Hindawi Publishing Corporation
Boundary Value Problems
Volume 2007, Article ID 47218, 12 pages
doi:10.1155/2007/47218

Research Article

Existence and Multiplicity Results for Degenerate Elliptic Equations with Dependence on the Gradient

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Received 17 October 2006; Revised 2 January 2007; Accepted 9 February 2007

Recommended by Shujie Li

We study the existence of positive solutions for a class of degenerate nonlinear elliptic equations with gradient dependence. For this purpose, we combine a blowup argument, the strong maximum principle, and Liouville-type theorems to obtain a priori estimates.

1. Introduction

We consider the following nonvariational problem:

$$-\Delta_m u = f(x,u,\nabla u) - a(x)g(u,\nabla u) + \tau \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad (P)_\tau$$

where Ω is a bounded domain with smooth boundary of $\mathbb{R}^N$, $N \geq 3$. Δ_m denotes the usual m-Laplacian operators, $1 < m < N$ and $\tau \geq 0$. We will obtain a priori estimate to positive solutions of problem $(P)_\tau$ under certain conditions on the functions f, g, a. This result implies nonexistence of positive solutions to τ large enough.

Also we are interested in the existence of a positive solutions to problem $(P)_0$, which does not have a clear variational structure. To avoid this difficulty, we make use of the blow-up method over the solutions to problem $(P)_\tau$, which have been employed very often to obtain a priori estimates (see, e.g., [1, 2]). This analysis allows us to apply a result due to [3], which is a variant of a Rabinowitz bifurcation result. Using this result, we obtain the existence of positive solutions.

Throughout our work, we will assume that the nonlinearities f and g satisfy the following conditions.

(H_1) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function.

(H_2) $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function.

(H_3) There exist $L > 0$ and $c_0 \geq 1$ such that $u^p - L|\eta|^\alpha \leq f(x,u,\eta) \leq c_0 u^p + L|\eta|^\alpha$ for all $(x,u,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, where $p \in (m-1, m_* - 1)$ and $\alpha \in (m-1, mp/(p+1))$. Here, we denote $m_* = m(N-1)/(N-m)$.

(H_4) There exist $M > 0$, $c_1 \geq 1$, $q > p$, and $\beta \in (m-1, mp/(p+1))$ such that $|u|^q - M|\eta|^\beta \leq g(u,\eta) \leq c_1|u|^q + M|\eta|^\beta$ for all $(u,\eta) \in \mathbb{R} \times \mathbb{R}^N$.

We also assume the following hypotheses on the function a.

(A_1) $a : \overline{\Omega} \to \mathbb{R}$ is a nonnegative continuous function.

(A_2) There is a subdomain Ω_0 with C^2-boundary so that $\overline{\Omega_0} \subset \Omega$, $a \equiv 0$ in $\overline{\Omega_0}$, and $a(x) > 0$ for $x \in \Omega \setminus \overline{\Omega_0}$.

(A_3) We assume that the function a has the following behavior near to $\partial\Omega_0$:

$$a(x) = b(x) d(x, \partial\Omega_0)^\gamma, \tag{1.1}$$

$x \in \Omega \setminus \overline{\Omega_0}$, where γ is positive constant and $b(x)$ is a positive continuous function defined in a small neighborhood of $\partial\Omega_0$.

Observe that particular situations on the nonlinearities have been considered by many authors. For instance, when $a \equiv 0$ and f verifies (H_3), Ruiz has proved that the problem $(P)_0$ has a bounded positive solution (see [2] and reference therein). On the other hand, when $f(x,u,\eta) = u^p$ and $g(x,u,\eta) = u^q$, $q > p$ and $m < p$, and $a \equiv 1$, a multiplicity of results was obtained by Takeuchi [4] under the restriction $m > 2$. Later, Dong and Chen [5] improve the result because they established the result for all $m > 1$. We notice that the Laplacian case was studied by Rabinowitz by combining the critical point theory with the Leray-Schauder degree [6]. Then, when $m \geq p$, since $(f(x,u) - g(x,u))/u^{m-1}$ becomes monotone decreasing for $0 < u$, we know that the solution to $(P)_0$ is unique (as far as it exists) from the Díaz and Saá's uniqueness result (see [7]). For more information about this type of logistic problems, see [1, 8–13] and references cited therein.

Our main results are the following.

Theorem 1.1. *Let $u \in C^1(\Omega)$ be a positive solution of problem $(P)_\tau$. Suppose that the conditions (H_1)–(H_4) and the hypotheses (A_1)–(A_3) are satisfied with $\gamma \neq m(q-p)/(1-m+p)$. Then, there is a positive constant C, depending only on the function a and Ω, such that*

$$0 \leq u(x) + \tau \leq C \tag{1.2}$$

for any $x \in \Omega$.

Moreover, if $\gamma = m(q-p)/(1-m+p)$, then there exists a positive constant $c_1 = c_1(p, \alpha, \beta, N, c_0)$ such that the conclusion of the theorem is true, provided that $\inf_{\partial\Omega_0} b(x) > c_1$.

Observe that this result implies in particular that there is no solution for $0 < \tau$ large enough. By using a variant of a Rabinowitz bifurcation result, we obtain an existence result for positive solutions.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, the problem $(P)_0$ has at least one positive solution.*

2. A priori estimates and proof of Theorem 1.1

We will use the following lemma which is an improvement of Lemma 2.4 by Serrin and Zou [14] and was proved in Ruiz [2].

Lemma 2.1. *Let u be a nonnegative weak solution to the inequality*

$$-\Delta_m u \geq u^p - M|\nabla u|^\alpha, \tag{2.1}$$

in a domain $\Omega \subset \mathbb{R}^N$, where $p > m-1$ and $m-1 \leq \alpha < mp/(p+1)$. Take $\lambda \in (0,p)$ and let $B(\cdot,R_0)$ be a ball of radius R_0 such that $B(\cdot,2R_0)$ is included in Ω.

Then, there exists a positive constant $C = C(N,m,q,\alpha,\lambda,R_0)$ such that

$$\int_{B(\cdot,R)} u^\lambda \leq CR^{(N-m\lambda)/(p+1-m)}, \tag{2.2}$$

for all $R \in (0,R_0]$.

We will also make use of the following weak Harnack inequality, which was proved by Trudinger [15].

Lemma 2.2. *Let $u \geq 0$ be a weak solution to the inequality $\Delta_m u \leq 0$ in Ω. Take $\lambda \in [1,m_* - 1)$ and $R > 0$ such that $B(\cdot,2R) \subset \Omega$. Then there exists $C = C(N,m,\lambda)$ (independent of R) such that*

$$\inf_{B(\cdot,R)} u \geq CR^{-N/\lambda}\left(\int_{B(\cdot,2R)} u^\lambda\right)^{1/\lambda}. \tag{2.3}$$

The following lemma allows us to control the parameter τ in the Blow-Up analysis. (See Section 2.1.)

Lemma 2.3. *Let u be a solution to the problem $(P)_\tau$. Then there is a positive constant k_0 which depends only on Ω_0 such that*

$$\tau \leq k_0\left(\max_{x\in\overline{\Omega}} u\right)^{m-1}. \tag{2.4}$$

Proof. Since u is a positive solution, the inequality holds if $\tau = 0$. Now if $\tau > 0$, then from (H_1) and (A_2) we get

$$-\Delta_m u = f(x,u,\nabla u) - a(x)g(u,\nabla u) + \tau \geq \tau \quad \forall x \in \Omega_0. \tag{2.5}$$

Let v be the positive solution to

$$\begin{aligned} -\Delta_m v &= 1 \quad \text{in } \Omega_0, \\ v &= 0 \quad \text{on } \partial\Omega_0 \end{aligned} \tag{2.6}$$

and $w = (\tau/2)^{1/(m-1)}v$ in Ω_0, then it follows that $-\Delta_m w = \tau/2 < -\Delta_m u$ in Ω_0 and $u > w$ on $\partial\Omega_0$. Thus, using the comparison lemma (see [16]), we obtain $u \geq w$ in Ω_0. Therefore, there is a positive constant k_0 such that

$$\tau \le k_0 u^{m-1} \tag{2.7}$$

at the maximum point of v and the conclusion follows. □

2.1. A priori estimates. We suppose that there is a sequence $\{(u_n,\tau_n)\}_{n\in\mathbb{N}}$ with u_n being a C^1-solution of $(P)_{\tau_n}$ such that $\|u_n\|+\tau_n \xrightarrow[n\to\infty]{} \infty$. By Lemma 2.3, we can assume that there exists $x_n \in \Omega$ such that $u_n(x_n) = \|u_n\| =: S_n \xrightarrow[n\to\infty]{} \infty$. Let $d_n := d(x_n,\partial\Omega)$, we define $w_n(y) = S_n^{-1}u_n(x)$, where $x = S_n^{-\theta}y + x_n$ for some positive θ that will be defined later. The functions w_n are well defined at least $B(0,d_nS_n^{\theta})$, and $w_n(0) = \|w_n\| = 1$. Easy computations show that

$$\begin{aligned} -\Delta_m w_n(y) = S_n^{1-(\theta+1)m}\big[& f(S_n^{-\theta}y+x_n, S_n w_n(y), S_n^{1-\theta}\nabla w_n(y)) \\ & - a(S_n^{-\theta}y+x_n) g(S_n w_n(y), S_n^{1-\theta}\nabla w_n(y)) + \tau_n\big]. \end{aligned} \tag{2.8}$$

From our conditions on the functions f and g, the right-hand side of (2.8) reads as

$$\begin{aligned} & S_n^{1-(\theta+1)m}\big[f(S_n^{-\theta}y+x_n, S_n w_n(y), S_n^{1-\theta}\nabla w_n(y)) \\ & \qquad - a(S_n^{-\theta}y+x_n) g(S_n w_n(y), S_n^{1-\theta}\nabla w_n(y)) + \tau_n\big] \\ & \quad \le S_n^{1-(\theta+1)m+q}\Big[c_0 S_n^{p-q} w_n(y)^p + M S_n^{(1-\theta)\alpha-q}\,|\nabla w_n(y)|^{\alpha} \\ & \qquad - a(S_n^{-\theta}y+x_n)\Big(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q}\,|\nabla w_n(y)|^{\beta}\Big)\Big] + S_n^{1-(\theta+1)m}\tau_n. \end{aligned} \tag{2.9}$$

We note that from Lemma 2.3 we have $S_n^{1-(\theta+1)m}\tau_n \le c_0 S_n^{1-(\theta+1)m} S_n^{m-1} \xrightarrow[n\to\infty]{} 0$.

We split this section into the following three steps according to location of the limit point x_0 of the sequence $\{x_n\}_n$.

(1) $x_0 \in \overline{\Omega}\setminus\overline{\Omega_0}$. Here, up to subsequence, we may assume that $\{x_n\}_n \subset \Omega\setminus\overline{\Omega_0}$. We define $\delta_n' = \min\{\operatorname{dist}(x_n,\partial\Omega), \operatorname{dist}(x_n,\partial\Omega_0)\}$ and $B = B(0,\delta_n' S_n^{\theta})$ if $\operatorname{dist}(x_0,\partial\Omega) > 0$, or $\delta_n' = \operatorname{dist}(x_n,\partial\Omega_0)$ and $B = B(0,\delta_n' S_n^{\theta})\cap\Omega$ if $\operatorname{dist}(x_0,\partial\Omega) = 0$. Then, w_n is well defined in B and satisfies

$$\sup_{y\in B} w_n(y) = w_n(0) = 1. \tag{2.10}$$

Now, taking $\theta = (q+1-m)/m$ in (2.9) and applying regularity theorems for the m-Laplacian operator, we can obtain estimates for w_n such that for a subsequence $w_n \to w$, locally uniformly, with w be a C^1-function defined in $\mathbb{R}^N$ or in a halfspace, if $\operatorname{dist}(x_0,\partial\Omega)$ is positive or zero, satisfying

$$-\Delta_m w \le -a(x_0) w^q, \quad w \ge 0,\ w(0) = \max w = 1, \tag{2.11}$$

which is a contradiction with the strong maximum principle (see [17]).

(2) $x_0 \in \Omega_0$. In this case, up to subsequence we may assume that $\{x_n\}_n \subset \Omega_0$. Let $d_n = \operatorname{dist}(x_n, \partial\Omega_0)$ and $\theta = (1+p-m)/m$. Then, w_n is well defined in $B(0, d_n S_n^{\theta})$ and satisfies

$$\sup_{y \in B(0, d_n S_n^{\theta})} w_n(y) = w_n(0) = 1. \tag{2.12}$$

On the other hand, for any $n \in \mathbb{N}$, we have $a(S_n^{-\theta} y + x_n) = 0$ and

$$-\Delta_m w_n(y) = S_n^{1-(\theta+1)m}\left[f\left(S_n^{-\theta} y + x_n, S_n w_n(y), S_n^{1-\theta} \nabla w_n(y)\right) + \tau_n\right]. \tag{2.13}$$

From the hypothesis (H_4),

$$\begin{aligned} -\Delta_m w_n(y) &= S_n^{1-(\theta+1)m}\left[f\left(S_n^{-\theta} y + x_n, S_n w_n(y), S_n^{1-\theta} \nabla w_n(y)\right) + \tau_n\right] \\ &\geq w_n(y)^p - M S_n^{\alpha(1-\theta)+1-(\theta+1)m} \left| \nabla w_n(y) \right|^{\alpha} + \tau_n S_n^{1-(\theta+1)m}. \end{aligned} \tag{2.14}$$

From our choice of the constants α and θ, we have $\alpha(1-\theta)+1-(\theta+1)m = \alpha(2m-(1+p))/m - p < 0$, that is, $S_n^{\alpha(1-\theta)+1-(\theta+1)m} |\nabla w_n(y)|^{\alpha}$ and $\tau_n S_n^{1-(\theta+1)m}$ tend to 0 as n goes to ∞. This implies that for a subsequence w_n converges to a solution of $-\Delta_m v \geq v^p$, $v \geq 0$ in $\mathbb{R}^N$, $v(0) = \max v = 1$. This is a contradiction with [14, Theorem III].

(3) $x_0 \in \partial\Omega_0$. Let $\delta_n = d(x_n, z_n)$, where $z_n \in \partial\Omega_0$. Denote by ν_n the unit normal of $\partial\Omega_0$ at z_n pointing to $\Omega \setminus \Omega_0$.

Up to subsequences, We may distinguish two cases: $x_n \in \partial\Omega_0$ for all n or $x_n \in \Omega \backslash \partial\Omega_0$ for all n.

Case 1 ($x_n \in \partial\Omega_0$ for all n). In this case, $x_n = z_n$. For ε sufficiently small but fixed take $\tilde{x}_n = z_n - \varepsilon \nu_n$. Then we have the following.

Claim 1. For any large n we have

$$u_n(\tilde{x}_n) < \frac{S_n}{4}. \tag{2.15}$$

Proof of Claim 1. In other cases, define for all n sufficiently large, passing to a subsequence if necessary, the following functions

$$\tilde{w}_n(y) = S_n^{-1} u_n\left(\tilde{x}_n + S_n^{-(p+1-m)/m} y\right), \tag{2.16}$$

which are well defined at least in $B(0, \varepsilon S_n^{(p+1-m)/m})$, $w_n(0) \geq 1/4$ and $\sup_{B(0, \varepsilon S_n^{(p+1-m)/m})} \tilde{w}_n \leq 1$.

Arguing as in the previous case $x_0 \in \Omega_0$, we arrive to a contradiction. □

Now, by continuity, for any large n there exist two points in $\Omega_0 x_n^* = x_n - t_n^* \nu_n$ and $x_n^{**} = x_n - t_n^{**} \nu_n$, $0 < t_n^* < t_n^{**} < \varepsilon$ such that

$$u_n(x_n^*) = \frac{S_n}{2}, \qquad u_n(x_n^{**}) = \frac{S_n}{4}. \tag{2.17}$$

Claim 2. There exists a number $\tilde{\delta}_n \in (0, \min\{d(x_n, x_n^*), d(x_n^*, x_n^{**})\})$ such that $S_n/4 < u_n(x) < S_n$ for all $x \in B(x_n^*, \tilde{\delta}_n)$. Moreover, there exists y_n satisfying $d(x_n^*, y_n) = \tilde{\delta}_n$ and either $u_n(y_n) = S_n/4$ or else $u_n(y_n) = S_n$.

Proof of Claim 2. Define $\widetilde{\delta}_n = \sup\{\delta > 0 : S_n/4 < u_n(x) < S_n \text{ for all } x \in B(x_n^*, \delta)\}$. It is easy to prove that $\widetilde{\delta}_n$ is well defined. Thus, the continuity of u_n ensures the existence of y_n. □

Now we will obtain an estimate from below of $\widetilde{\delta}_n S_n^{(p+1-m)/m}$.

Claim 3. There exists a positive constant $c = c(p,\alpha,\beta,N,c_0)$ such that

$$\widetilde{\delta}_n S_n^{(p+1-m)/m} \geq c, \tag{2.18}$$

for any n sufficiently large.

Proof of Claim 3. Assume, passing to a subsequence if necessary, that $\widetilde{\delta}_n S_n^{(p+1-m)/m} < 1$ for any n. We have that the functions $\widetilde{w}_n(y) = S_n^{-1} u_n(x_n^* + S_n^{-(p+1-m)/m} y)$ are well defined in $B(0,1)$ for n sufficiently large and satisfy

$$-\Delta_m \widetilde{w}_n \leq c_0 \widetilde{w}_n^p + |\nabla \widetilde{w}_n|^\alpha + |\nabla \widetilde{w}_n|^\beta. \tag{2.19}$$

Applying Lieberman's regularity (see [18]), we obtain that there exists a positive constant $k = k(p,\alpha,\beta,N,c_0)$ such that $|\nabla \widetilde{w}_n| \leq k$ in $B(0,1)$. Assume for example that $u_n(y_n) = S_n/4$. By the generalized mean value theorem, we have

$$\frac{1}{4} = \frac{1}{2} - \frac{1}{4} = \widetilde{w}_n(0) - \widetilde{w}_n(S_n^\theta(y_n - x_n^*)) \leq |\nabla \widetilde{w}_n(\xi)| \widetilde{\delta}_n S_n^\theta. \tag{2.20}$$

□

Claim 4. For any n sufficiently large, we have $B(x_n^*, \widetilde{\delta}_n) \subset B(\widetilde{x}_n, \varepsilon)$.

Proof of Claim 4. Take $x \in B(x_n^*, \widetilde{\delta}_n)$, by Claim 2 we get

$$\begin{aligned} d(x,\widetilde{x}_n) &\leq d(x,x_n^*) + d(x_n^*,\widetilde{x}_n) < \widetilde{\delta}_n + d(x_n^*,\widetilde{x}_n) \\ &\leq d(x_n,x_n^*) + d(x_n^*,\widetilde{x}_n) = d(x_n,\widetilde{x}_n) \leq \varepsilon. \end{aligned} \tag{2.21}$$

So, $x \in B(\widetilde{x}_n, \varepsilon)$.

Let λ be a number such that $N(p+1-m)/m < \lambda < p$ (this is possible because $p < m_* - 1$). By Claims 3 and 4, and by Lemma 2.2, we get

$$\begin{aligned} \left(\inf_{B(\widetilde{x}_n,\varepsilon/2)} u_n\right)^\lambda &\geq c\varepsilon^{-N} \int_{B(\widetilde{x}_n,\varepsilon)} u_n^\lambda \geq \int_{B(x_n^*,\widetilde{\delta}_n)} u_n^\lambda \\ &\geq C\widetilde{\delta}_n^N S_n^\lambda/4 \geq C_1 S_n^{N(m-1-p)/m+\lambda} \xrightarrow[n\to\infty]{} \infty. \end{aligned} \tag{2.22}$$

Therefore, the last inequality tells us that

$$\int_{B(\widetilde{x}_n,\varepsilon/2)} u_n^\lambda \xrightarrow[n\to\infty]{} \infty, \tag{2.23}$$

which contradicts Lemma 2.1. □

Now, we will analyze the other case.

Case 2 ($x_n \in \Omega\backslash\partial\Omega_0$ for all n). Define $2d = \operatorname{dist}(x_0, \partial\Omega) > 0$. Since Ω_0 has C^2-boundary as in [19], we have

$$d(x_n + S_n^{-\theta} y, \partial\Omega_0) = |\delta_n + S_n^{-\theta}\nu_n \cdot y + o(S_n^{-\theta})|,$$

$$a(x_n + S_n^{-\theta} y) = \begin{cases} b(x_n + S_n^{-\theta} y) S_n^{-\gamma\theta} |\delta_n S_n^{\theta} + \nu_n \cdot y + o(1)|^{\gamma}, & \text{if } x_n + S_n^{-\theta} y \in \Omega \setminus \Omega_0, \\ 0, & \text{if } x_n + S_n^{-\theta} y \in \Omega_0. \end{cases} \tag{2.24}$$

We define $b_n(x_n + S_n^{-\theta} y) = S_n^{\gamma\theta} a(x_n + S_n^{-\theta} y)$.

For n large enough, w_n is well defined in $B(0, dS_n^{\theta})$ and we get

$$\sup_{y \in B(0, dS_n^{\theta})} w_n(y) = w_n(0) = 1. \tag{2.25}$$

By (2.9), we obtain

$$\begin{aligned} -\Delta_m w_n(y) \le S_n^{1-(\theta+1)m+q} \Big[& c_0 S_n^{p-q} w_n(y)^p + M S_n^{(1-\theta)\alpha - q} |\nabla w_n(y)|^{\alpha} \\ & - b_n(x_n + S_n^{-\theta} y) S_n^{-\gamma\theta} \left(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q} |\nabla w_n(y)|^{\beta} \right) \Big] \\ & + S_n^{1-(\theta+1)m} \tau_n. \end{aligned} \tag{2.26}$$

Now we need to consider the following cases.

If $0 < \gamma < m(q-p)/(1-m+p)$, we choose $\theta = (1-m+q)/(\gamma+m)$.

We first assume that $\{\delta_n S_n^{\theta}\}_{n \in \mathbb{N}}$ is bounded. Up to subsequence, we may assume that $\delta_n S_n^{\theta} \xrightarrow[n\to\infty]{} d_0 \ge 0$, from (2.26) we get

$$\begin{aligned} -\Delta_m w_n(y) &\le S_n^{\gamma\theta} \Big[c_0 S_n^{p-q} w_n(y)^p + M S_n^{(1-\theta)\alpha - q} |\nabla w_n(y)|^{\alpha} \\ &\quad - b_n(x_n + S_n^{-\theta} y) S_n^{-\gamma\theta} \left(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q} |\nabla w_n(y)|^{\beta} \right) \Big] + S_n^{1-(\theta+1)m} \tau_n \\ &= c_0 S_n^{p-q+\gamma\theta} w_n(y)^p + M S_n^{\gamma\theta + (1-\theta)\alpha - q} |\nabla w_n(y)|^{\alpha} \\ &\quad - b_n(x_n + S_n^{-\theta} y) \left(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q} |\nabla w_n(y)|^{\beta} \right) + S_n^{1-(\theta+1)m} \tau_n. \end{aligned} \tag{2.27}$$

Thus, up to a subsequence, we may assume that w_n converges to a C^1 function w defined in $\mathbb{R}^N$ and satisfying $w \ge 0$, $w(0) = \max w = 1$ in $\mathbb{R}^N$, and

$$-\Delta_m w(y) \le \begin{cases} -b(x_0) |d_0 + \nu_0 \cdot y|^{\gamma} w^q(y), & \text{if } \nu_0 \cdot y > \sigma, \\ 0, & \text{if } \nu_0 \cdot y < \sigma, \end{cases} \tag{2.28}$$

where $\sigma = -d_0$ if $x_n \in \Omega \setminus \overline{\Omega}_0$ or $\sigma = d_0$ if $x_n \in \overline{\Omega}_0$ and ν_0 is a unitary vector in $\mathbb{R}^N$. This is impossible by the strong maximum principles.

Suppose now that $\{\delta_n S_n^{\theta}\}$ is unbounded, we may assume that $\beta_n = (\delta_n^{-1} S_n^{-\theta})^{\gamma/m} \xrightarrow[n\to\infty]{} 0$ for any $r > 0$. Let us introduce $z = y/\beta_n$ and $v_n(z) = w_n(\beta_n z)$, using (2.26) we see that v_n satisfies

$$\begin{aligned}
-\Delta_m v_n(z) \le \beta_n^m S_n^{\gamma\theta} \Big[& c_0 S_n^{p-q} v_n(z)^p + M S_n^{(1-\theta)\alpha-q} \beta_n^{-\alpha} \left| \nabla v_n(z) \right|^{\alpha} \\
& - b_n (x_n + S_n^{-\theta}\beta_n z) S_n^{-\gamma\theta} \left(v_n(z)^q - g_0 S_n^{\beta(1-\theta)-q} \beta_n^{-\beta} \left| \nabla v_n(z) \right|^{\beta} \right) \Big] \\
& + S_n^{1-(\theta+1)m} \tau_n \\
= {} & c_0 \beta_n^m S_n^{\gamma\theta+p-q} v_n(z)^p + M S_n^{\gamma\theta+(1-\theta)\alpha-q} \beta_n^{m-\alpha} \left| \nabla v_n(z) \right|^{\alpha} \\
& - \beta_n^m b_n (x_n + S_n^{-\theta}\beta_n z) \left(v_n(z)^q - g_0 S_n^{\beta(1-\theta)-q} \beta_n^{m-\beta} \left| \nabla v_n(z) \right|^{\beta} \right) + S_n^{1-(\theta+1)m} \tau_n.
\end{aligned} \tag{2.29}$$

On the other hand,

$$\beta_n^m b_n (x_n + S_n^{-\theta}\beta_n z) = b(x_n + S_n^{-\theta}\beta_n z) \left[1 + \beta_n^{(m+\gamma)/\gamma} \nu_n \cdot z + o(\beta_n^{m/\gamma}) \right]^{\gamma} \xrightarrow[n\to\infty]{} b(x_0). \tag{2.30}$$

Thus, since $\gamma < m(q-p)/(1-m+p)$ and our choice of θ and β_n, it is easy to see that $S_n^{\gamma\theta+p-q}$, $S_n^{\gamma\theta+(1-\theta)\alpha-q} \beta_n^{m-\alpha}$ and $S_n^{\beta(1-\theta)-q} \beta_n^{m-\beta}$ tend to 0 as n goes to $+\infty$. Therefore, we obtain a limit function v that satisfies $-\Delta_m v \le -b(x_0) v^q$, $v \ge 0$, $v(0) = \max v = 1$ in $\mathbb{R}^N$ which is again impossible.

If $\gamma = m(q-p)/(1-m+p)$, in this case, by our assumptions on the function b, we obtain for $\theta = (1-m+p)/m$

$$\begin{aligned}
-\Delta_m w_n(y) \le {} & c_0 w_n(y)^p + M S_n^{(1-\theta)\alpha-p} \left| \nabla w_n(y) \right|^{\alpha} \\
& - b_n (x_n + S_n^{-\theta} y) \left(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q} \left| \nabla w_n(y) \right|^{\beta} \right) + S_n^{1-(\theta+1)m} \tau_n.
\end{aligned} \tag{2.31}$$

Arguing as in the proof of Claim 3 in the above case $x_n \in \partial\Omega_0$ for all n, we may assume that $\delta_n S_n \theta \ge d_0 = d_0(p,\alpha,\beta,N,c_0) > 0$. Therefore, the limit w of the sequence w_n satisfies

$$-\Delta_m w(y) \le c_0 w(y)^p - b(x_0) \left| d_0 - \left| \nu_0 \cdot y + o(1) \right| \right|^{\gamma} w(y)^q. \tag{2.32}$$

Now, evaluating in $x = 0$, the last inequality reads as

$$-\Delta_m w(0) \le c_0 - b(x_0) d_0^{\gamma} < 0, \tag{2.33}$$

provided that $b(x_0) > c_0/d_0^{\gamma}$. This contradicts the strong maximum principle.

If $\gamma > m(q-p)/(1-m+p)$, we choose $\theta = (p-m+1)/m$, then we get

$$\begin{aligned}
-\Delta_m w_n(y) \ge {} & w_n(y)^p - M S_n^{(1-\theta)\alpha-p} \left| \nabla w_n(y) \right|^{\alpha} \\
& - S_n^{q-p-\gamma\theta} b_n (x_n + S_n^{-\theta} y) \left(g_1 w_n(y)^q + g_2 S_n^{\beta(1-\theta)-q} \left| \nabla w_n(y) \right|^{\beta} \right) + S_n^{1-(\theta+1)m} \tau_n.
\end{aligned} \tag{2.34}$$

Arguing as seen before, that is, $\{\delta_n S_n^{-\theta}\}$ is whether bounded or unbounded, we obtain that the limit equation of the last inequality becomes

$$-\Delta_m v \geq v^p, \quad v \geq 0 \text{ in } \mathbb{R}^N,\ v(0) = \max v = 1, \tag{2.35}$$

which is a contradiction with [14, Theorem III].

3. Proof of Theorem 1.2

The following result is due to Azizieh and Clément (see [3]).

Lemma 3.1. *Let $\mathbb{R}^+ := [0,+\infty)$ and let $(E,\|\cdot\|)$ be a real Banach space. Let $G : \mathbb{R}^+ \times E \to E$ be continuous and map bounded subsets on relatively compact subsets. Suppose moreover that G satisfies the following:*

(a) $G(0,0) = 0$,

(b) *there exists $R > 0$ such that*

(i) *$u \in E$, $\|u\| \leq R$, and $u = G(0,u)$ imply that $u = 0$,*

(ii) $\deg(\mathrm{Id} - G(0,\cdot), B(0,R), 0) = 1$.

Let J denote the set of the solutions to the problem

$$u = G(t,u) \tag{$\mathfrak{P}$}$$

in $\mathbb{R}^+ \times E$. Let $\mathfrak{C}$ denote the component (closed connected maximal subset with respect to the inclusion) of J to which $(0,0)$ belongs. Then if

$$\mathfrak{C} \cap (\{0\} \times E) = \{(0,0)\}, \tag{3.1}$$

then $\mathfrak{C}$ is unbounded in $\mathbb{R}^+ \times E$.

Proof of Theorem 1.2. First, we consider the following problem:

$$\begin{aligned} -\Delta_m u = f(x,u^+,\nabla u^+) - a(x)g(u^+,\nabla u^+) + \tau \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{$(P)_\tau^+$}$$

and let u be a nontrivial solution to the problem above, then u is nonnegative and so is solution for the problem $(P)_\tau$. In fact, suppose that $U = \{x \in \Omega : u(x) < 0\}$ is nonempty. Then u is a weak solution to

$$\begin{aligned} -\Delta_m u = \tau \geq 0 \quad \text{in } U, \\ u = 0 \quad \text{on } \partial U. \end{aligned} \tag{3.2}$$

Using Lemma 2.3, we obtain that $u(x) \geq 0$, which is a contradiction with the definition of U.

Consider $T : L^\infty(\Omega) \to C^1(\overline{\Omega})$ as the unique weak solution $T(v)$ to the problem

$$\begin{aligned} -\Delta_m T(v) = v \quad \text{in } \Omega, \\ T(v) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

It is well known that the function T is continuous and compact (e.g., see [3, Lemma 1.1]).

Next, denote by $G(\tau,u):=T(f(x,u^+,\nabla u^+)-a(x)g(u^+,\nabla u^+)+\tau)$, then $G:\mathbb{R}^+\times C^1(\overline{\Omega})\to C^1(\overline{\Omega})$ is continuous and compact. Now, we will verify the hypotheses of Lemma 3.1. It is clear that $G(0,0)=0$. On the other hand, consider the compact homotopy $H(\lambda,u):[0,1]\times C^1(\overline{\Omega})\to C^1(\overline{\Omega})$ given by $H(\lambda,u)=u-\lambda G(0,u)$. We will show that

$$\text{if } u \text{ is a nontrivial solution to } H(\lambda,u)=0, \text{ then } \|u\|>R>0. \tag{3.4}$$

This fact implies that condition (i) of (b) holds. Moreover, (3.4) also implies that $\deg(H(\lambda,\cdot)B(0,R),0)$ is well defined since there is not solution on $\partial B(0,R)$. By the invariance property of the degree, we have

$$\deg\left(\mathrm{Id}-\lambda G(0,\cdot),B(0,R),0\right)=\deg\left(\mathrm{Id},B(0,R),0\right)=1,\quad \forall\lambda\in(0,1] \tag{3.5}$$

and (ii) of (b) holds.

In order to prove (3.4), note that $H(\lambda,u)=0$ implies that u is a solution to the problem

$$\begin{gathered}-\Delta_m u=\lambda\left(f(x,u^+,\nabla u^+)-a(x)g(u^+,\nabla u^+)\right)\quad \text{in } \Omega,\\ u=0\quad \text{on } \partial\Omega.\end{gathered} \tag{3.6}$$

Multiplying (3.6) by u, integrating over Ω the equation obtained, and applying Hölder's and Poincare's inequalities, we have that

$$\begin{aligned}\int_\Omega|\nabla u|^m&\le c_0\int_\Omega u^{p+1}+M_1\left[\int_\Omega|\nabla u|^\alpha u+\int_\Omega|\nabla u|^\beta u\right]\\ &\le C\left(\int_\Omega|\nabla u|^m\right)^{(p+1)/m}+M_1\left(\int_\Omega|\nabla u|^m\right)^{\alpha/m}\left(\int_\Omega u^{m/(m-\alpha)}\right)^{(m-\alpha)/m}\\ &\quad+M_1\left(\int_\Omega|\nabla u|^m\right)^{\beta/m}\left(\int_\Omega u^{m/(m-\beta)}\right)^{(m-\beta)/m}\\ &\le C\left(\int_\Omega|\nabla u|^m\right)^{(p+1)/m}+C_1\left(\int_\Omega|\nabla u|^m\right)^{(\alpha+1)/m}+C_1\left(\int_\Omega|\nabla u|^m\right)^{(\beta+1)/m}.\end{aligned} \tag{3.7}$$

This inequality implies that $\int_\Omega|\nabla u|^m>c>0$. Hence, we have $\|u\|>R>0$.

Now, we note that Theorem 1.1 and $C^{1,\rho}$ estimates imply that the component $\mathfrak{C}$ which contains $(0,0)$ is bounded. So, applying Lemma 3.1, we obtain that $\mathfrak{C}\cap(\{0\}\times C^1(\overline{\Omega}))\neq(0,0)$. Therefore, we have a positive solution u to the problem $(P)_0$. □

Acknowledgments

The first author would like to thank the hospitality of Departamento de Matemáticas, Universidad de Tarapacá. He also wants to thank Professors Heriberto Roman and Yurilev

Chalco for their comments and the fruitful discussions. The first author was partially supported by FONDECYT no. 3060061 and FONDAP Matemáticas Aplicadas, Chile. The second author was supported by FONDECYT no. 1051055.

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