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Research Article

Existence and Uniqueness of Solutions for Boundary Value Problems to the Singular One-Dimension p-Laplacian

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In this paper, We study the existence and uniqueness of solutions for boundary value problems to the singular one-dimension p-Laplacian by using mixed monotone method. Our results improve several recent results established in the literature.

1. Introduction

In this paper, we discuss the existence and uniqueness of solution to the boundary value problem

$$\begin{gathered}(\phi(x'))' + \lambda q(t) f(x) = 0, \quad t \in (0,1),\ \lambda > 0,\\ x(0) = x(1) = 0,\end{gathered} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$, and f may be singular at $x = 0$.

By a solution x to (1.1) we mean a function $x \in C^1[0,1]$, $\phi(x') \in AC[0,1]$ such that x satisfies (1.1) and the boundary condition; here $AC[0,1]$ denotes the space of absolutely continuous functions defined on $[0,1]$.

It is of interest to note here that the existence of positive solutions to problem (1.1) has been studied in great detail in the literature, see [1–10]. However, there are few works on the uniqueness of solutions for boundary problems to the singular one-dimension p-Laplacian. In this paper, we present a new existence and uniqueness theory by using mixed monotone method which has been used in [11, 12].

2. Preliminaries

Let $E = C[0,1]$, with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$, so E is a Banach space. Also, we define

$$P = \{u \in E : u \text{ is concave on } [0,1] \text{ and } u(0) = u(1) = 0\}. \tag{2.1}$$

One may readily verify that P is a cone in E. $e \in P$, with $\|e\| \le 1, e \ne \theta$. Define

$$Q_e = \{x \in P \mid x \ne \theta, \text{ there exist constants } m,\ M > 0 \text{ such that } m\,e \le x \le Me\}. \tag{2.2}$$

Now we give a definition (see [13]).

Definition 2.1. Assume $A : Q_e \times Q_e \to Q_e$. A is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y, that is, if $x_1 \le x_2$ $(x_1, x_2 \in Q_e)$ implies $A(x_1, y) \le A(x_2, y)$ for any $y \in Q_e$, and $y_1 \le y_2$ $(y_1, y_2 \in Q_e)$ implies $A(x, y_1) \ge A(x, y_2)$ for any $x \in Q_e$. $x^* \in Q_e$ is said to be a fixed point of A if $A(x^*, x^*) = x^*$.

Theorem 2.2 (see [11]). *Suppose that $A : Q_e \times Q_e \to Q_e$ is a mixed monotone operator and there exists a constant β, $0 \le \beta < 1$ such that*

$$A\left(tx, \frac{1}{t}y\right) \ge t^{\beta} A(x, y), \quad \forall x, y \in Q_e,\ 0 < t < 1. \tag{2.3}$$

Then A has a unique fixed point $x^* \in Q_e$. Moreover, for any $(x_0, y_0) \in Q_e \times Q_e$,

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \tag{2.4}$$

satisfy

$$x_n \longrightarrow x^*, \qquad y_n \longrightarrow x^*, \tag{2.5}$$

where

$$\|x_n - x^*\| = o\left(1 - r^{\beta^n}\right), \qquad \|y_n - x^*\| = o\left(1 - r^{\beta^n}\right), \tag{2.6}$$

$0 < r < 1$, *r is a constant from (x_0, y_0).*

Theorem 2.3 (see [11, 13]). *Suppose that $A : Q_e \times Q_e \to Q_e$ is a mixed monotone operator and there exists a constant $\beta \in (0, 1)$ such that* (2.3) *holds. If x^*_λ is a unique solution of equation*

$$A(x, x) = \lambda x \quad (\lambda > 0) \tag{2.7}$$

in Q_e, then $\|x^*_\lambda - x^*_{\lambda_0}\| \to 0$, $\lambda \to \lambda_0$. If $0 < \beta < 1/2$, then $0 < \lambda_1 < \lambda_2$ implies $x^*_{\lambda_1} \ge x^*_{\lambda_2}$, $x^*_{\lambda_1} \ne x^*_{\lambda_2}$, and

$$\lim_{\lambda\to+\infty} \|x^*_\lambda\| = 0, \qquad \lim_{\lambda\to 0^+} \|x^*_\lambda\| = +\infty. \tag{2.8}$$

3. Existence and uniqueness

In this section, we discuss the singular one-dimension p-Laplacian

$$\begin{gathered}(\phi(x'))' + q(t)f(x) = 0, \quad t \in (0,1), \\ x(0) = x(1) = 0.\end{gathered} \tag{3.1}$$

Throughout this section we assume that

$$f(x) = g(x) + h(x), \tag{3.2}$$

where

$$\begin{gathered}g : [0,+\infty) \longrightarrow [0,+\infty) \text{ is continuous and nondecreasing;} \\ h : (0,+\infty) \longrightarrow (0,+\infty) \text{ is continuous and nonincreasing.}\end{gathered} \tag{3.3}$$

Theorem 3.1. *Suppose that there exists $\alpha \in (0,p-1)$ such that*

$$g(tx) \geq t^{\alpha} g(x), \tag{3.4}$$

$$h(t^{-1}x) \geq t^{\alpha} h(x), \tag{3.5}$$

for any $t \in (0,1)$ and $x > 0$, and $q \in C((0,1),(0,\infty))$ satisfies

$$\int_0^1 t^{-\alpha}(1-t)^{-\alpha} q(t)dt < +\infty, \quad 0 < \alpha < p-1. \tag{3.6}$$

Then (3.1) *has a unique positive solution $x_\lambda^*(t)$.*
And moreover, $0 < \lambda_1 < \lambda_2$ implies that $x_{\lambda_1}^* \leq x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$. If $\alpha/(p-1) \in (0,1/2)$, then

$$\lim_{\lambda \to 0^+} \|x_\lambda^*\| = 0, \qquad \lim_{\lambda \to +\infty} \|x_\lambda^*\| = +\infty. \tag{3.7}$$

Lemma 3.2. *Let u, v be solutions to*

$$\begin{gathered}(\phi(u'))' + \lambda q(t) t^{-\alpha}(1-t)^{-\alpha} = 0, \quad t \in (0,1),\ \lambda > 0,\ q(t)t^{-\alpha}(1-t)^{-\alpha} \in L^1(0,1), \\ u(0) = u(1) = 0, \\ (\phi(v'))' + \lambda q(t) t^{\alpha}(1-t)^{\alpha} = 0, \quad t \in (0,1),\ \lambda > 0,\ q(t)t^{\alpha}(1-t)^{\alpha} \in L^1(0,1), \\ v(0) = v(1) = 0,\end{gathered} \tag{3.8}$$

then there exist positive constants C_u, C_v such that

$$t(1-t)\|u\| \leq u(t) \leq C_u t(1-t), \qquad t(1-t)\|v\| \leq v(t) \leq C_v t(1-t). \tag{3.9}$$

Proof. Because $q(t)t^{-\alpha}(1-t)^{-\alpha} q(t)t^{\alpha}(1-t)^{\alpha} \geq 0$, then u, v is concave and positive on $(0,1)$. As $u, v \in C^1[0,1]$, thus

$$\lim_{t \to 0} \frac{u(t)}{t} = u'(0), \qquad \lim_{t \to 1} \frac{u(t)}{1-t} = -u'(1), \qquad \lim_{t \to 0} \frac{v(t)}{t} = v'(0), \qquad \lim_{t \to 1} \frac{v(t)}{1-t} = -v'(1). \tag{3.10}$$

Let

$$C_u = \sup_{t\in(0,1)} \frac{u(t)}{t(1-t)}, \qquad C_v = \sup_{t\in(0,1)} \frac{v(t)}{t(1-t)}, \tag{3.11}$$

then $0 < C_u,\ C_v < \infty$, and

$$u(t) \le C_u\, t(1-t), \qquad v(t) \le C_v\, t(1-t). \tag{3.12}$$

Since u, v is concave, then

$$u(t) \ge t(1-t)\|u\|, \qquad v(t) \ge t(1-t)\|v\|. \tag{3.13}$$

So we have

$$t(1-t)\|u\| \le u(t) \le C_u t(1-t), \qquad t(1-t)\|v\| \le v(t) \le C_v t(1-t). \tag{3.14}$$

□

Lemma 3.3 (see [7]). *If $u, v \in C^1[0,1]$ satisfies*

$$\begin{gathered} -(\phi(u'))' \ge -(\phi(v'))', \quad a.e.\ t \in [0,1], \\ u(0) \ge v(0), \qquad u(1) \ge v(1), \end{gathered} \tag{3.15}$$

then $u(t) \ge v(t)$ for all $t \in [0,1]$.

Proof of Theorem 3.1. Since (3.5) holds, let $t^{-1}x = y$, one has

$$h(y) \ge t^{\alpha} h(ty). \tag{3.16}$$

Then

$$h(ty) \le \frac{1}{t^{\alpha}} h(y), \quad \forall t \in (0,1),\ y > 0. \tag{3.17}$$

Let $y = 1$. The above inequality is

$$h(t) \le \frac{1}{t^{\alpha}} h(1), \quad \forall t \in (0,1). \tag{3.18}$$

From (3.5), (3.17), and (3.18), one has

$$h(t^{-1}x) \ge t^{\alpha} h(x), \quad h\left(\frac{1}{t}\right) \ge t^{\alpha} h(1), \quad h(tx) \le \frac{1}{t^{\alpha}} h(x), \quad h(t) \le \frac{1}{t^{\alpha}} h(1), \quad t \in (0,1),\ x > 0. \tag{3.19}$$

Similarly, from (3.4), one has

$$g(tx) \ge t^{\alpha} g(x), \quad g(t) \ge t^{\alpha} g(1), \quad t \in (0,1),\ x > 0. \tag{3.20}$$

Let $t = 1/x$, $x > 1$, one has

$$g(x) \le x^{\alpha} g(1), \quad x \ge 1. \tag{3.21}$$

Let $e(t) = t(1-t)$, and we define

$$Q_e = \left\{ x \in C[0,1] \mid \frac{1}{M} t(1-t) \le x(t) \le Mt(1-t),\ t \in [0,1] \right\}, \tag{3.22}$$

where $M > 1$ is chosen such that

$$M > \max\left\{ C_u^{(p-1)/(p-1-\alpha)} \left(g(1)+h(1)\right)^{1/(p-1-\alpha)},\ \|v\|^{-(p-1)/(p-1-\alpha)} \left(g(1)+h(1)\right)^{-1/(p-1-\alpha)} \right\}. \tag{3.23}$$

For any fixed $x, y \in Q_e$,consider the following boundary value problem:

$$\begin{gathered} \phi(w'(t))' + \lambda q(t)\left[g(x(t)) + h(y(t))\right] = 0, \quad t \in (0,1),\ \lambda > 0; \\ w(0) = w(1) = 0. \end{gathered} \tag{3.24}$$

By (3.18)–(3.21), for $x, y \in Q_e$, we can obtain

$$\begin{aligned} g(x(t)) &\le g(Mt(1-t)) \le g(M) \le M^{\alpha} g(1), \quad t \in (0,1), \\ h(y(t)) &\le h\left(\frac{1}{M} t(1-t)\right) \le t^{-\alpha}(1-t)^{-\alpha} h\left(\frac{1}{M}\right) \\ &\le M^{\alpha} t^{-\alpha}(1-t)^{-\alpha} h(1), \quad t \in (0,1). \end{aligned} \tag{3.25}$$

So,

$$\begin{aligned} g(x(t)) + h(y(t)) &\le M^{\alpha}\left[g(1) + t^{-\alpha}(1-t)^{-\alpha} h(1)\right] \\ &\le M^{\alpha} t^{-\alpha}(1-t)^{-\alpha}(g(1)+h(1)), \quad t \in (0,1), \end{aligned} \tag{3.26}$$

then $\lambda q(t)[g(x(t))+h(y(t))] \in L^1(0,1)$. It follows from [7] that, for each fixed $x, y \in Q_e$, problem (3.22) has a solution $w \in C^1[0,1]$, and (3.24) is equivalent to

$$w(t) = \lambda^{1/(p-1)} \int_0^t \phi^{-1}\left(\tau + \int_s^1 q(r)\left[g(x(r)) + h(y(r))\right] dr\right) ds, \quad 0 \le t \le 1, \tag{3.27}$$

where $\tau = \phi(w'(1))$ is a solution of the equation

$$\int_0^1 \phi^{-1}\left(\tau + \int_s^1 q(r)\left[g(x(r)) + h(y(r))\right] dr\right) ds = 0. \tag{3.28}$$

For any $x, y \in Q_e$, we define

$$A(x,y)(t) = w(t) = \lambda^{1/(p-1)} \int_0^t \phi^{-1}\left(\tau + \int_s^1 q(r)\left[g(x(r)) + h(y(r))\right] dr\right) ds, \tag{3.29}$$

then $A(x,y)(t)$ is concave on $(0,1)$, for any $(x,y) \in Q_e \times Q_e$, $q(t)[g(x(t) + h(y(t)))] \in L^1(0,1)$.

First, we show for any $(x,y)\in Q_e$, $A(x,y)\in Q_e$.

Let $x,y\in Q_e$, from (3.19) and (3.20), we have

$$\begin{aligned} g(x(t)) &\geq g\left(\frac{1}{M}t(1-t)\right) \geq t^{\alpha}(1-t)^{\alpha} g\left(\frac{1}{M}\right) \geq t^{\alpha}(1-t)^{\alpha}\frac{1}{M^{\alpha}}g(1), \\ h(y(t)) &\geq h(Mt(1-t)) \geq h(M) = h\left(\frac{1}{1/M}\right) \geq \frac{1}{M^{\alpha}}h(1), \quad t\in(0,1). \end{aligned} \tag{3.30}$$

Thus, we have

$$g(x(t)) + h(y(t)) \geq \frac{1}{M^{\alpha}}\left[g(1)t^{\alpha}(1-t)^{\alpha} + h(1)\right]. \tag{3.31}$$

So, we can obtain

$$\begin{aligned} (*)\; g(x(t)) + h(y(t)) &\leq M^{\alpha}\left[g(1) + t^{-\alpha}(1-t)^{-\alpha}h(1)\right] \\ &\leq M^{\alpha}t^{-\alpha}(1-t)^{-\alpha}(g(1)+h(1)), \quad t\in(0,1), \\ (**)\; g(x(t)) + h(y(t)) &\geq \frac{1}{M^{\alpha}}\left[g(1)t^{\alpha}(1-t)^{\alpha} + h(1)\right] \\ &\geq M^{-\alpha}t^{\alpha}(1-t)^{\alpha}(g(1)+h(1)), \quad t\in(0,1). \end{aligned} \tag{3.32}$$

So

$$-\left[\phi(w')\right]' \leq q(t)M^{\alpha}t^{-\alpha}(1-t)^{-\alpha}(g(1)+h(1)), \tag{3.33}$$

that is,

$$-\left[\phi(w')\right]' \leq -M^{\alpha}(g(1)+h(1))\left[\phi(u')\right]'. \tag{3.34}$$

Similarly,

$$-\left[\phi(w')\right]' \geq q(t)M^{-\alpha}t^{\alpha}(1-t)^{\alpha}(g(1)+h(1)), \tag{3.35}$$

that is,

$$-\left[\phi(w')\right]' \geq -M^{-\alpha}(g(1)+h(1))\left[\phi(v')\right]'. \tag{3.36}$$

By Lemma 3.3,

$$\begin{aligned} w(t) &\leq M^{\alpha/(p-1)}(g(1)+h(1))^{1/(p-1)}u(t) \\ &\leq C_u(g(1)+h(1))^{1/(p-1)}M^{\alpha/(p-1)}t(1-t) \leq Mt(1-t), \\ w(t) &\geq M^{-\alpha/(p-1)}(g(1)+h(1))^{1/(p-1)}v(t) \\ &\geq \|v\|M^{-\alpha/(p-1)}(g(1)+h(1))^{1/(p-1)}t(1-t) \geq \frac{1}{M}t(1-t). \end{aligned} \tag{3.37}$$

So, the operator A is well defined.

Next, for any $l \in (0,1)$, one has

$$q(t)\left[g(lx(t)) + h(l^{-1}y(t))\right] \ge l^{\alpha} q(t)\left[g(x(t)) + h(y(t))\right]. \tag{3.38}$$

Then by Lemma 3.3 we have

$$A(lx, l^{-1}y)(t) \ge l^{\alpha/(p-1)} A(x,y)(t) \quad \left(0 < \beta = \frac{\alpha}{p-1} < 1\right). \tag{3.39}$$

So the conditions of Theorems 2.2 and 2.3 hold. Therefore, there exists a unique $x_\lambda^* \in Q_e$ such that $A_\lambda(x^*, x^*) = x_\lambda^*$. It is easy to check that x_λ^* is a unique positive solution of (3.1) for given $\lambda > 0$. Moreover, Theorem 2.3 means that if $0 < \lambda_1 < \lambda_2$, then $x_{\lambda_1}^*(t) \le x_{\lambda_2}^*(t)$, $x_{\lambda_1}^*(t) \ne x_{\lambda_2}^*(t)$ and if $\alpha/(p-1) \in (0,1/2)$, then

$$\lim_{\lambda \to 0^+} \|x_\lambda^*\| = 0, \qquad \lim_{\lambda \to +\infty} \|x_\lambda^*\| = +\infty. \tag{3.40}$$

This completes the proof. □

Example 3.4. Consider the following singular p-Laplace boundary value problem:

$$\begin{gathered} (\phi(x'))' + \lambda q(t)\left(\mu x^a + x^{-b}\right) = 0, \quad t \in (0,1); \\ x(0) = x(1) = 0, \end{gathered} \tag{3.41}$$

where $\lambda, a, b > 0,\ \mu \ge 0,\ q \in C(0,1),\ q > 0,\ t \in (0,1)$, and

$$\int_0^1 q(t) t^{-\alpha}(1-t)^{-\alpha} dt < +\infty, \quad 0 < \alpha = \max\{a,b\} < p-1. \tag{3.42}$$

Applying Theorem 3.1, we can find that (3.41) has a unique positive solution $x_\lambda^*(t)$. In addition, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^* \le x_{\lambda_2}^*$, $x_{\lambda_1}^* \ne x_{\lambda_2}^*$. If $\alpha/(p-1) \in (0,1/2)$, then

$$\lim_{\lambda \to 0^+} \|x_\lambda^*\| = 0, \qquad \lim_{\lambda \to +\infty} \|x_\lambda^*\| = +\infty. \tag{3.43}$$

To see that, we put

$$\beta = \frac{\alpha}{p-1}, \qquad g(x) = \mu x^a, \qquad h(x) = x^{-b}. \tag{3.44}$$

Thus $0 < \beta < 1$ and

$$g(tx) = t^a g(x) \ge t^{\alpha} g(x), \qquad h(t^{-1}x) = t^b h(x) \ge t^{\alpha} h(x), \tag{3.45}$$

for any $t \in (0,1)$ and $x > 0$, thus all conditions in Theorem 3.1 are satisfied.

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