Hindawi Publishing Corporation
Boundary Value Problems
Volume 2007, Article ID 64012, 16 pages
doi:10.1155/2007/64012

Research Article

The Shooting Method and Nonhomogeneous Multipoint BVPs of Second-Order ODE

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Received 25 May 2007; Revised 20 August 2007; Accepted 23 August 2007

Recommended by Kanishka Perera

In a recent paper, Sun et al. (2007) studied the existence of positive solutions of nonhomogeneous multipoint boundary value problems for a second-order differential equation. It is the purpose of this paper to show that the shooting method approach proposed in the recent paper by the first author can be extended to treat this more general problem.

1. Introduction

In a previous paper [1], the first author demonstrated that the classical shooting method could be effectively used to establish existence and multiplicity results for boundary value problems of second-order ordinary differential equations. This approach has an advantage over the traditional method of using fixed point theorems on cones by Krasnosel'skiĭ [2]. It has come to our attention after the publication of [1] that Baxley and Haywood [3] had also used similar ideas to study Dirichlet boundary value problems.

In this article, we continue our exposition by further extending this shooting method approach to treat multipoint boundary value problems with a nonhomogeneous boundary condition at the right endpoint, and homogeneous boundary condition at the left endpoint of the most general type, that is, the Robin boundary condition which includes both Dirichlet and Neumann boundary conditions as special cases.

The study of multipoint boundary value problems for linear second-order differential equations was initiated by Il'in and Moiseev [4, 5]. Nonlinear second-order boundary value problems with three-point boundary conditions were first studied by Gupta [6, 7] followed by many others, notably Marano [8]. Please consult the articles cited in the References Section.

Symmetric positive solutions for Dirichlet boundary value problems, which are related to second-order elliptic partial differential equations, were studied by Constantian [9], Avery [10], and Henderson and Thompson [11]. We defer a discussion of these results in relation to ours to the last section of this paper.

We will first establish two existence results (Theorems 3.1 and 3.2) on multipoint problems for the second-order differential equation

$$u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

where the nonlinear term is in a separable format, and a and f are continuous functions satisfying

$$\begin{gathered} a : [0,1] \longrightarrow [0,\infty), \quad a(t) \not\equiv 0, \\ f : [0,\infty) \longrightarrow [0,\infty), \quad f(u) > 0 \text{ for } u > 0. \end{gathered} \tag{1.2}$$

Note that the assumption that $f(u)$ does not vanish for $u > 0$ is a technical assumption imposed for convenience. Without this assumption, the second inequality sign in (1.8) and (1.9) below may not be strict.

Analogous results (Theorems 3.3 and 3.4) are then formulated and extended to nonlinear equations of the more general form

$$y''(t) + F(t, y(t)) = 0, \quad t \in (0,1), \tag{1.3}$$

where the nonlinear term may not be in a separable format.

In both [12, 13], the Neumann boundary condition

$$u'(0) = 0 \tag{1.4}$$

is imposed on the left endpoint. Some other authors use the Dirichlet condition

$$u(0) = 0. \tag{1.5}$$

The results in this paper are applicable to the most general Robin boundary condition of the form

$$(\sin\theta)u(0) - (\cos\theta)u'(0) = 0, \tag{1.6}$$

where θ is a given number in $[0, 3\pi/4)$. The choices $\theta = 0$ and $\pi/2$ correspond, respectively, to the Neumann and Dirichlet conditions (1.4) and (1.5). We leave out those θ in $[3\pi/4, \pi]$ as solutions satisfying the corresponding Robin's condition cannot furnish a positive solution for our boundary value problem. To see this, note that if $\theta \in [3\pi/4, \pi]$, then $u'(0) = u(0)\tan\theta \le -u(0)$. Since $u(t)$ is concave, $u(t)$ must lie below the line joining the points $(0, u(0))$ and $(1, 0)$, so $u(t)$ cannot be positive in $[0, 1]$.

The second boundary condition we impose involves $m-2$ given points $\xi_i \in (0,1)$, $i = 1, \ldots, m-2$, together with $t = 1$. Let k_i, $i = 1, \ldots, m-2$ be another set of $m-2$ given positive numbers, and $b \geq 0$. We require the solution to satisfy

$$u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = b \geq 0. \tag{1.7}$$

The boundary value problem for the differential equation (1.1) with boundary conditions (1.6) and (1.7) is often referred to as the m-point problem. When $b = 0$, the multipoint boundary condition is said to be homogeneous. Otherwise, it is called nonhomogeneous. In the special case when $m = 3$, only one interior point $\xi = \xi_1$ is used and the boundary value problem is called a three-point problem.

In the case of left Neumann problem, it is known that a necessary condition for the existence of a positive solution is

$$0 < \sum k_i < 1. \tag{1.8}$$

To see this, we put $b = 0$ in (1.7) and use the fact that $u(1) < u(\xi_i)$ for all i, because $u(t)$ is a concave function in $[0,1]$.

In the case of the left Dirichlet problem, the corresponding necessary condition is

$$0 < \sum k_i \xi_i < 1. \tag{1.9}$$

To see this, we use the fact that $u(t)$ is a concave function, and so $u(t)$ lies strictly above the straight line joining the origin $(0,0)$ with the point $(1,u(1))$. Therefore, $u(\xi_i) > \xi_i u(1)$ for all i. Plugging these inequalities and $b = 0$ into (1.7) gives (1.9).

We will state and prove the corresponding necessary condition for the general Robin condition in the next Section, see Lemma 2.2.

In [12], Ma proved the following existence result for the homogeneous three-point problem. Define

$$f_0 = \lim_{u \to 0+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \tag{1.10}$$

Theorem 1.1. *The three-point problem (1.1), (1.5), and (1.7) (with $m = 3$ and $b = 0$) has at lease a positive solution if either*

(a) *$f_0 = 0$ and $f_\infty = \infty$ (the superlinear case) or*

(b) *$f_0 = \infty$ and $f_\infty = 0$ (the sublinear case).*

For the nonhomogeneous problem, Ma [14] has the following result for the superlinear case.

Theorem 1.2. *Suppose that $f(u)$ is superlinear as in case (a) of Theorem 1.1. There exists a positive number b^* such that for all $b \in (0,b^*)$, the nonhomogeneous three-point problem (1.1), (1.5), and (1.7) has at least one positive solution. Furthermore, for $b > b^*$, there is no positive solution.*

In a recent paper by Sun et al. [13], Theorem 1.2 was extended to the multipoint Neumann problem (1.4) and (1.7). The authors also stated an analogue for the sublinear case (i.e., when $f_0 = \infty$ and $f_\infty = 0$ as in case (b) of Theorem 1.1) without providing a proof. However, the simple counterexample

$$u''(t) + 1 = 0, \quad u'(0) = 0, \quad u(1) - \frac{u(1/2)}{2} = b \tag{1.11}$$

has the solution $u(t) = -t^2/2 + 2b + 7/8$ for all $b > 0$, showing that the result as stated in [13, Theorem 1.2] is false.

Since our technique of proof uses the shooting method, the issues of continuability and uniqueness of initial value problems for the differential equations (1.1) or (1.3) arise naturally. In fact, these issues have already been discussed in [1]. The readers can be referred to that paper for more details. We only give a brief summary below. It is well known that continuability and uniqueness may not always hold for initial value problems of general nonlinear equations. In particular, it is known, see, for example, Coffman and Wong [15], that solutions of superlinear equation may not be continuable to a solution defined on the entire interval $[0,1]$. This is not a problem for our study because in our technique, we only need to be able to extend the solution up to its first zero. Since the solution is concave, this poses no problem at all. We also know that solutions of initial value problems may not be unique if $f(u)$ is not Lipschitz continuous. In such a situation, we can approximate $f(u)$ by Lipschitz continuous functions, obtain existence for the smoothed equation, and then use a compactness (passing to limit) argument to derive solutions for the original equation.

2. Auxiliary lemmas

Our first Lemma has already been presented in [1]. It is repeated here for the sake of easy reference. It is a simple consequence of a well-known fact in the Sturm Comparison theory of linear differential equations.

Lemma 2.1. *Let $Y(t)$ and $Z(t)$ be, respectively, positive solutions of the two linear differential equations*

$$\begin{aligned} Y''(t) + b(t)Y(t) &= 0, \\ Z''(t) + B(t)Z(t) &= 0, \end{aligned} \tag{2.1}$$

in the interval $[0,1]$ such that $Y'(0)/Y(0) \geq Z'(0)/Z(0)$, and we assume that $b(t) \leq B(t)$ for all $t \in [0,1]$. Let $\xi_i \in (0,1)$ and $k_i > 0$, $i = 1,\ldots,m-2$ be $2m-4$ given constants, and let $\tau \in [0,1]$ be any constant greater than all the ξ_i, then

$$\sum_{i=1}^{m-2} \frac{k_i Y(\xi_i)}{Y(\tau)} \leq \sum_{i=1}^{m-2} \frac{k_i Z(\xi_i)}{Z(\tau)}. \tag{2.2}$$

If we assume, furthermore, that $b(t) \not\equiv B(t)$, then strict inequality holds in (2.2).

Proof. The classical Sturm comparison theorem has a strong form that yields the inequality

$$\frac{Y'(t)}{Y(t)} \geq \frac{Z'(t)}{Z(t)}, \quad t \in [0,1], \tag{2.3}$$

where strict inequality will hold if we know, in addition, that $b \not\equiv B$ in $[0,t]$. One way to prove this is to note that the function $r(t) = Y'(t)/Y(t)$ satisfies a Riccati equation of the form

$$r'(t) + b(t) + r^2(t) = 0. \tag{2.4}$$

The function $s(t) = Z'(t)/Z(t)$ satisfies an analogous Riccati equation. The inequality $r(t) \geq s(t)$ follows by applying results in differential inequalities to compare the two Riccati equations.

Let ξ be any point in $(0,\tau)$. By integrating over $[\xi,\tau]$, we see that

$$\log\left(\frac{Y(\xi)}{Y(\tau)}\right) = -\int_\xi^\tau \frac{Y'(t)}{Y(t)}dt \leq -\int_\xi^\tau \frac{Z'(t)}{Z(t)}dt = \log\left(\frac{Z(\xi)}{Z(\tau)}\right). \tag{2.5}$$

Hence, $Y(\xi)/Y(\tau) \leq Z(\xi)/Z(\tau)$. In particular, the inequality is true for $\xi = \xi_i$, and the conclusion of the lemma follows by taking the appropriate linear combination of the various fractions. □

Lemma 2.2. *A necessary condition for the homogeneous Robin multipoint boundary value problem, with $\theta \neq \pi/2$, to have a positive solution is*

$$\sum_{i=1}^{m-2} \frac{k_i(1+\xi_i \tan\theta)}{1+\tan\theta} < 1. \tag{2.6}$$

Proof. Let S be the tangent line to the solution curve $u(t)$ at the initial point $(0,u(0))$. Let $Y(t)$ be the function that is represented by S. Then Y satisfies the simple differential equation $Y''(t) = 0$. We can use Lemma 2.1 to compare $u(t)$ with $Y(t)$ to get

$$\frac{u(\xi_i)}{u(1)} > \frac{Y(\xi_i)}{Y(1)} = \frac{1+\xi_i \tan\theta}{1+\tan\theta}. \tag{2.7}$$

Substituting these inequalities into the homogeneous Robin boundary condition gives (2.6). □

The next lemma is reminiscent of the eigenvalue problem of a linear equation.

Lemma 2.3. *Consider the homogeneous linear multipoint boundary value problem*

$$y''(t) + \lambda a(t) y(t) = 0, \quad t \in (0,1), \tag{2.8}$$

$$\textit{(see (1.6))}, \quad y(1) - \sum_{i=1}^{m-2} k_i y(\xi_i) = 0, \tag{2.9}$$

where λ is a positive parameter, and $a(t) \geq 0$, $a(t) \not\equiv 0$. Furthermore, assume that (2.6) holds. Then there exists a unique constant $L_\theta > 0$ for which the problem, with $\lambda = L_\theta$, has a positive nontrivial solution.

Proof. Let $y(t;\lambda)$ be the "shooting" solution of the initial value problem for (2.8) with $y(0,\lambda) = 1$ and $y'(0,\lambda) = \tan\theta$, when $\theta \neq \pi/2$. In the Dirichlet case $\theta = \pi/2$, we let $y(0,\lambda) = 0$ and $y'(0,\lambda) = 1$. Let us increase λ continuously from 0 to the first value $\lambda = \Lambda_\theta$ for which $y(1;\Lambda_\theta) = 0$. The assumption that $a(t) \not\equiv 0$ is needed here to ensure that Λ_θ exists.

For $\lambda \in [0,\Lambda_\theta)$, $y(1,\lambda) > 0$, and we can define

$$\phi(\lambda) = \sum_{i=1}^{m-2} \frac{k_i y(\xi_i;\lambda)}{y(1,\lambda)}, \tag{2.10}$$

which is a continuous function of λ. Condition (2.6) implies that $\phi(0) < 1$. On the other hand, $\lim_{\lambda\to\Lambda_\theta} \phi(\lambda) = \infty$. Hence, by the intermediate value theorem, there exists a value $\lambda = L_\theta$ such that $\phi(\lambda) = 1$, and this yields a solution of the boundary value problems (2.8) and (2.9).

The uniqueness of L_θ follows from the fact that $\phi(\lambda)$ is a strictly increasing function of λ, which is a simple corollary of Lemma 2.1. □

In the proof of Lemma 2.3, we see that if $y(1,\lambda) > 0$, then $\phi(\lambda)$ is defined and finite. Later in Section 3, we have occasions to make use of the inverse of this simple fact, namely, that if $\phi(\lambda)$ (or a similar function) is defined and finite, then $y(1;\lambda)$ (or the value at $t = 1$ of a similar function) is positive.

3. Main results

To study the multipoint problem, we use the shooting solution $u(t;h)$, which satisfies the initial condition

$$u(0;h) = h, \qquad u'(0;h) = h\tan\theta, \tag{3.1}$$

for $\theta \neq \pi/2$, and

$$u(0;h) = 0, \qquad u'(0;h) = h, \tag{3.2}$$

for the Dirichlet case.

The function $u(t;h)$ concaves downwards. It can happen that $u(t;h)$ intersects the t-axis at some point $t = \tau \leq 1$. Such a function cannot be a positive solution to our boundary value problem. In the contrary case, suppose that $u(t;h)$ remains positive in $[0,1]$.

We define two functions

$$\overline{\phi}(h) = \sum_{i=1}^{m-2} \frac{k_i u(\xi_i;h)}{u(1,h)} \tag{3.3}$$

$$\psi(h) = \max\left(u(1;h) - \sum_{i=1}^{m-2} k_i u(\xi_i;h), 0\right), \tag{3.4}$$

which are continuous in h (when restricted to where the functions are defined). The first function is similar to $\phi(\lambda)$ defined in (2.10), except that we use $u(t;h)$ instead of $y(t;\lambda)$. Note that $\psi(h) = 0$ if and only if $\overline{\phi}(h) \geq 1$.

The second function ψ can be extended to include all $h \geq 0$ by simply defining $\psi(h) = 0$ if $u(t;h)$ vanishes at some $t \leq 1$. The extended function $\psi(h)$ becomes a continuous function of $h \in [0,\infty)$.

It is obvious from the definition that for $b > 0$, $u(t;\kappa)$ furnishes a solution to our multipoint problem if and only if $\psi(\kappa) = b$. For $b = 0$, $u(t;\kappa)$ is a nontrivial solution if and only if $\kappa \neq 0$ and is a boundary point of the set of points $\{h > 0 \mid \psi(h) = 0\}$ (in other words, $\psi(\kappa) = 0$, and every neighborhood of κ contains points for which $\psi(h) > 0$).

We can now state our first result.

Theorem 3.1. *Suppose that (1.2) hold, and*

$$\limsup_{u\to 0+} \frac{f(u)}{u} < L_\theta, \qquad \liminf_{u\to\infty} \frac{f(u)}{u} > L_\theta, \tag{3.5}$$

where L_θ is the positive constant guaranteed by Lemma 2.3. Then there exists a constant $b^* > 0$ such that the BVP (1.1), (1.6), and (1.7) has

(1) *at least two positive solutions for $b \in (0,b^*)$,*
(2) *at least one positive solution for $b = 0$ or b^*,*
(3) *and no positive solution for $b > b^*$.*

Proof. The first condition means that when u is sufficiently small, the nonlinear term $a(t)f(u(t))$ is dominated by the linear function $La(t)u(t)$. More precisely, let L_1 be any number such that

$$\limsup_{u\to 0+} \frac{f(u)}{u} < L_1 < L_\theta. \tag{3.6}$$

Then there exists a $u_1 > 0$ such that for all $u \in [0,u_1]$,

$$f(u) < L_1 u < L_\theta u. \tag{3.7}$$

Let us shoot a solution $u(t;h)$ with a sufficiently small h. Since $u(t;h)$ concaves downwards, its curve lies below the straight line that is tangent to the curve at the point $(0,h)$. By choosing h sufficiently small, say for $h < h_1$ for some $h_1 > 0$, we can guarantee that $u(t;h) \leq u_1$ for all $t \in [0,1]$. The inequality (3.7), therefore, holds for all t close to $t = 0$, up to the first zero of $u(t,h)$ if there is one before $t = 1$. This allows us to compare $u(t;h)$ with solutions of

$$z''(t) + L_1 a(t)z(t) = 0, \quad z(0) = h \leq h_1, \quad z'(0) = h\tan\theta, \tag{3.8}$$

at least in the neighborhood of 0 before the first zero of $u(t;h)$. It is easy to see that, in fact, $z(t) = hy(t;L_1)$, where $y(t;\lambda)$ is the solution of (2.8) defined in the proof of Lemma 2.3. By the Sturm comparison theorem, $u(t;h) \geq z(t) \geq hy(t;L_\theta)$ for all t. Since $y(t;L_\theta)$ satisfies the boundary condition (2.9), we see that $y(t;L_\theta)$ does not vanish in $[0,1]$. Hence, $u(t;h)$ does not vanish in $[0,1]$, and the comparison of $u(t;h)$ with $z(t)$ is actually valid on the entire interval $[0,1]$. Another implication is that $\psi(h)$ will now be determined by (3.4) instead of being set simply to 0 in the case when the solution vanishes somewhere in $[0,1]$.

Using Lemma 2.1, we have

$$\overline{\phi}(h) \geq \phi(L_1) > \phi(L_\theta) = 1. \tag{3.9}$$

It follows that $u(1,h) - \sum k_i u(\xi_i;h) > 1$ and consequently $\psi(h) > 0$. Recall that this fact is proved for all $h \in (0,h_1)$.

Next, let us study the function $\psi(h)$ when h is large. The second condition of (3.5) is similar to the first one, and suggests an analogous situation. Let L_2 be any number between L_θ and $\liminf_{u\to\infty} f(u)/u$. By hypothesis, we can find a u_2 large enough such that

$$f(u) \geq L_2 u > L_\theta u \tag{3.10}$$

for all $u \geq u_2$. This allows us to compare solutions of (1.1) with solutions of

$$w''(t) + L_2 a(t) w(t) = 0, \quad w(0) = h \geq u_2, \quad w'(0) = h \tan\theta \tag{3.11}$$

(note that $w(t)$ is simply $hy(t;L_2)$) as long as $u(t;h)$ remains above u_2. This last requirement complicates our arguments because we have no guarantee that $u(t;h) \geq u_2$ when t is near 1. The Dirichlet case has an additional complication because $u(t;0) = 0$, and we have to deal with those points that are near $t = 0$.

In the following, we present the detailed proof for the Neumann case. The proof for the general case is similar, with an appropriate modification of the value of τ. We leave the Dirichlet case to the readers.

We now assume only the Neumann case with $u(0;h) = h$ and $u'(0,h) = 0$. Suppose that $u(\tau;h) = u_2$ for some $t = \tau$. We claim that if $\tau \leq 1 - u_2/h$, then $u(t;h)$ must vanish somewhere in $[\tau,1]$. In such cases, by definition, $\psi(h) = 0$. To prove the claim, in the *tu*-plane, we draw a straight line S joining the points $(0,h)$ and $(1,0)$. The point $(1 - u_2/h, u_2)$ lies on S. The solution curve $u(t;h)$ intersects the straight line S at the initial point $(0,h)$ but stays above S at least for a neighborhood near $t = 0$. If $\tau \leq 1 - u_2/h$, then the point on the curve at $t = \tau$ is below S. Therefore, the solution curve intersects S at a second point somewhere before τ. Since $u(t;h)$ is concave, it cannot intersect S at a third point, so $u(t;h)$ must lie strictly below S in $[t,1]$, forcing it to vanish somewhere before reaching $t = 1$.

It, therefore, remains to find what $\psi(h)$ is when $\tau > 1 - u_2/h$. By choosing h sufficiently large, we can make τ as close to 1 as we please. Let us determine how close it should be in order to work for us. We know that $\phi(L_2) > \phi(L_\theta) = 1$. By continuity, we can pick a point τ_1 close to, but distinct from, 1 such that

$$\sum_{i=1}^{m-2} \frac{k_i w(\xi_i)}{w(\tau_1)} = \sum_{i=1}^{m-2} \frac{k_i y(\xi_i;L_2)}{y(\tau_1;L_2)} \geq 1. \tag{3.12}$$

Now, we let h_2 be chosen such that $\tau_1 = 1 - u_2/h_2$.

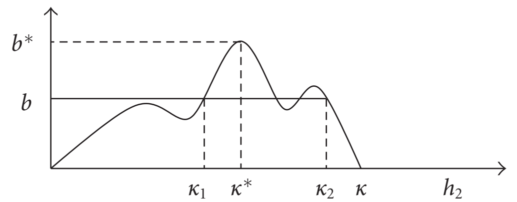

FIGURE 3.1. Graph of $\psi(h)$ (Theorem 3.1).

We claim that $\psi(h) = 0$ for all $h > h_2$. Let us consider all shooting solutions $u(t;h)$ with initial height $h \geq h_2$. If $u(\tau_1;h) \leq u_2$, then $u(t;h)$ must have reached u_2 before $\tau_1 < 1 - u_2/h$. By the above claim, we know that $\psi(h) = 0$. So we assume that $u(\tau_1;h) > u_2$. In the interval $[0,\tau_1]$, comparing (1.1) with w is valid and Lemma 2.1 gives

$$\sum_{i=1}^{m-2} \frac{k_i u(\xi_i;h)}{u(\tau_1;h)} \geq \sum_{i=1}^{m-2} \frac{k_i w(\xi_i)}{w(\tau_1)} \geq 1. \tag{3.13}$$

Even though we do not have precise information on how $u(t;h)$ behaves in the interval $[\tau_1,1]$, we can still determine $\psi(h)$. It can happen that $u(t;h)$ has a zero in this interval. Then $\psi(h) = 0$, by definition. If $u(t;h)$ has no zero in $[\tau,1]$, we know that it is a decreasing function, and so $u(\tau_1;h) > u(1;h)$. Hence,

$$\phi(h) = \sum_{i=1}^{m-2} \frac{k_i u(\xi_i;h)}{u(1;h)} > \sum_{i=1}^{m-2} \frac{k_i u(\xi_i;h)}{u(\tau_1;h)} \geq 1. \tag{3.14}$$

It then follows that $\psi(h) = 0$.

To summarize, the continuous function $\psi(h)$ is positive in a right neighborhood of $h = 0$, and 0 for all $h > h_2$. It, therefore, is bounded above. Let the least upper bound be b^*, which is obviously positive, and suppose that it is attained at a point $\kappa^* > 0$, $\psi(\kappa^*) = b^*$. Figure 3.1 illustrates a concrete example.

Let $b \in (0,b^*)$. Then, by continuity, $\psi(h)$ must assume the value b at least twice: once at a point κ_1 in $(0,\kappa^*)$ and once at a point κ_2 in (κ^*,∞). Each of these furnishes a solution to the multipoint problem. It is, of course, possible that there may be other solutions, in particular when the function $\psi(h)$ has multiple local maxima and local minima. If $b = b^*$, then $h = \kappa^*$ gives a solution to the multipoint problem. If $b = 0$, then the first value κ in (κ^*,∞), for which $\psi(\kappa) = 0$, gives a solution to the multipoint problem. There may or may not be a solution for h in $(0,\kappa^*)$ because it can happen that the only value h that solves $\psi(h) = 0$ is $h = 0$, which corresponds to the trivial solution. For $b > b^*$, $\psi(h) = b$ has no solution and neither does the multipoint boundary value problem. □

THEOREM 3.2. *Suppose that (1.2) hold, and*

$$\limsup_{u \to \infty} \frac{f(u)}{u} < L_\theta. \tag{3.15}$$

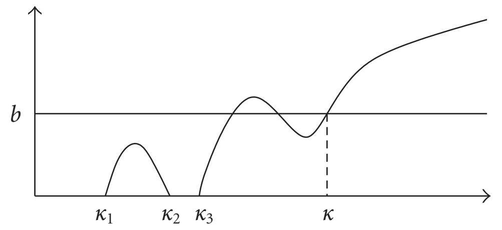

FIGURE 3.2. Graph of $\psi(h)$ (Theorem 3.2).

Then for all $b > 0$, the boundary value problem (1.1), (1.6), and (1.7) has at least one positive solution. If, in addition,

$$\liminf_{u \to 0+} \frac{f(u)}{u} > L_\theta, \tag{3.16}$$

then the same multipoint problem with $b = 0$ has at least one positive solution.

Proof. The arguments are the same as those used to prove Theorem 3.1, except that we interchange the parts regarding large and small h, respectively, and the conclusions are different.

Assume first that only (3.15) holds. Let L_1 be a number between $\limsup_{u\to\infty} f(u)/u$ and L_θ, and let u_1 be so large that $f(u)/u \le L_1 < L_\theta$, for $u \ge u_1$.

We can compare the shooting solution $u(t;h)$ with $z(t) = hy(t;L_1)$ as we do in the proof of Theorem 3.1. Since $z(1) > 0$, we can take $h > u_1/z(1)$. This ensures that $u(t;h)$ remains greater than u_1 for all $t \in [0,1]$ so that the comparison is valid in the entire interval $[0,1]$. In particular, we have

$$u(1;h) \ge z(1) = hy(1;L_\theta). \tag{3.17}$$

We see that

$$\lim_{h\to\infty} u(1;h) = \infty. \tag{3.18}$$

Now, Lemma 2.1 gives $\overline{\phi}(h) \le \phi(L_1) < \phi(L_\theta) = 1$. This implies that

$$\psi(h) = u(1;h) - \sum_{i=1}^{m-2} k_i u(\xi_i;h) \ge (1 - \overline{\phi}(h))u(1;h) \longrightarrow \infty \tag{3.19}$$

as $h \to \infty$. We thus see that $\psi(h)$ is a continuous function on $[0,\infty)$, with the properties that $\psi(0) = 0$ and $\psi(h) \to \infty$. This is depicted in Figure 3.2. Note also that the function $\psi(h)$ may vanish on a subinterval of $[0,\infty)$, and this situation is also illustrated in Figure 3.2.

On account of this, any $b > 0$ is in the range of $\psi(h)$ and the corresponding boundary value problem has at least one positive solution.

To prove the part concerning $b = 0$, we have to show that $\psi(h) = 0$ for all h that are sufficiently small. This is done by using the second condition (3.16) to compare $u(t;h)$

with $w(t) = hy(t;L_2)$ as in the proof of Theorem 3.1. In fact, the argument is easier this time since the comparison condition is now satisfied for all $t \in [0,1]$, and we do not need to find special treatments for a set of t such as those near $t = 1$ in the proof of Theorem 3.1. □

We remark that Theorem 3.2 does not assert the uniqueness of the positive solution when $b = 0$. In fact, in the example shown in Figure 3.2, there are three positive solutions, corresponding to the points κ_1, κ_2, and κ_3.

It is interesting to note the asymmetry between Theorems 3.1 and 3.2. In Theorem 3.1, we need both asymptotic conditions to prove the existence of positive solutions to the boundary value problem, while in Theorem 3.2, we need only one asymptotic condition to get the existence of the solutions. In addition, Theorem 3.1 gives at least two positive solutions to the boundary value problem for $b < b^*$, while Theorem 3.2 can only guarantee at least one positive solution for all $b > 0$.

By examining the proof of Theorem 3.1 more closely, it is not difficult to see that the only places where the separable format of the nonlinear term is needed are to enable us to compare the nonlinear term of (1.1) with the linear terms of the comparing equations (3.8) and (3.11). If we include that as part of the hypotheses, we can obtain the following analogous results concerning the more general nonlinear equivalent (1.3).

Theorem 3.3. *Suppose that (1.2) hold, and there exist four constants L_1, L_2, u_1, and u_2 such that*

$$\begin{gathered} 0 < L_1 < L_\theta < L_2, \qquad 0 < u_1 < u_2, \\ F(t,u) \le L_1 a(t) u, \quad \forall u < u_1, \\ F(t,u) \ge L_2 a(t) u, \quad \forall u > u_2. \end{gathered} \tag{3.20}$$

Then the conclusions of Theorem 3.1 hold for the BVP (1.3), (1.6), and (1.7).

Likewise, we have the following.

Theorem 3.4. *If we replace condition (3.15) by the existence of two constants L_1 and u_1 such that*

$$F(t,u) \le L_1 a(t) u, \quad \forall u > u_1, \tag{3.21}$$

and condition (3.16) by the existence of two constants L_2 and u_2 such that

$$F(t,u) \ge L_2 a(t) u, \quad \forall u < u_2, \tag{3.22}$$

then the conclusions of Theorem 3.2 hold for the BVP (1.3), (1.6), and (1.7).

Let us assume, in addition, that $a(t) > 0$ for all $t \in [0,1]$, and $b(t) \ge 0$ is a given function on $[0,1]$. Theorem 3.4 obviously implies the following extension of Theorem 3.2 to the "forced" equation

$$u''(t) + a(t) f(u(t)) + b(t) = 0, \quad t \in (0,1). \tag{3.23}$$

Theorem 3.5. *Suppose that (1.2) hold, and $a(t) > 0$ and $b(t) \geq 0$ are in $[0,1]$. Then Theorem 3.2 continues to hold for (3.23).*

Proof. Let $F(t,u) = a(t)f(u) + b(t)$. Then the hypotheses of Theorem 3.5 imply the hypotheses of Theorem 3.4, and the same conclusions of Theorem 3.2 hold. □

Note again the asymmetry, Theorem 3.1 does not appear to have a similar extension to the forced equation.

4. Discussion and examples

(1) Theorem 1.1 has been extended by Raffoul [16] by showing that it still holds if f_0 and f_∞ are positive finite constants satisfying certain bounds. The constants given by Raffoul are not the best possible. Our Theorems 3.1 and 3.2 include Raffoul's results and give the best possible constant L_θ. A discussion of Raffoul's results can be found in [1]. In [17], Liu and Yu proved results similar to that of Theorem 1.1 for the homogeneous three-point problem. Liu [18] further improved Theorem 1.1 by considering cases when both of f_0 and f_∞ are finite or zero. He also proved existence theorems when both f_0 and f_∞ are finite. In all these cases, he also assumed that $f(u)$ is either bounded above or below by a constant multiple of $|u|$ in certain specified intervals. The techniques used in this paper can easily be extended to generalize these results.

(2) With the exception of a few, for example, [1, 3], most of the results on the existence of solutions to multipoint boundary value problems are based upon fixed-point theorems of Krasnoselskii's type; see Krasnoselskii [2], Guo and Lakshmikantham [19], and recent articles of Ma [12, 14, 20], Liu [18], and Sun et al. [13]. Other methods in nonlinear functional analysis such as Leray-Schauder continuation theorem, nonlinear alternative of Leray-Schauder fixed point theorem, and coincidence degree theory have also been used, see Mawhin [21, 22]. In most cases, the conditions imposed are stronger than those required by using the shooting method and Sturm comparison theorem as discussed in the previous work [1] and in this paper. Nevertheless, the abstract methods in Banach spaces have the important advantage over the shooting method; in that, it can be applied to higher-order equations, in higher dimensions, and to equations with deviating arguments.

(3) The shooting method and Sturm's comparison theorem are effective when the nonlinear function is bounded by a linear growth. In the case of (1.1), the corresponding linear differential operator is invertible on a suitable function space. This is known as the nonresonance case and Leray-Schauder degree theory has been applied to study m-point problems. For results on three-point boundary conditions, see Gupta et al. [23, 24] and Gupta and Trofimchuk [25]. Gupta [6, 7] further employed a nonlinear alternative of Leray-Schauder fixed point theorem to obtain sharper conditions where the coefficient function in (1.1) is not required to be continuous but only integrable in $[0,1]$. For results related to multipoint boundary conditions, see Guo et al. [26]. For the resonance case with nonlinear function still subject to linear growth, see Gupta [27, 28] and Feng and Webb [29].

(4) In Feng and Webb [29, 30] and Feng [31], more general nonlinear functions subject to certain sign conditions and quadratic growth are studied. It is interesting to see if the shooting method and Sturm comparison theorem can be extended to such cases.

(5) Multipoint problems have been studied for p-Laplacian equations; see, for example, [32]. The shooting method can be easily adapted to get similar results for such equations because the Sturm comparison theorem remains valid for p-Laplacian equations.

(6) Another generalization of the multipoint problem can be obtained by replacing (1.7) by an integral condition (sometimes called a nonlocal condition) such as

$$u(1) = \int_0^1 k(t)u(t)\,dt \tag{4.1}$$

and the corresponding nonhomogeneous form. This can be considered as the continuous analogue of the discrete m-point condition. The shooting method can also be easily adapted to treat such generalizations.

(7) Numerous authors have derived results guaranteeing multiple solutions to various boundary value problems. Typically, assumptions are imposed on the nonlinear function so that it is alternatively large and then small in successive subintervals of $[0,\infty)$. See [1] for a discussion of such results.

We close our discussion with three examples.

Example 4.1. The forced equation (3.23) in Theorem 3.5 is an example that is covered by the general case (1.3) and cannot be put into the separable format (1.1). A similar example that is covered by Theorem 3.3 but not Theorem 3.1 is

$$u''(t) + a(t)f(u(t)) + b(t)g(u(t)) = 0, \tag{4.2}$$

where $a(t) > 0$, $b(t) \geq 0$ in $[0,1]$, and $g(u)$ is a continuous nonnegative function such that $g(0) = 0$.

Example 4.2. Consider the second-order nonlinear differential equation

$$u''(t) + \frac{u^2(t)}{1+u(t)} = 0, \quad t \in (0,1), \tag{4.3}$$

subject to the Robin boundary condition at the left end-point

$$u'(0) = u(0), \qquad u(1) - \frac{1}{3}u\left(\frac{1}{2}\right) = b. \tag{4.4}$$

Here, $f_0 = 0$ and $f_\infty = 1$ as defined by (1.10).

Theorem 3.1 shows that there exists a positive number b^* such that the boundary value problem (4.3) and (4.4) has at least two positive solutions for all b, $0 \leq b < b^*$, at least one positive solution for $b = b^*$, and has no solution for $b^* > 0$. Here, Theorem 1.2 and other similar results such as Raffoul [16], Liu [18], and Liu and Yu [17] are not applicable.

Example 4.3. Consider the second-order nonlinear equation

$$u''(t)+\frac{|u(t)|^{\gamma}}{1+u(t)}=0,\quad t\in(0,1),\ 0<\gamma<1, \tag{4.5}$$

subject to the Dirichlet boundary condition at the left end-point

$$u(0)=0,\qquad u(1)-\frac{1}{2}u\left(\frac{1}{3}\right)-\frac{1}{3}u\left(\frac{1}{2}\right)=b\geq 0. \tag{4.6}$$

Here, $f_0=\infty$ and $f_\infty=0$, which is typical in the sublinear case. Theorem 3.2 is applicable (by choosing any $L_\theta>0$) and we conclude that the boundary value problem (4.5) and (4.6) always has a positive solution for all $b\geq 0$. This example is not covered by either Theorem 1.1 or any other previously known results applicable to the sublinear case.

Acknowledgments

The authors would like to thank two anonymous referees for their careful reading of this manuscript and for suggesting useful rhetorical changes. They are thankful to one of the referees who pointed out that the shooting method has also been used by Naito and Tanaka [33] and by Kong [34] for the study of two-point boundary value problems.

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