \in C^0[0,T]$, and therefore v is a sequential solution of problem (1.1), (1.2) with $\lambda = \lambda_2$ in (1.1). From (3.15) we get $v(t) = \lim_{n'\to\infty} v_{k_{n'}}(t) \leq \lim_{n'\to\infty} u_{k_{n'}}(t) = u(t)$ for $t \in (0,T]$, which shows that (3.14) is true since $u, v \in C^0[0,T]$. □

Example 3.5. Let ϕ satisfy (H_1). Consider the differential equation

$$(\phi(u'))' + \frac{r(u')}{t^\nu} = \lambda\left(\frac{h_1(t)}{u^\beta} + h_2(t)(\phi(u'))^\gamma\right), \tag{3.17}$$

where $\nu,\gamma \in (0,\infty)$, $\beta \in (0,1)$, $r \in C^0[0,\infty)$ is positive on $(0,\infty)$, $r(0)=0$, $h_1,h_2 \in L^1[0,T]$, $0<\varepsilon \le h_1(t) \le H$, $0 \le h_2(t) \le H$ for a.e. $t \in [0,T]$ and $\int_0^\infty \phi^{-1}(s)/(1+s^\gamma)ds = \infty$. The differential equation (3.17) is the special case of (1.1) with $f(t,y) = r(y)/t^\nu$ satisfying (H_2) and $g(t,x,y) = h_1(t)/x^\beta + h_2(t)(\phi(y))^\gamma$. It follows from the inequalities $g(t,x,y) \le H(1+1/x^\beta)(1+(\phi(y))^\gamma)$ for a.e. $t \in [0,T]$ and all $(x,y) \in (0,\infty)\times[0,\infty)$ and $g(t,x,y) \ge \varepsilon(\alpha/A)^\beta$ for a.e. $t \in [0,T]$ and all $(x,y) \in (0,A/\alpha]\times[0,\infty)$ that assumptions (H_3) and (H_4) are satisfied with $p(x) = H(1+1/x^\beta)$, $\omega(y) = 1+(\phi(y))^\gamma$, and $m_B = \varepsilon(\alpha/A)^\beta$. Hence, by Theorem 3.1, problem (3.17), (1.2) has a sequential solution for every $\lambda > 0$. This sequential solution is either a positive solution or a dead core solution or a pseudo dead core solution. If λ is sufficiently small, all sequential solutions of problem (3.17), (1.2) are positive solutions by Corollary 3.2. Corollary 3.3 guarantees that all sequential solutions of problem (3.17), (1.2) are dead core solutions for sufficiently large λ.

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