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Research Article

Removable Singularities of $\mathcal{WT}$-Differential Forms and Quasiregular Mappings

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A theorem on removable singularities of $\mathcal{WT}$-differential forms is proved and applied to quasiregular mappings.

1. Main theorem

We recall some facts on differential forms and quasiregular mappings. Our notation is as in [1]. Let $\mathcal{M}$ be a Riemannian manifold of the class C^3, $\dim \mathcal{M} = n$, without boundary. Each differential form α can be written in terms of the local coordinates $x_1,\dots,x_n$ as the linear combination

$$\alpha = \sum_{1\le i_1<\cdots<i_k\le n} \alpha_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{1.1}$$

Let α be a differential form defined on an open set $D \subset \mathcal{M}$. If $\mathcal{F}(D)$ is a class of functions defined on D, then we say that the differential form α is in this class provided that $\alpha_{i_1\cdots i_k} \in \mathcal{F}(D)$. For instance, the differential form α is in the class $L^p(D)$ if all its coefficients are in this class.

A differential form α of degree k on the manifold $\mathcal{M}$ with coefficients $\alpha_{i_1\cdots i_k} \in L^p_{\text{loc}}(\mathcal{M})$ is called *weakly closed* if for each differential form β, $\deg\beta = k+1$, with compact support $\operatorname{supp}\beta = \overline{\{m \in \mathcal{M} : \beta \neq 0\}}$ in $\mathcal{M}$ and with coefficients in the class $W^1_{q,\text{loc}}(\mathcal{M})$, $1/p+1/q = 1$, $1 \le p, q \le \infty$, we have

$$\int_{\mathcal{M}} \langle \alpha, \delta\beta\rangle * \mathbb{1}_{\mathcal{M}} = 0. \tag{1.2}$$

Here the operator $*$ and the exterior differentiation d define the codifferential operator δ by the formula

$$\delta\alpha = (-1)^k *^{-1} d * \alpha \tag{1.3}$$

for a differential form α of degree k.

Clearly, $\delta\alpha$ is a differential form of degree $k-1$. For smooth differential forms α condition (1.2) agrees with the traditional condition of closedness $d\alpha = 0$.

For an arbitrary simple form of degree k,

$$w = w_1 \wedge \cdots \wedge w_k, \tag{1.4}$$

we set

$$\|w\| = \left(\sum_{i=1}^{k} |w_i|^2\right)^{1/2}. \tag{1.5}$$

For a simple form w we have Hadamard's inequality

$$|w| \le \prod_{i=1}^{k} |w_i|. \tag{1.6}$$

Taking these into account and using the inequality between geometric and arithmetic means

$$\left(\prod_{i=1}^{k} |w_i|\right)^{1/k} \le \frac{1}{k}\sum_{i=1}^{k} |w_i| \le \left(\frac{1}{k}\sum_{i=1}^{k} |w_i|^2\right)^{1/2} \tag{1.7}$$

we obtain

$$|w| \le k^{-k/2}\|w\|^k. \tag{1.8}$$

Let

$$w = w_1 \wedge \cdots \wedge w_k, \qquad \theta = \theta_1 \wedge \cdots \wedge \theta_{n-k} \tag{1.9}$$

be simple weakly closed differential forms on $\mathcal{M}$.

We say that the pair of forms (1.9) satisfies a $\mathcal{WT}$-condition on $\mathcal{M}$ if there exist constants $\nu_1, \nu_2 > 0$ such that almost everywhere on $\mathcal{M}$

$$\nu_1\|w\|^{kp} \le \langle w, *\theta\rangle, \qquad \|\theta\| \le \nu_2\|w\|. \tag{1.10}$$

Our main removability result for differential forms is the following.

Theorem 1.1. *Let $\mathcal{M}$ be a Riemannian C^3-manifold, $\dim M = n \ge 2$, and let $E \subset \mathcal{M}$ be a compact set of p-capacity zero, $1 \le p \le n$. Let Z and θ be simple forms on $\mathcal{M} \setminus E$ of degrees $k-1$, $n-k$, respectively, $\|dZ\| \in L^{kp}_{\text{loc}}$. Suppose that the pair dZ and θ satisfies a $\mathcal{WT}$-condition on $\mathcal{M} \setminus E$.*

If

$$\operatorname*{ess\,sup}_{m\in\mathcal{M}\setminus E} |Z(m)| < \infty, \tag{1.11}$$

then there exist forms $\widetilde{Z}$, $\widetilde{\theta}$ *such that* $\|d\widetilde{Z}\|, \|\widetilde{\theta}\| \in L^{kp}$ *on* $\mathcal{M}$, *the pair* $d\widetilde{Z}$, $\widetilde{\theta}$ *satisfies the* $\mathcal{WT}$*-condition on* $\mathcal{M}$ *and their restrictions to* $\mathcal{M} \setminus E$ *coincide with* Z, θ, *respectively.*

2. p-capacity

First we recall some basic facts about condensers. Let D be an open set on $\mathcal{M}$ and let $A, B \subset D$ be such that $\overline{A}$ and $\overline{B}$ are compact in D and $\overline{A} \cap \overline{B} = \varnothing$. Each triple $(A,B;D)$ is called a *condenser* on $\mathcal{M}$.

We fix $p \geq 1$. The p-capacity of the condenser $(A,B;D)$ is defined by

$$\operatorname{cap}_p(A,B;D) = \inf \int_D |\nabla\varphi|^p * \mathcal{M}, \tag{2.1}$$

where the infimum is taken over the set of all continuous functions φ of class $W^1_{p,\mathrm{loc}}(D)$ such that $\varphi|_A = 0$, $\varphi|_B = 1$. It is easy to see that for a pair $(A,B;D)$ and $(A_1,B_1;D)$ with $A_1 \subset A$, $B_1 \subset B$ we have

$$\operatorname{cap}_p(A_1,B_1;D) \leq \operatorname{cap}_p(A,B;D). \tag{2.2}$$

A standard approximation argument shows that the quantity $\operatorname{cap}_p(A,B;D)$ does not change if one restricts the class of functions in the variational problem (2.1) to smooth functions φ equal to 0 and 1 in the sets A and B, respectively, and $\nabla\varphi \neq 0$ a.e. on $\mathcal{M}\setminus(A\cup B)$.

We say that a compact set $E \subset \mathcal{M}$ is of p-capacity zero, if $\operatorname{cap}_p(E,U;\mathcal{M}) = 0$ for all open sets $U \subset \mathcal{M}$ such that $E \cap \overline{U} = \varnothing$.

We will need the following lemma.

Lemma 2.1. *A set* $E \subset \mathcal{M}$ *is of 1-capacity zero if and only if*

$$\mathcal{H}^{n-1}(E) = 0. \tag{2.3}$$

Proof. Fix $\varepsilon > 0$ and an open set $U \subset \mathcal{M}$ such that $\operatorname{cap}_1(E,U;M) = 0$. Choose a smooth function $\varphi : \mathcal{M} \to [0,1]$ such that $\varphi|_E = 0$, $\varphi|_U = 1$, $\nabla\varphi \neq 0$ a.e. on $\mathcal{M} \setminus (E \cup U)$ and

$$\int_{\mathcal{M}} |\nabla\varphi| * \mathcal{M} \leq \varepsilon. \tag{2.4}$$

By the coarea formula we have

$$\int_{\mathcal{M}} |\nabla\varphi| * \mathcal{M} = \int_0^1 dt \int_{G_t} d\mathcal{H}^{n-1} = \int_0^1 \mathcal{H}^{n-1}(G_t), \tag{2.5}$$

where $G_t = \{m \in \mathcal{M} : \varphi(m) = t\}$ is a level set of φ [2, Section 3.2].

Thus we obtain

$$\inf_t \mathcal{H}^{n-1}(G_t) \leq \varepsilon \tag{2.6}$$

and there exist sets G_t of arbitrarily small $(n-1)$-measure.

Since U is open it is possible only for the set E of $(n-1)$-measure zero. □

If a compact set $E \subset \mathcal{M}$ is of p-capacity zero, then E is of q-capacity zero for all $q \in [1, p]$. By Lemma 2.1 we conclude that a set E of p-capacity zero, $p \geq 1$, satisfies $\mathcal{H}^{n-1}(E) = 0$. In particular, such a set has n-measure zero.

3. Applications to quasiregular mappings

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of dimension n. It is convenient to use the following definition [3, Section 14]. A continuous mapping $F : \mathcal{M} \to \mathcal{N}$ of the class $W^1_{n,\mathrm{loc}}(\mathcal{M})$ is called a *quasiregular* mapping if F satisfies

$$|F'(m)|^n \leq K J_F(m) \tag{3.1}$$

almost everywhere on $\mathcal{M}$. Here $F'(m) : T_m(\mathcal{M}) \to T_{F(m)}(\mathcal{N})$ is the formal derivative of $F(m)$, further, $|F'(m)| = \max_{|h|=1} |F'(m)h|$. We denote by $J_F(m)$ the Jacobian of F at the point $m \in \mathcal{M}$, that is, the determinant of $F'(m)$.

For the following statement, see [1, Theorem 6.15, page 90].

Lemma 3.1. *If $F = (F_1, \dots, F_n) : \mathcal{M} \to \mathbb{R}^n$ is a quasiregular mapping and $1 \leq k < n$, then the pair of forms*

$$w = dF_1 \wedge \cdots \wedge dF_k, \qquad \theta = dF_{k+1} \wedge \cdots \wedge dF_n \tag{3.2}$$

satisfies a $\mathcal{W}\mathcal{T}$-condition on $\mathcal{M}$ with the structure constants $\nu_1 = \nu_1(n,k,K)$, $\nu_2 = \nu_2(n,k,K)$, and $p = n/k$.

We point out some special cases of Theorem 1.1.

Theorem 3.2. *Let $D \subset \mathbb{R}^n$ be a domain, $1 \leq k \leq n$, and let $E \subset D$ be a compact set of the n/k-capacity zero. Suppose that a quasiregular mapping*

$$F = (F_1, \dots, F_k, F_{k+1}, \dots, F_n) : D \setminus E \longrightarrow \mathbb{R}^n \tag{3.3}$$

satisfies (1.11) with

$$Z(x) = \sum_{i=1}^{k} (-1)^{i-1} c_i F_i dF_1 \wedge dF_2 \wedge \cdots \wedge \widetilde{dF_i} \wedge \cdots \wedge dF_k, \tag{3.4}$$

where the symbol $\widetilde{dF_i}$ means that this factor is omitted and $c_i = \mathrm{const}$, $\sum_{i=1}^{k} c_i = 1$.

Then there exists a quasiregular mapping $\widetilde{F} : D \to \mathbb{R}^n$ for which $\widetilde{F}|_{D \setminus E} = F$.

Proof. Since the statement is a special case of Theorem 1.1, it suffices to show that Z and θ satisfy the assumptions of the theorem. We have

$$dZ = \sum_{i=1}^{k}(-1)^{i-1}c_i dF_i \wedge dF_1 \wedge dF_2 \wedge \cdots \wedge \widetilde{dF_i} \wedge \cdots \wedge dF_k = dF_1 \wedge \cdots \wedge dF_k. \tag{3.5}$$

If we put

$$\theta = dF_{k+1} \wedge \cdots \wedge dF_n, \tag{3.6}$$

then by Lemma 3.1 the pair of forms $w = dZ$ and θ satisfies (1.10) on $D \setminus E$. Using Theorem 1.1 we can conclude that forms Z and θ have extensions to D. Moreover for an arbitrary subdomain D', $E \subset D' \subset\subset D$, it follows

$$\begin{aligned}\int_{D'\setminus E} J_F(x)dx_1 \cdots dx_n &= \int_{D'\setminus E} dF_1 \wedge \cdots \wedge dF_n = \int_{D'\setminus E} dZ \wedge \theta \\ &\le C\int_{D'\setminus E} |dZ||\theta|dx_1 \cdots dx_n \le C\|dZ\|_{L^p(D'\setminus E)}\|\theta\|_{L^q(D'\setminus E)},\end{aligned} \tag{3.7}$$

where $C = \text{const} < \infty$ [2, Section 1.7] and $p = n/k$, $q = n/(n-k)$.

From this it is easy to see that the vector function F belongs to $W^1_{n,\text{loc}}$ in D and E is removable for the quasiregular mapping F. Note that in the definition of a quasiregular mapping continuity is not needed, see [4, Section 3, Chapter II]. This property has a local character and its proof for subdomains of $\mathbb{R}^n$ implies its correctness for manifolds. □

The case $k = 1$ reduces to the well-known case, see Miklyukov [5].

Corollary 3.3. *Let $D \subset \mathbb{R}^n$ be a domain, and let $E \subset D$ be a compact set of n-capacity zero. Suppose that*

$$F = (F_1, F_2, \ldots, F_n) : D \setminus E \longrightarrow \mathbb{R}^n \tag{3.8}$$

is a quasiregular mapping such that

$$\sup_{x\in D\setminus E} |F_1(x)| < \infty. \tag{3.9}$$

Then there exists a quasiregular mapping $\widetilde{F} : D \to \mathbb{R}^n$ for which $\widetilde{F}|_{D\setminus E} = F$.

For $k = n$ we have the following result.

Corollary 3.4. *Let $D \subset \mathbb{R}^n$ be a domain, and let $E \subset D$ be a compact set of Hausdorff $(n-1)$-measure zero. Suppose that*

$$F = (F_1, F_2, \ldots, F_n) : D \setminus E \longrightarrow \mathbb{R}^n \tag{3.10}$$

is a quasiregular mapping such that

$$\operatorname*{ess\,sup}_{x\in D\setminus E} J_F(x) < \infty. \tag{3.11}$$

Then there exists a quasiregular mapping $f^* : D \to \mathbb{R}^n$ for which $f^*|_{D\setminus E} = f$.

Proof. Since the Jacobian determinant of F is bounded and E is of $(n-1)$-measure zero, the quasiregularity of F implies that F and the form

$$\sum_{i=1}^{n}(-1)^i F_i dF_1 dF_2 \wedge \cdots \widetilde{dF_i} \cdots \wedge dF_n \tag{3.12}$$

belong to $L^{\infty}_{\mathrm{loc}}(D)$. Hence the corollary follows from Theorem 3.2. □

Remark 3.5. Observe that Corollary 3.4 has an easy alternative proof. Since $J_F(x)$ is bounded and E is of $(n-1)$-measure zero, the quasiregularity of F implies that the derivative of F belongs to $L^{\infty}_{\mathrm{loc}}(D)$ and F is a Lipschitz mapping in $D \setminus E$. This shows that F can be extended to a Lipschitz mapping on D. It is clear that the extended mapping is quasiregular in D.

Corollary 3.4 gives the following version of the well-known Painlevé theorem.

Corollary 3.6. *Let $E \subset D \subset \mathbb{C}$ be a compact set of linear measure zero. Let $F : D \setminus E \to \mathbb{C}$ be a holomorphic function. The set E is removable for F if and only if*

$$\sup_{z \in K \setminus E} \left| F'(z) \right| < \infty, \tag{3.13}$$

for each compact set $K \subset D$.

4. Proof of Theorem 1.1

We will need the following integration by parts formula for differential forms [1].

Lemma 4.1. *Let $\alpha \in W^1_{p,\mathrm{loc}}(\mathcal{M})$ and $\beta \in W^1_q(\mathcal{M})$ be differential forms,* $\deg\alpha + \deg\beta = n-1$, $1/p + 1/q = 1$, $1 \le p, q \le \infty$, *and let β have a compact support. Then*

$$\int_{\mathcal{M}} d\alpha \wedge \beta = (-1)^{\deg\alpha+1} \int_{\mathcal{M}} \alpha \wedge d\beta. \tag{4.1}$$

In particular, the form α is weakly closed if and only if $d\alpha = 0$ a.e. on $\mathcal{M}$.

Let $D \subset \mathcal{M}$ be a domain containing E and with a compact closure in $\mathcal{M}$. Let $\{U_k\}_{k=1}^{\infty}$ be a sequence of open sets $U_k \subset \mathcal{M}$ such that

$$E \subset U_k, \qquad \overline{U}_k \subset D, \qquad \cap_{k=1}^{\infty} U_k = E. \tag{4.2}$$

Fix a nonnegative smooth function $\psi : \mathcal{M} \to \mathbb{R}$, $0 \le \psi \le 1$, with a compact support and $\psi \equiv 1$ on D. Fix a $k = 1, 2, \ldots$ and a smooth function $\varphi : \mathcal{M} \to \mathbb{R}$, $0 \le \varphi \le 1$, with the properties

$$\varphi|_E = 0, \quad \operatorname{supp}\varphi \subset U_k, \quad \varphi = 1 \quad \forall m \in \mathcal{M} \setminus U_k. \tag{4.3}$$

The form $\psi^p \varphi^p Z \wedge \theta$ has a compact support in $\mathcal{M} \setminus E$. This yields

$$\int_{\mathcal{M}\setminus E} d(\psi^p \varphi^p Z \wedge \theta) = 0. \tag{4.4}$$

Using (4.1) we have

$$\int_{\mathcal{M}\setminus E}\psi^p\varphi^p dZ\wedge\theta+(-1)^{\deg Z}\int_{\mathcal{M}\setminus E}\psi^p\varphi^p Z\wedge d\theta=-\int_{\mathcal{M}\setminus E}d(\psi^p\varphi^p)\wedge Z\wedge\theta. \tag{4.5}$$

Observe that

$$dZ\wedge\theta=\langle dZ,*\theta\rangle *\ \ _{\mathcal{M}}. \tag{4.6}$$

The form θ is closed and, consequently, from (1.10) we get

$$\begin{aligned}\nu_1\int_{\mathcal{M}\setminus E}\psi^p\varphi^p\|dZ\|^{kp} * \quad &\le \int_{\mathcal{M}\setminus E}\psi^p\varphi^p\langle dZ,*\theta\rangle * \quad = -\int_{\mathcal{M}\setminus E}d(\psi^p\varphi^p)\wedge Z\wedge\theta\\ &= -\int_{\mathcal{M}\setminus E}\langle d(\psi^p\varphi^p)\wedge Z,*\theta\rangle *\\ &\le \int_{\mathcal{M}\setminus E}|d(\psi^p\varphi^p)\wedge Z||*\theta| * \quad .\end{aligned} \tag{4.7}$$

But $\deg\theta=n-k$ and by (1.8) we have

$$|*\theta|=|\theta|\le(n-k)^{(n-k)/2}\|\theta\|^{n-k}. \tag{4.8}$$

Thus from the second condition of (1.10), it follows that

$$\nu_1\int_{\mathcal{M}\setminus E}\psi^p\varphi^p\|dZ\|^{kp} * \quad \le \nu_3\int_{\mathcal{M}\setminus E}|d(\psi^p\varphi^p)\wedge Z|\,\|dZ\|^{p-1} * \quad , \tag{4.9}$$

where $\nu_3=(n-k)^{(n-k)/2}\nu_2$.

By (1.11) there exists a constant $0<M<\infty$ such that

$$|Z(m)|<M \quad \text{for a.e. in } \mathcal{M}\setminus E. \tag{4.10}$$

Thus, we obtain

$$\nu_1\int_{\mathcal{M}\setminus E}\psi^p\varphi^p\|dZ\|^{kp} * \quad \le \nu_3 M\int_{\mathcal{M}\setminus E}|d(\psi^p\varphi^p)|\,\|dZ\|^{p-1} * \quad . \tag{4.11}$$

However,

$$|d(\psi^p\varphi^p)|\le p\varphi^p\psi^{p-1}|\nabla\psi|+p\varphi^{p-1}\psi^p|\nabla\varphi|, \tag{4.12}$$

$$\begin{aligned}&\nu_1\int_{\mathcal{M}\setminus E}\psi^p\varphi^p\|dZ\|^{kp} *\\ &\quad\le p\nu_3 M\int_{\mathcal{M}\setminus E}\varphi^p\psi^{p-1}|\nabla\psi|\|dZ\|^{p-1} * \quad + p\nu_3 M\int_{\mathcal{M}\setminus E}\psi^p\varphi^{p-1}|\nabla\varphi|\|dZ\|^{p-1} * \quad .\end{aligned} \tag{4.13}$$

Next we use the Cauchy inequality

$$ab^{p-1} \leq \frac{\varepsilon^{kp}}{kp}a^p + \frac{p-1}{kp}\varepsilon^{kp/(1-p)}b^{kp} \tag{4.14}$$

for $a, b, \varepsilon > 0$, $p \geq 1$.

For $\varepsilon > 0$ this implies two estimates

$$\begin{aligned}
&\int_{\mathcal{M}\setminus E} \varphi^p \psi^{p-1} |\nabla \psi| \, \|dZ\|^{n-k} * \\
&\quad \leq \frac{n-k}{kp}\varepsilon^{kp/(k-n)} \int_{\mathcal{M}\setminus E} \varphi^p \psi^p \|dZ\|^{kp} * \quad + \frac{\varepsilon^{kp}}{kp}\int_{\mathcal{M}\setminus E} \varphi^p |\nabla \psi|^p * \quad , \\
&\int_{\mathcal{M}\setminus E} \varphi^{p-1} \psi^{p} |\nabla \varphi| \, \|dZ\|^{n-k} * \\
&\quad \leq \frac{n-k}{kp}\varepsilon^{kp/(k-n)} \int_{\mathcal{M}\setminus E} \varphi^p \psi^p \|dZ\|^{kp} * \quad + \frac{\varepsilon^{kp}}{kp}\int_{\mathcal{M}\setminus E} \psi^p |\nabla \varphi|^p * \quad .
\end{aligned} \tag{4.15}$$

Now from (4.13) it follows

$$\begin{aligned}
&\nu_1 \int_{\mathcal{M}\setminus E} \psi^p \varphi^p \|dZ\|^{kp} * \\
&\quad \leq C_1 \int_{\mathcal{M}\setminus E} \psi^p \varphi^p \|dZ\|^{kp} * \quad + C_2 \int_{\mathcal{M}\setminus E} \varphi^p |\nabla \psi|^p * \quad + C_2 \int_{\mathcal{M}\setminus E} \psi^p |\nabla \varphi|^p * \quad ,
\end{aligned} \tag{4.16}$$

where

$$C_1 = \frac{n-k}{k}\nu_3 M \varepsilon^{kp/(k-n)}, \qquad C_2 = \nu_3 M \frac{\varepsilon^{kp}}{k}. \tag{4.17}$$

Choose $\varepsilon = \varepsilon_0 > 0$ such that $C_1 = \nu_1/2$. Then we obtain

$$\begin{aligned}
&\frac{1}{2}\nu_1 \int_{\mathcal{M}\setminus E} \psi^p \varphi^p \|dZ\|^{kp} * \\
&\quad \leq \nu_3 M \frac{\varepsilon_0^{kp}}{k} \int_{\mathcal{M}\setminus E} \varphi^p |\nabla \psi|^p * \quad + \nu_3 M \frac{\varepsilon_0^{kp}}{k} \int_{\mathcal{M}\setminus E} \psi^p |\nabla \varphi|^p * \\
&\quad = \nu_3 M \frac{\varepsilon_0^{kp}}{k} \int_{U_k\setminus E} |\nabla \varphi|^p * \quad + \nu_3 M \frac{\varepsilon_0^{kp}}{k} \int_{\mathcal{M}\setminus D} |\nabla \psi|^p *
\end{aligned} \tag{4.18}$$

and since $0 \leq \psi$, $\varphi \leq 1$,

$$\frac{1}{2}\nu_1 \int_{D\setminus U_k} \|dZ\|^{kp} * \quad \leq \nu_3 M \frac{\varepsilon_0^{kp}}{k} \left(\int_{U_k\setminus E} |\nabla \varphi|^p * \quad + \int_{\mathcal{M}\setminus D} |\nabla \psi|^p * \quad \right). \tag{4.19}$$

The special choice of φ and ψ permits to take the infimum over φ and ψ such that

$$\frac{1}{2}\nu_1 \int_{D\setminus U_k} \|dZ\|^{kp} * \quad \leq \nu_3 M \frac{\varepsilon_0^{kp}}{k} \operatorname{cap}_p (E, U_k; \mathcal{M}) + \nu_3 M \frac{\varepsilon_0^{kp}}{k} \operatorname{cap}_p (D, \mathcal{M}; \mathcal{M}). \tag{4.20}$$

However, $\operatorname{cap}_p(E,\mathcal{M}\setminus U_k;\mathcal{M})=0$ and thus we arrive at the estimates

$$\frac{1}{2}\nu_1\int_{D\setminus U_k}\|dZ\|^{kp}* \quad \le \nu_3 M\frac{\varepsilon_0^{kp}}{k}\operatorname{cap}_p(D,\mathcal{M};\mathcal{M}), \tag{4.21}$$

$$\frac{1}{2}\nu_1\int_{D}\|dZ\|^{kp}* \quad \le \nu_3 M\frac{\varepsilon_0^{kp}}{k}\operatorname{cap}_p(D,\mathcal{M};\mathcal{M}) \tag{4.22}$$

because by Lemma 2.1 the set E is of $(n-1)$-measure zero.

Next by Lemma 2.1, the coefficients of Z can be extended to $W^1_{p,\mathrm{loc}}$-functions in $\mathcal{M}$. This is due to the estimate (4.22) and to the ACL-property of W^1_p-functions; note that the ACL-property can be easily transformed to the manifold $\mathcal{M}$ since $\mathcal{M}$ is in the class C^3.

Thus, Z can be extended up to some form $\widetilde{Z}$. Moreover clearly, $\|d\widetilde{Z}\|\in L^{kp}_{\mathrm{loc}}(\mathcal{M})$. The extension of θ is analogous. Theorem 1.1 is completely proved.

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