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*Research Article*

# Existence of Solutions to a Nonlocal Boundary Value Problem with Nonlinear Growth

**Xiaojie Lin**

*School of Mathematical Sciences, Xuzhou Normal University, Xuzhou, Jiangsu 221116, China*

Correspondence should be addressed to Xiaojie Lin, linxiaojie1973@163.com

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This paper deals with the existence of solutions for the following differential equation: $x''(t) = f(t, x(t), x'(t))$, $t \in (0,1)$, subject to the boundary conditions: $x(0) = \alpha x(\xi)$, $x'(1) = \int_0^1 x'(s)dg(s)$, where $\alpha \geq 0$, $0 < \xi < 1$, $f : [0,1] \times R^2 \to R$ is a continuous function, $g : [0,1] \to [0,\infty)$ is a nondecreasing function with $g(0) = 0$. Under the resonance condition $g(1) = 1$, some existence results are given for the boundary value problems. Our method is based upon the coincidence degree theory of Mawhin. We also give an example to illustrate our results.

## 1. Introduction

In this paper, we consider the following second-order differential equation:

$$x''(t) = f(t, x(t), x'(t)), \quad t \in (0,1), \tag{1.1}$$

subject to the boundary conditions:

$$x(0) = \alpha x(\xi), \qquad x'(1) = \int_0^1 x'(s)dg(s), \tag{1.2}$$

where $\alpha \geq 0$, $0 < \xi < 1$, $f : [0,1] \times R^2 \to R$ is a continuous function, $g : [0,1] \to [0,\infty)$ is a nondecreasing function with $g(0) = 0$. In boundary conditions (1.2), the integral is meant in the Riemann-Stieltjes sense.

We say that BVP (1.1), (1.2) is a problem at resonance, if the linear equation

$$x''(t) = 0, \quad t \in (0,1), \tag{1.3}$$

with the boundary condition (1.2) has nontrivial solutions. Otherwise, we call them a problem at nonresonance.

Nonlocal boundary value problems were first considered by Bicadze and Samarskiĭ [1] and later by Il'pin and Moiseev [2, 3]. In a recent paper [4], Karakostas and Tsamatos studied the following nonlocal boundary value problem:

$$\begin{gathered} x''(t) + q(t) f\left(x(t), x'(t)\right) = 0, \quad t \in (0,1), \\ x(0) = 0, \qquad x'(1) = \int_0^1 x'(s)\,dg(s). \end{gathered} \tag{1.4}$$

Under the condition $0 = g(0) \leq g(1) < 1$ (i.e., nonresonance case), they used Krasnosel'skii's fixed point theorem to show that the operator equation $x = Ax$ has at least one fixed point, where operator $A$ is defined by

$$(Ax)(t) = \frac{t}{1 - g(1)} \int_0^1 \int_s^1 q(r) f\left(x(r), x'(r)\right) dr\, dg(s) + \int_0^t \int_s^1 q(r) f\left(x(r), x'(r)\right) dr\, ds. \tag{1.5}$$

However, if $g(1) = 1$ (i.e., resonance case), then the method in [4] is not valid.

As special case of nonlocal boundary value problems, multipoint boundary value problems at resonance case have been studied by some authors [5–11].

The purpose of this paper is to study the existence of solutions for nonlocal BVP (1.1), (1.2) at resonance case (i.e., $g(1) = 1$) and establish some existence results under nonlinear growth restriction of $f$. Our method is based upon the coincidence degree theory of Mawhin [12].

## 2. Main Results

We first recall some notation, and an abstract existence result.

Let $Y$, $Z$ be real Banach spaces, let $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator which is Fredholm map of index zero (i.e., $\operatorname{Im} L$, the image of $L$, $\operatorname{Ker} L$, the kernel of $L$ are finite dimensional with the same dimension as the $Z/\operatorname{Im} L$), and let $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible; we denote the inverse by $K_P$. Let $\Omega$ be an open bounded, subset of $Y$ such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ is said to be $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded, and $K_P(I - Q)N : \overline{\Omega} \to Y$ is compact. Let $J : \operatorname{Im} Q \to \operatorname{Ker} L$ be a linear isomorphism.

The theorem we use in the following is Theorem IV.13 of [12].

**Theorem 2.1.** *Let $L$ be a Fredholm operator of index zero, and let $N$ be $L$-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$,

(ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$,

(iii) $\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$,

*where* $Q : Z \to Z$ *is a projection with* $\operatorname{Im} L = \operatorname{Ker} Q$. *Then the equation* $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

*For* $x \in C^1[0,1]$, *we use the norms* $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$ *and* $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$ *and denote the norm in* $L^1[0,1]$ *by* $\|\cdot\|_1$. *We will use the Sobolev space* $W^{2,1}(0,1)$ *which may be defined by*

$$W^{2,1}(0,1) = \left\{x : [0,1] \longrightarrow R \mid x, x' \text{ are absolutely continuous on } [0,1] \text{ with } x'' \in L^1[0,1]\right\}. \tag{2.1}$$

*Let* $Y = C^1[0,1]$, $Z = L^1[0,1]$. $L : \operatorname{dom} L \subset Y \to Z$ *is a linear operator defined by*

$$Lx = x'', \quad x \in \operatorname{dom} L, \tag{2.2}$$

*where*

$$\operatorname{dom} L = \left\{x \in W^{2,1}(0,1) : x(0) = \alpha x(\xi), x'(1) = \int_0^1 x'(s)dg(s)\right\}. \tag{2.3}$$

*Let* $N : Y \to Z$ *be defined as*

$$Nx = f(t, x(t), x'(t)), \quad t \in (0,1). \tag{2.4}$$

*Then BVP* (1.1), (1.2) *is* $Lx = Nx$.

We will establish existence theorems for BVP (1.1), (1.2) in the following two cases:

case (i): $\alpha = 0, g(1) = 1, \int_0^1 s\,dg(s) \neq 1$;

case (ii): $\alpha = 1, g(1) = 1, \int_0^1 s\,dg(s) \neq 1$.

**Theorem 2.2.** *Let* $f : [0,1] \times R^2 \to R$ *be a continuous function and assume that*

(H1) *there exist functions* $a, b, c, r \in L^1[0,1]$ *and constant* $\theta \in [0,1)$ *such that for all* $(x, y) \in R^2$, $t \in [0,1]$, *it holds that*

$$|f(t,x,y)| \leq a(t)|x| + b(t)|y| + c(t)\left(|x|^\theta + |y|^\theta\right) + r(t), \tag{2.5}$$

(H2) *there exists a constant* $M > 0$*, such that for* $x \in \operatorname{dom} L$*, if* $|x'(t)| > M$*, for all* $t \in [0,1]$*, then*

$$\int_0^1 f(s, x(s), x'(s))ds - \int_0^1 \int_0^s f(v, x(v), x'(v))dv\, dg(s) \neq 0, \tag{2.6}$$

(H3) *there exists a constant* $M^* > 0$*, such that either*

$$d \cdot \left[\int_0^1 f(s, ds, d)ds - \int_0^1 \int_0^s f(v, dv, d)dv\, dg(s)\right] < 0, \quad \text{for any } |d| > M^*, \tag{2.7}$$

*or else*

$$d \cdot \left[\int_0^1 f(s, ds, d)ds - \int_0^1 \int_0^s f(v, dv, d)dv\, dg(s)\right] > 0, \quad \text{for any } |d| > M^*. \tag{2.8}$$

*Then BVP* (1.1), (1.2) *with* $\alpha = 0$, $g(1) = 1$, *and* $\int_0^1 s\, dg(s) \neq 1$ *has at least one solution in* $C^1[0,1]$ *provided that*

$$\|a\|_1 + \|b\|_1 < \frac{1}{2}. \tag{2.9}$$

**Theorem 2.3.** *Let* $f : [0,1] \times R^2 \to R$ *be a continuous function. Assume that assumption (H1) of Theorem 2.2 is satisfied, and*

(H4) *there exists a constant* $M > 0$*, such that for* $x \in \operatorname{dom} L$*, if* $|x(t)| > M$*, for all* $t \in [0,1]$*, then*

$$\int_0^1 f(s, x(s), x'(s))ds - \int_0^1 \int_0^s f(v, x(v), x'(v))dv\, dg(s) \neq 0, \tag{2.10}$$

(H5) *there exists a constant* $M^* > 0$*, such that either*

$$e \cdot \left[\int_0^1 f(s, e, 0)ds - \int_0^1 \int_0^s f(v, e, 0)dv\, dg(s)\right] < 0, \quad \text{for any } |e| > M^*, \tag{2.11}$$

*or else*

$$e \cdot \left[\int_0^1 f(s, e, 0)ds - \int_0^1 \int_0^s f(v, e, 0)dv\, dg(s)\right] > 0, \quad \text{for any } |e| > M^*. \tag{2.12}$$

*Then BVP* (1.1), (1.2) *with* $\alpha = 1, g(1) = 1$, *and* $\int_0^1 s\,dg(s) \neq 1$ *has at least one solution in* $C^1[0,1]$ *provided that*

$$\|a\|_1 + \|b\|_1 < \frac{1}{2}. \tag{2.13}$$

## 3. Proof of Theorems 2.2 and 2.3

We first prove Theorem 2.2 via the following Lemmas.

**Lemma 3.1.** *If* $\alpha = 0$, $g(1) = 1$, *and* $\int_0^1 s\,dg(s) \neq 1$, *then* $L : \operatorname{dom} L \subset Y \to Z$ *is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator* $Q : Z \to Z$ *can be defined by*

$$Qy = \frac{1}{1 - \int_0^1 s\,dg(s)} \left[ \int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\,dg(s) \right], \tag{3.1}$$

*and the linear operator* $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be written by*

$$K_P y = \int_0^t \int_0^s y(v)dv\,ds. \tag{3.2}$$

*Furthermore,*

$$\|K_P y\| \le \|y\|_1, \quad \text{for every } y \in \operatorname{Im} L. \tag{3.3}$$

*Proof.* It is clear that

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x = dt, d \in R, t \in [0,1]\}. \tag{3.4}$$

Obviously, the problem

$$x'' = y \tag{3.5}$$

has a solution $x(t)$ satisfying $x(0) = 0$, $x'(1) = \int_0^1 x'(s)dg(s)$, if and only if

$$\int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\,dg(s) = 0, \tag{3.6}$$

which implies that

$$\operatorname{Im} L = \left\{ y \in Z : \int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\,dg(s) = 0 \right\}. \tag{3.7}$$

In fact, if (3.5) has solution $x(t)$ satisfying $x(0) = 0$, $x'(1) = \int_0^1 x'(s)dg(s)$, then from (3.5) we have

$$x(t) = x'(0)t + \int_0^t \int_0^s y(v)dv\, ds. \tag{3.8}$$

According to $x'(1) = \int_0^1 x'(s)dg(s)$, $g(1) = 1$, we obtain

$$\begin{aligned} x'(1) = x'(0) + \int_0^1 y(s)ds &= \int_0^1 x'(s)dg(s) \\ &= \int_0^1 \left[x'(0) + \int_0^s y(v)dv\right] dg(s) \\ &= x'(0)g(1) + \int_0^1 \int_0^s y(v)dv\, dg(s), \end{aligned} \tag{3.9}$$

then

$$\int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\, dg(s) = 0. \tag{3.10}$$

On the other hand, if (3.6) holds, setting

$$x(t) = dt + \int_0^t \int_0^s y(v)dv\, ds, \tag{3.11}$$

where $d$ is an arbitrary constant, then $x(t)$ is a solution of (3.5), and $x(0) = 0$, and from $g(1) = 1$ and (3.6), we have

$$\begin{aligned} x'(1) - \int_0^1 x'(s)dg(s) &= d + \int_0^1 y(s)ds - \int_0^1 \left[d + \int_0^s y(v)dv\right] dg(s) \\ &= d(1 - g(1)) + \int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\, dg(s) \\ &= 0. \end{aligned} \tag{3.12}$$

Then $x'(1) = \int_0^1 x'(s)dg(s)$. Hence (3.7) is valid.

For $y \in Z$, define

$$Qy = \frac{1}{1 - \int_0^1 s\, dg(s)} \left[\int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\, dg(s)\right], \quad 0 \le t \le 1. \tag{3.13}$$

Let $y_1 = y - Qy$, and we have

$$\begin{aligned}\left[1-\int_0^1 s\,dg(s)\right]Qy_1 &= \int_0^1 (y-Qy)(s)ds - \int_0^1\int_0^s (y-Qy)(v)dv\,dg(s)\\ &= \int_0^1 y(s)ds - Qy - \int_0^1\int_0^s y(v)dv\,dg(s) + Qy\int_0^1 s\,dg(s)\\ &= \int_0^1 y(s)ds - \int_0^1\int_0^s y(v)dv\,dg(s) - Qy\left[1-\int_0^1 s\,dg(s)\right]\\ &= 0,\end{aligned} \tag{3.14}$$

then $Qy_1 = 0$, thus $y_1 \in \operatorname{Im} L$. Hence, $Z = \operatorname{Im} L + Z_1$, where $Z_1 = \{x(t) \equiv d : t \in [0,1], d \in R\}$, also $\operatorname{Im} L \cap Z_1 = \{0\}$. So we have $Z = \operatorname{Im} L \oplus Z_1$, and

$$\dim \operatorname{Ker} L = \dim Z_1 = \operatorname{co}\dim \operatorname{Im} L = 1. \tag{3.15}$$

Thus, $L$ is a Fredholm operator of index zero.

We define a projector $P : Y \to \operatorname{Ker} L$ by $(Px)(t) = x'(0)t$. Then we show that $K_P$ defined in (3.2) is a generalized inverse of $L : \operatorname{dom} L \cap Y \to Z$.

In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_P)y(t) = \left[(K_P y)(t)\right]'' = y(t), \tag{3.16}$$

and, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we know

$$(K_P L)x(t) = \int_0^t\int_0^s x''(v)dv\,ds = x(t) - x(0) - x'(0)t. \tag{3.17}$$

In view of $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $x(0) = 0$, and $Px = 0$, thus

$$(K_P L)x(t) = x(t). \tag{3.18}$$

This shows that $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. Also we have

$$\|K_P y\|_\infty \le \iint_0^1 |y(v)|dv\,ds = \|y\|_1, \qquad \|(K_P y)'\|_\infty \le \|y\|_1, \tag{3.19}$$

then $\|K_P y\| \le \|y\|_1$. The proof of Lemma 3.1 is finished. □

**Lemma 3.2.** *Under conditions* (2.5) *and* (2.9), *there are nonnegative functions* $\overline{a}, \overline{b}, \overline{r} \in L^1[0,1]$ *satisfying*

$$|f(t,x,y)| \le \overline{a}(t)|x| + \overline{b}(t)|y| + \overline{r}(t). \tag{3.20}$$

*Proof.* Without loss of generality, we suppose that $\|c\|_1 = \int_0^1 |c(t)|dt = \beta > 0$. Take $\gamma \in (0, (1/2\beta)(1/2 - (\|a\|_1 + \|b\|_1)))$, then there exists $\overline{M} > 0$ such that

$$|x|^\theta \le \gamma |x| + \overline{M}, \qquad |y|^\theta \le \gamma |y| + \overline{M}. \tag{3.21}$$

Let

$$\overline{a}(t) = a(t) + \gamma c(t), \qquad \overline{b}(t) = b(t) + \gamma c(t), \qquad \overline{r}(t) = r(t) + 2\overline{M}c(t). \tag{3.22}$$

Obviously, $\overline{a}, \overline{b}, \overline{r} \in L^1[0,1]$, and

$$\begin{aligned} \|\overline{a}\|_1 &\le \|a\|_1 + \gamma \|c\|_1, \\ \left\|\overline{b}\right\|_1 &\le \|b\|_1 + \gamma \|c\|_1. \end{aligned} \tag{3.23}$$

Then

$$\|\overline{a}\|_1 + \left\|\overline{b}\right\|_1 \le \|a\|_1 + \|b\|_1 + 2\beta\gamma < \frac{1}{2}, \tag{3.24}$$

and from (2.5) and (3.21), we have

$$\begin{aligned} |f(t,x,y)| &\le [a(t) + \gamma c(t)]|x| + [b(t) + \gamma c(t)]|y| + 2\overline{M}c(t) + r(t) \\ &= \overline{a}(t)|x| + \overline{b}(t)|y| + \overline{r}(t). \end{aligned} \tag{3.25}$$

Hence we can take $\overline{a}$, $\overline{b}$, 0, and $\overline{r}$ to replace $a$, $b$, $c$, and $r$, respectively, in (2.5), and for the convenience omit the bar above $a$, $b$, and $r$, that is,

$$|f(t,x,y)| \le a(t)|x| + b(t)|y| + r(t). \tag{3.26}$$

□

**Lemma 3.3.** *If assumptions (H1), (H2) and $\alpha = 0$, $g(1) = 1$, and $\int_0^1 s\, dg(s) \ne 1$ hold, then the set $\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda Nx$ for some $\lambda \in [0,1]\}$ is a bounded subset of $Y$.*

*Proof.* Suppose that $x \in \Omega_1$ and $Lx = \lambda Nx$. Thus $\lambda \ne 0$ and $QNx = 0$, so that

$$\int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\, dg(s) = 0, \tag{3.27}$$

thus by assumption (H2), there exists $t_0 \in [0,1]$, such that $|x'(t_0)| \le M$. In view of

$$x'(0) = x'(t_0) - \int_0^{t_0} x''(t)dt, \tag{3.28}$$

then, we have

$$\left|x'(0)\right| \leq M + \left\|x''\right\|_1 = M + \|Lx\|_1 \leq M + \|Nx\|_1. \tag{3.29}$$

Again for $x \in \Omega_1$, $x \in \operatorname{dom} L \setminus \operatorname{Ker} L$, then $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $LPx = 0$ thus from Lemma 3.1, we know

$$\|(I-P)x\| = \|K_P L(I-P)x\| \leq \|L(I-P)x\|_1 = \|Lx\|_1 \leq \|Nx\|_1. \tag{3.30}$$

From (3.29) and (3.30), we have

$$\|x\| \leq \|Px\| + \|(I-P)x\| = \left|x'(0)\right| + \|(I-P)x\| \leq 2\|Nx\|_1 + M. \tag{3.31}$$

If (2.5) holds, from (3.31), and (3.26), we obtain

$$\|x\| \leq 2\left[\|a\|_1\|x\|_\infty + \|b\|_1\left\|x'\right\|_\infty + \|r\|_1 + \frac{M}{2}\right]. \tag{3.32}$$

Thus, from $\|x\|_\infty \leq \|x\|$ and (3.32), we have

$$\|x\|_\infty \leq \frac{2}{1-2\|a\|_1}\left[\|b\|_1\left\|x'\right\|_\infty + \|r\|_1 + \frac{M}{2}\right]. \tag{3.33}$$

From $\|x'\|_\infty \leq \|x\|$, (3.32), and (3.33), one has

$$\begin{aligned}
\left\|x'\right\|_\infty \leq \|x\| &\leq 2\left[1 + \frac{2\|a\|_1}{1-2\|a\|_1}\right]\left[\|b\|_1\left\|x'\right\|_\infty + \|r\|_1 + \frac{M}{2}\right] \\
&= \frac{2}{1-2\|a\|_1}\left[\|b\|_1\left\|x'\right\|_\infty + \|r\|_1 + \frac{M}{2}\right],
\end{aligned} \tag{3.34}$$

that is,

$$\left\|x'\right\|_\infty \leq \frac{2}{1-2(\|a\|_1 + \|b\|_1)}\left[\|r\|_1 + \frac{M}{2}\right] := M_1. \tag{3.35}$$

From (3.35) and (3.33), there exists $M_2 > 0$, such that

$$\|x\|_\infty \leq M_2. \tag{3.36}$$

Thus

$$\|x\| = \max\left\{\|x\|_\infty, \left\|x'\right\|_\infty\right\} \leq \max\{M_1, M_2\}. \tag{3.37}$$

Again from (2.5), (3.35), and (3.36), we have

$$\|x''\|_1 = \|Lx\|_1 \le \|Nx\|_1 \le \|a\|_1 M_2 + \|b\|_1 M_1 + \|r\|_1. \tag{3.38}$$

Then we show that $\Omega_1$ is bounded. □

**Lemma 3.4.** *If assumption (H2) holds, then the set $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$ is bounded.*

*Proof.* Let $x \in \Omega_2$, then $x \in \operatorname{Ker} L = \{x \in \operatorname{dom} L : x = dt, d \in R, t \in [0,1]\}$ and $QNx = 0$; therefore,

$$\int_0^1 f(s, ds, d)ds - \int_0^1 \int_0^s f(v, dv, d)dv\, dg(s) = 0, \tag{3.39}$$

From assumption (H2), $\|x\|_\infty = |d| \le M$, so $\|x\| = |d| \le M$, clearly $\Omega_2$ is bounded. □

**Lemma 3.5.** *If the first part of condition (H3) of Theorem 2.2 holds, then*

$$d \cdot \frac{1}{1 - \int_0^1 s\, dg(s)} \left[\int_0^1 f(s, ds, d)ds - \int_0^1 \int_0^s f(v, dv, d)dv\, dg(s)\right] < 0, \tag{3.40}$$

*for all $|d| > M^*$. Let*

$$\Omega_3 = \{x \in \operatorname{Ker} L : -\lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\}, \tag{3.41}$$

*where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is the linear isomorphism given by $J(d) = dt$, for all $d \in R$, $t \in [0,1]$. Then $\Omega_3$ is bounded.*

*Proof.* Suppose that $x = d_0 t \in \Omega_3$, then we obtain

$$\lambda d_0 t = \frac{(1-\lambda)t}{1 - \int_0^1 s\, dg(s)} \left[\int_0^1 f(s, d_0 s, d_0)ds - \int_0^1 \int_0^s f(v, d_0 v, d_0)dv\, dg(s)\right], \quad 0 \le t \le 1, \tag{3.42}$$

or equivalently

$$\lambda d_0 = \frac{1-\lambda}{1 - \int_0^1 s\, dg(s)} \left[\int_0^1 f(s, d_0 s, d_0)ds - \int_0^1 \int_0^s f(v, d_0 v, d_0)dv\, dg(s)\right]. \tag{3.43}$$

If $\lambda = 1$, then $d_0 = 0$. Otherwise, if $|d_0| > M^*$, in view of (3.40), one has

$$\lambda d_0^2 = \frac{d_0(1-\lambda)}{1 - \int_0^1 s\, dg(s)} \left[\int_0^1 f(s, d_0 s, d_0)ds - \int_0^1 \int_0^s f(v, d_0 v, d_0)dv\, dg(s)\right] < 0, \tag{3.44}$$

which contradicts $\lambda d_0^2 \geq 0$. Then $|x| = |d_0 t| \leq |d_0| \leq M^*$ and we obtain $\|x\| \leq M^*$; therefore, $\Omega_3 \subset \{x \in \operatorname{Ker} L : \|x\| \leq M^*\}$ is bounded. □

The proof of Theorem 2.2 is now an easy consequence of the above lemmas and Theorem 2.1.

*Proof of Theorem 2.2.* Let $\Omega = \{x \in Y : \|x\| < \delta\}$ such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. By the Ascoli-Arzela theorem, it can be shown that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact; thus $N$ is $L$-compact on $\overline{\Omega}$. Then by the above Lemmas, we have the following.

(i) $Lx \neq \lambda N x$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$.

(ii) $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.

(iii) Let $H(x, \lambda) = -\lambda x + (1-\lambda) JQNx$, with $J$ as in Lemma 3.5. We know $H(x, \lambda) \neq 0$, for $x \in \operatorname{Ker} L \cap \partial\Omega$. Thus, by the homotopy property of degree, we get

$$\begin{aligned} \deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0) \\ &= \deg(-I, \Omega \cap \operatorname{Ker} L, 0). \end{aligned} \tag{3.45}$$

According to definition of degree on a space which is isomorphic to $R^n$, $n < \infty$, and

$$\Omega \cap \operatorname{Ker} L = \{dt : |d| < \delta\}. \tag{3.46}$$

We have

$$\begin{aligned} \deg(-I, \Omega \cap \operatorname{Ker} L, 0) &= \deg\left(-J^{-1} I J, J^{-1}(\Omega \cap \operatorname{Ker} L), J^{-1}\{0\}\right) \\ &= \deg(-I, (-\delta, \delta), 0) = -1 \neq 0, \end{aligned} \tag{3.47}$$

and then

$$\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0. \tag{3.48}$$

Then by Theorem 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, so that the BVP (1.1), (1.2) has at least one solution in $C^1[0, 1]$. The proof is completed. □

*Remark 3.6.* If the second part of condition (H3) of Theorem 2.2 holds, that is,

$$d \cdot \frac{1}{1 - \int_0^1 s\, dg(s)} \left[ \int_0^1 f(s, ds, d) ds - \int_0^1 \int_0^s f(v, dv, d) dv\, dg(s) \right] > 0, \tag{3.49}$$

for all $|d| > M^*$, then in Lemma 3.5, we take

$$\Omega_3 = \{x \in \operatorname{Ker} L : \lambda x + (1-\lambda) JQNx = 0, \lambda \in [0, 1]\}, \tag{3.50}$$

and exactly as there, we can prove that $\Omega_3$ is bounded. Then in the proof of Theorem 2.2, we have

$$\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) = \deg(I, \Omega \cap \operatorname{Ker} L, 0) = 1, \tag{3.51}$$

since $0 \in \Omega \cap \operatorname{Ker} L$. The remainder of the proof is the same.

By using the same method as in the proof of Theorem 2.2 and Lemmas 3.1–3.5, we can show Lemma 3.7 and Theorem 2.3.

**Lemma 3.7.** *If $\alpha = 1$, $g(1) = 1$, and $\int_0^1 s\,dg(s) \neq 1$, then $L : \operatorname{dom} L \subset Y \to Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator $Q : Z \to Z$ can be defined by*

$$Qy = \frac{1}{1 - \int_0^1 s\,dg(s)} \left[ \int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\,dg(s) \right], \tag{3.52}$$

*and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by*

$$K_P y = -\frac{t}{\xi} \int_0^{\xi} \int_0^s y(v)dv\,ds + \int_0^t \int_0^s y(v)dv\,ds. \tag{3.53}$$

*Furthermore,*

$$\|K_P y\| \leq 2\|y\|_1, \quad \forall y \in \operatorname{Im} L. \tag{3.54}$$

*Notice that*

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x = e, e \in R\},$$
$$\operatorname{Im} L = \left\{ y \in Z : \int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\,dg(s) = 0 \right\}. \tag{3.55}$$

*Proof of Theorem 2.3.* Let

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\}. \tag{3.56}$$

Then, for $x \in \Omega_1$, $Lx = \lambda Nx$; thus $\lambda \neq 0$, $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$; hence

$$\int_0^1 y(s)ds - \int_0^1 \int_0^s y(v)dv\,dg(s) = 0, \tag{3.57}$$

thus, from assumption (H4), there exists $t_0 \in [0,1]$, such that $|x(t_0)| < M$ and in view of $x(0) = x(t_0) - \int_0^{t_0} x'(t)dt$, we obtain

$$|x(0)| \leq M + \|x'\|_\infty. \tag{3.58}$$

From $x(0) = x(\xi)$, there exists $t_1 \in (0,\xi)$, such that $x'(t_1) = 0$. Thus, from $x'(t) = x'(t_1) + \int_{t_1}^{t} x''(t)dt$, one has

$$\|x'\|_\infty \leq \|x''\|_1. \tag{3.59}$$

We let $Px = x(0)$; hence from (3.58) and (3.59), we have

$$\begin{aligned} \|Px\| = |x(0)| &\leq M + \|x'\|_\infty \leq M + \|x''\|_1 \\ &= M + \|Lx\|_1 \leq M + \|Nx\|_1, \end{aligned} \tag{3.60}$$

thus, by using the same method as in the proof of Lemmas 3.2 and 3.3, we can prove that $\Omega_1$ is bounded too. Similar to the other proof of Lemmas 3.4–3.7 and Theorem 2.2, we can verify Theorem 2.3. □

Finally, we give two examples to demonstrate our results.

*Example 3.8.* Consider the following boundary value problem:

$$\begin{gathered} x'' = t^3 + 8 + \sin(x)^3 + \frac{1}{9}(t+1)x', \quad t \in (0,1), \\ x(0) = 0, \qquad x'(1) = \int_0^1 x'(s)dg(s), \end{gathered} \tag{3.61}$$

where $\alpha = 0$,

$$f(t,x,y) = t^3 + 8 + \sin(x)^3 + \frac{1}{9}(t+1)y, \quad t \in (0,1), \tag{3.62}$$

and $g(s) = s^2$ satisfying $g(0) = 0$, $g(1) = 1$, and $\int_0^1 s\,dg(s) = 2/3 \neq 1$, then we can choose $a(t) = 0$, $b(t) = 2/9$, and $r(t) = 10$, for $t \in [0,1]$; thus

$$\begin{gathered} |f(t,x,y)| \leq \frac{2}{9}|y| + 10, \\ \|a\|_1 + \|b\|_1 = \frac{2}{9} < \frac{1}{2}. \end{gathered} \tag{3.63}$$

Since

$$\begin{aligned} &\int_0^1 f(s,x(s),x'(s))ds - \int_0^1 \int_0^s f(v,x(v),x'(v))dv\,dg(s) \\ &\quad = \iint_0^1 f(v,x(v),x'(v))dv\,dg(s) - \int_0^1 \int_0^s f(v,x(v),x'(v))dv\,dg(s) \\ &\quad = \int_0^1 \int_s^1 f(v,x(v),x'(v))dv\,dg(s), \end{aligned} \tag{3.64}$$

and $f$ has the same sign as $x'(t)$ when $|x'(t)| > 90$, we may choose $M = M^* = 90$, and then the conditions (H1)–(H3) of Theorem 2.2 are satisfied. Theorem 2.2 implies that BVP (3.61) has at least one solution, $x \in C^1[0,1]$.

*Example 3.9.* Consider the following boundary value problem:

$$\begin{gathered} x'' = t^2 + 4 + \frac{1}{7}(t+2)x + \cos\left(x'\right)^3, \quad t \in (0,1), \\ x(0) = x(1), \qquad x'(1) = \int_0^1 x'(s)dg(s), \end{gathered} \tag{3.65}$$

where $\alpha = 1$,

$$f(t,x,y) = t^2 + 4 + \frac{1}{7}(t+2)x + \cos\left(y\right)^3, \quad t \in (0,1), \tag{3.66}$$

and $g(s) = s^2$ satisfying $g(0) = 0, g(1) = 1$, and $\int_0^1 s\,dg(s) = 2/3 \neq 1$, then we can choose $a(t) = 3/7$, $b(t) = 0$, and $r(t) = 6$, for $t \in [0,1]$; thus

$$\begin{gathered} \left|f(t,x,y)\right| \le \frac{3}{7}|x| + 6, \\ \|a\|_1 + \|b\|_1 = \frac{3}{7} < \frac{1}{2}. \end{gathered} \tag{3.67}$$

Similar to Example 3.8, we have

$$\int_0^1 f(s,x(s),x'(s))ds - \int_0^1 \int_0^s f(v,x(v),x'(v))dv\,dg(s) = \int_0^1 \int_s^1 f(v,x(v),x'(v))dv\,dg(s), \tag{3.68}$$

and $f$ has the same sign as $x(t)$ when $|x(t)| > 21$, we may choose $M = M^* = 21$, and then all conditions of Theorem 2.3 are satisfied. Theorem 2.3 implies that BVP (3.65) has at least one solution $x \in C^1[0,1]$.

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