Research Article

Existence Results of Three-Point Boundary Value Problems for Second-order Ordinary Differential Equations

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We establish existence results of the following three-point boundary value problems: $u''(t) + f(t, u(t)) = 0$, $t \in (0,1)$, $(BC)u(0) = 0$ and $u(1) = \delta u(\eta)$, where $0 < \eta < 1$ and $0 < \delta \leq 1$. The approach applied in this paper is upper and lower solution method associated with basic degree theory or Schauder's fixed point theorem. We deal with this problem with the function f which is Carathéodory or singular on its domain.

1. Introduction

In this paper, we consider three-point boundary value problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \qquad u(1) = \delta u(\eta), \tag{1.2}$$

where $0 < \eta < 1$ and $0 < \delta \leq 1$.

In the mathematical literature, a number of works have appeared on nonlocal boundary value problems, and one of the first of these was [1]. Il'in and Moiseev initiated the research of multipoint boundary value problems for second-order linear ordinary differential equations, see [2, 3], motivated by the study [4–6] of Bitsadze and Samarskii.

Recently, nonlinear multipoint boundary value problems have been receiving considerable attention, and have been studied extensively by using iteration scheme

(e.g., [7]), fixed point theorems in cones (e.g., [8]), and the Leray-Schauder continuation theorem (e.g., [9]). We refer more detailed treatment to more interesting research [10, 11] and the references therein.

The theory of upper and lower solutions is also a powerful tool in studying boundary value problems. For the existence results of two-point boundary value problem, there already are lots of interesting works by applying this essential technique (see [12, 13]). Recently, it is shown that this method plays an important role in proving the existence of solutions for three-point boundary value problems (see [14–16]).

Last but not least, as the singular source term appearing in two-point problems, singular three-point boundary value problems also attract more attention (e.g., [17]).

In this paper, we will discuss the existence of solutions of some general types on three-point boundary value problems by using upper and lower solution method associated with basic degree theory or Schauder's fixed point theorem.

This paper is organized as follows. In Section 2, we give two lemmas which will be extensively used later. In Section 3, when the source term f is a Carathéodory function, we consider the Sobolev space $W^{2,1}(0,1)$ defined by

$$W^{2,1}(0,1) := \left\{ u \in C^1[0,1] \mid u'' \in L^1(0,1) \right\}, \tag{1.3}$$

and obtain the existence of $W^{2,1}$-solution in Theorems 3.5 and 3.11. In Section 4, we discuss the singular case, that is, f maybe singular at the end points $t = 0$ or $t = 1$, or at $u = 0$. We will introduce the $\mathcal{A}$-class of functions and another space $W^{2,\mathcal{A}}$ (see [18, 19]) as follows:

$$\mathcal{A} := \left\{ h \in L^1_{\text{loc}}(0,1) \mid s(1-s)h(s) \in L^1(0,1) \right\},$$
$$W^{2,\mathcal{A}}(0,1) := \left\{ u \in W^{1,1}(0,1) \mid u'' \in \mathcal{A} \right\}, \tag{1.4}$$

and prove the existence of $W^{2,\mathcal{A}}$-solution in Theorems 4.1 and 4.4. Some sufficient conditions for constructing upper and lower solutions are given in each section for applications.

2. Preliminaries

Define $G : [0,1] \times [0,1] \to (-\infty, \infty)$ by

$$G(t,s) := \frac{1}{1-\delta\eta}t(1-s) - U(t,s) - \frac{\delta}{1-\delta\eta}V(t,s), \quad 0 \le t, \ s \le 1, \tag{2.1}$$

where δ and η are given as (1.2) and

$$U(t,s) = \begin{cases} t - s, & s \le t, \\ 0, & t \le s, \end{cases}$$
$$V(t,s) = \begin{cases} t(\eta - s), & s \le \eta, \\ 0, & \eta \le s. \end{cases} \tag{2.2}$$

By direct computations, we get the following results.

Lemma 2.1. (i) *The function $G : [0,1] \times [0,1] \to (-\infty, \infty)$ defined by (2.1), is the Green function corresponding for the problem*

$$u''(t) = 0,$$
$$u(0) = 0, \qquad u(1) = \delta u(\eta). \tag{2.3}$$

(ii) *The function $G : [0,1] \times [0,1] \to (-\infty, \infty)$ defined by (2.1), is continuous.*
(iii) *In the case $0 < \delta\eta < 1$, we have*

$$Q^{-1} := \max_{0 \leq t \leq 1} \int_0^1 G(t,s)ds = \begin{cases} \dfrac{1}{8}\left(\dfrac{1-\delta\eta^2}{1-\delta\eta}\right)^2, & \delta\eta(2-\eta) \leq 1, \\[4mm] \dfrac{\delta\eta(1-\eta)}{2(1-\delta\eta)}, & \delta\eta(2-\eta) \geq 1. \end{cases} \tag{2.4}$$

Lemma 2.2. *If $h \in \mathcal{A}$, then the problem*

$$u''(t) + h(t) = 0 \tag{2.5}$$

with boundary condition (1.2) has a unique solution $u \in W^{2,\mathcal{A}}(0,1)$ such that

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.6}$$

where $G(t,s)$ is defined by (2.1).

3. Carathéodory Case

In this section we first introduce the Carathéodory function as follows.

Definition 3.1. A function $f(t,u)$ defined on $E \subseteq [a,b] \times \mathbb{R}$ is called a Carathéodory function on E if

    (i) for almost every $t \in [a,b]$, $f(t,\cdot)$ is continuous on $\mathbb{R}$;

    (ii) for any $u \in \mathbb{R}$, the function $f(\cdot,u)$ is measurable on $[a,b]$;

    (iii) for any $r > 0$, there exists $h_r \in L(a,b)$ such that for any $u \in [-r,r]$ and for almost every $t \in [a,b]$ with $(t,u) \in E$, we have $|f(t,u)| \leq h_r(t)$.

We in this section assume that f is a Carathéodory function and discuss the existence of $W^{2,1}$-solution by assuming the existence of upper and lower solutions.

3.1. Existence of $W^{2,1}$-Solutions

We first introduce the definitions of $W^{2,1}$-upper and lower solutions as below.

Definition 3.2. A function $\alpha \in C[0,1]$ is called a $W^{2,1}$-lower solution of problem (1.1) and (1.2) if it satisfies

   (i) $\alpha(0) \leq 0$, $\alpha(1) \leq \delta\alpha(\eta)$, and

   (ii) for any $t_0 \in (0,1)$, either $D^-\alpha(t_0) < D_+\alpha(t_0)$, or there exists an open interval $I_0 \subseteq (0,1)$ containing t_0 such that $\alpha \in W^{2,1}(I_0)$ and, for almost every $t \in I_0$, we have

$$\alpha''(t) + f(t, \alpha(t)) \geq 0. \tag{3.1}$$

Definition 3.3. A function $\beta \in C[0,1]$ is called a $W^{2,1}$-upper solution of problem (1.1) and (1.2) if it satisfies

   (i) $\beta(0) \geq 0$, $\beta(1) \geq \delta\beta(\eta)$, and

   (ii) for any $t_0 \in (0,1)$, either $D_-\beta(t_0) > D^+\beta(t_0)$, or there exists an open interval $I_0 \subseteq (0,1)$ containing t_0 such that $\beta \in W^{2,1}(I_0)$ and, for almost every $t \in I_0$, we have

$$\beta''(t) + f(t, \beta(t)) \leq 0. \tag{3.2}$$

Before proving our main results, we first consider such a modified problem given as follows:

$$u''(t) + f(t, \gamma(t, u(t))) + \frac{\gamma(t, u(t)) - u(t)}{1 + |u(t)|} = 0, \quad t \in (0,1), \tag{3.3}$$

with boundary condition (1.2), where $\gamma : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\gamma(t, u) = \begin{cases} \alpha(t) & \text{if } u < \alpha(t), \\ u & \text{if } \alpha(t) \leq u \leq \beta(t), \\ \beta(t) & \text{if } u > \beta(t). \end{cases} \tag{3.4}$$

Proposition 3.4. *Let $\alpha(t)$ and $\beta(t)$ be respective $W^{2,1}$-lower and upper solutions of problem (1.1) and (1.2) with $\alpha(t) \leq \beta(t)$ on $[0,1]$. If $u \in W^{2,1}(0,1)$ is a solution of problem (3.3) and (1.2), then $\alpha(t) \leq u(t) \leq \beta(t)$, for any $t \in [0,1]$.*

Proof. Suppose there exists $t_0 \in [0,1]$ such that

$$\min_{t \in [0,1]} (u(t) - \alpha(t)) = u(t_0) - \alpha(t_0) < 0. \tag{3.5}$$

Case 1. If $t_0 \in (0,1)$, we have $u'(t_0) - D_-\alpha(t_0) \leq u'(t_0) - D^+\alpha(t_0)$, which implies $D^-\alpha(t_0) \geq D_+\alpha(t_0)$. Hence, by Definition 3.2 and the continuity of $u - \alpha$ at t_0, there exist an open interval $I_0 \subseteq (0,1)$ with $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and a neighborhood N of t_0 contained in I_0 such that for almost every $t \in I_0 \cap N$,

$$\alpha''(t) + f(t, \alpha(t)) \geq 0. \tag{3.6}$$

Furthermore, it follows from $u'(t_0) - \alpha'(t_0) = 0$ that for $t \geq t_0$, $t \in N$, we have

$$
\begin{aligned}
u'(t) - \alpha'(t) &= \int_{t_0}^{t} (u''(s) - \alpha''(s)) ds \\
&\leq \int_{t_0}^{t} \left[-f(s, \gamma(s, u(s))) - \frac{\gamma(s, u(s)) - u(s)}{1 + |u(s)|} + f(s, \alpha(s)) \right] ds \\
&= \int_{t_0}^{t} \left[-f(s, \alpha(s)) - \frac{\alpha(s) - u(s)}{1 + |u(s)|} + f(s, \alpha(s)) \right] ds \\
&< 0.
\end{aligned}
\tag{3.7}
$$

This implies that the minimum of $u - \alpha$ cannot occur at t_0, a contradiction.

Case 2. If $t_0 = 0$, by the definition of $W^{2,1}$-lower solution $\alpha(0) \leq 0$, we then have

$$
0 = u(0) \leq u(0) - \alpha(0) < 0.
\tag{3.8}
$$

And we get a contradiction.

Case 3. If $t_0 = 1$, it follows from the conclusion of Case 1 that

$$
u(1) - \alpha(1) \geq \delta(u(\eta) - \alpha(\eta)) > \delta(u(1) - \alpha(1)) \geq u(1) - \alpha(1),
\tag{3.9}
$$

which is impossible.

Consequently, we obtain $\alpha(t) \leq u(t)$ on $[0, 1]$. By the similar arguments as above, we also have

$$
u(t) \leq \beta(t), \quad \text{on } [0, 1].
\tag{3.10}
$$

$\square$

Theorem 3.5. *Let $\alpha(t)$ and $\beta(t)$ be $W^{2,1}$-lower and upper solutions of problem (1.1) and (1.2) such that $\alpha(t) \leq \beta(t)$ on $[0, 1]$ and let f be a Carathéodory function on E, where*

$$
E := \{(t, u) \in [0, 1] \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t), \ t \in [0, 1]\}.
\tag{3.11}
$$

Then problem (1.1) and (1.2) has at least one solution $u \in W^{2,1}(0, 1)$ such that, for all $t \in [0, 1]$,

$$
\alpha(t) \leq u(t) \leq \beta(t).
\tag{3.12}
$$

Proof. We consider the modified problem (3.3) and (1.2) with respect to the given $\alpha(t)$ and $\beta(t)$. Consider the Banach space $C[0,1]$ with supremum and the operator $T : C[0,1] \to C[0,1]$ by

$$(Tu)(t) := \int_0^1 G(t,s) \left[f(s, \gamma(s, u(s))) + \frac{\gamma(s, u(s)) - u(s)}{1 + |u(s)|} \right] ds, \tag{3.13}$$

for $u \in C[0,1]$, where $G(t,s)$ is defined as (2.1). Since f is a Carathéodory function on E, for almost all $t \in [0,1]$ and for all $x \in [\alpha(t), \beta(t)]$, there exists a function $h \in L(0,1)$, we have

$$|f(t,u)| \leq h(t). \tag{3.14}$$

Define

$$K := \{ u \in C[0,1] \mid \|u\| \leq M \}, \tag{3.15}$$

where

$$M := \max_{t \in [0,1]} \int_0^1 |G(t,s)|[h(s) + M_1] ds < \infty, \tag{3.16}$$

$$M_1 := \max_{(t,u) \in [0,1] \times \mathbb{R}} \frac{\gamma(t,u) - u}{1 + |u|}. \tag{3.17}$$

It is clear that K is a closed, bounded and convex set in $C[0,1]$ and one can show that $T : K \to K$ is a completely continuous mapping by Arzelà-Ascoli theorem and Lebesgue dominated convergence theorem. By applying Schauder's fixed point theorem, we obtain that T has a fixed point in K which is a solution of problem (3.3) and (1.2). From Proposition 3.4, this fixed point is also a solution of problem (1.1) and (1.2). Hence, we complete the proof. □

We further illustrate the use of Theorem 3.11 in the following second-order differential equation:

$$u''(t) + f(t, u(t)) + h(t) = 0 \tag{3.18}$$

with the boundary condition (1.2).

Corollary 3.6. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $f(t,u)/u$ is essentially bounded for $|u| \geq M$, where M is a constant large enough. Assume further that $h \in L(0,1)$ and there exists a constant $0 < \sqrt{A} < \pi/2$ such that*

$$\limsup_{|u| \to \infty} \max_{t \in [0,1]} \frac{f(t,u)}{u} \leq A. \tag{3.19}$$

Then, problem (3.18) and (1.2) has at least one solution.

Proof. By hypothesis, for any given $\epsilon > 0$ small enough such that $\sqrt{A + \epsilon} \le \pi/2$ and for almost all $t \in [0, 1]$, for any u large enough, we have

$$f(t, u) \le (A + \epsilon)u. \tag{3.20}$$

We now choose an upper solution $\beta(t)$ of the form

$$\beta(t) = w(t) + s\psi(t) \ge 0. \tag{3.21}$$

To this end, we compute

$$\beta'' + f(t, \beta) + h(t) \le \beta'' + (A + \epsilon)\beta + h(t)$$
$$= w'' + (A + \epsilon)w + h(t) + s[\psi'' + (A + \epsilon)\psi]. \tag{3.22}$$

Clearly, one can choose w such that

$$w'' + (A + \epsilon)w + h(t) = 0,$$
$$w(0) = 0, \qquad w(1) = \delta w(\eta), \tag{3.23}$$

that is,

$$w(t) = \frac{\sin\left(\sqrt{A + \epsilon}\, t\right)}{\delta \sin\left(\sqrt{A + \epsilon}\, \eta\right) - \sin\left(\sqrt{A + \epsilon}\right)} \int_0^1 \frac{\sin\left(\sqrt{A + \epsilon}\,(s - 1)\right)}{\sqrt{A + \epsilon}} h(s)\,ds$$

$$+ \frac{\delta \sin\left(\sqrt{A + \epsilon}\, t\right)}{\delta \sin\left(\sqrt{A + \epsilon}\, \eta\right) - \sin\left(\sqrt{A + \epsilon}\right)} \int_0^\eta \frac{\sin\left(\sqrt{A + \epsilon}\,(\eta - s)\right)}{\sqrt{A + \epsilon}} h(s)\,ds \tag{3.24}$$

$$+ \int_0^t \frac{\sin\left(\sqrt{A + \epsilon}\,(s - t)\right)}{\sqrt{A + \epsilon}} h(s)\,ds,$$

and choose $\psi(t) = l \sin \sqrt{A + \epsilon}\, t$, where $l > 0$, which is a positive solution of

$$\psi'' + (A + \epsilon)\psi = 0,$$
$$\psi(0) = 0. \tag{3.25}$$

Hence, if s is large enough, we can show that $\beta(0) = 0$ and $\beta(1) \ge \delta\beta(\eta)$, where $\delta \le 1$, which implies that $\beta(t)$ is a positive $W^{2,1}$-upper solution. In the same way we construct a $W^{2,1}$-lower solution $\alpha = w(t) - s\psi \le 0$ on $[0, 1]$. $\qquad \square$

3.2. Nontangency Solution

In this subsection, we afford another stronger lower and upper solutions to get a strict inequality of the solution between them.

Definition 3.7. A function $\alpha \in C[0,1]$ is a strict $W^{2,1}$-lower solution of problem (1.1) and (1.2), if it is not a solution of problem (1.1) and (1.2), $\alpha(0) < 0$, $\alpha(1) \leq \delta\alpha(\eta)$ and for any $t_0 \in (0,1)$, one of the following is satisfied:

(i) $D^-\alpha(t_0) < D_+\alpha(t_0)$;

(ii) there exist an interval $I_0 \subseteq [0,1]$ and $\epsilon > 0$ such that $t_0 \in \text{int}(I_0)$, $\alpha \in W^{2,1}(I_0)$ and for almost every $t \in I_0$, for all $u \in [\alpha(t), \alpha(t) + \epsilon]$ we have

$$\alpha''(t) + f(t,u) \geq 0. \tag{3.26}$$

Definition 3.8. A function $\beta \in C[0,1]$ is a strict $W^{2,1}$-upper solution of problem (1.1) and (1.2), if it is not a solution of problem (1.1) and (1.2), $\beta(0) > 0$, $\beta(1) \geq \delta\beta(\eta)$ and for any $t_0 \in (0,1)$, one of the following is satisfied:

(i) $D_-\beta(t_0) > D^+\beta(t_0)$,

(ii) there exist an interval $I_0 \subseteq [0,1]$ and $\epsilon > 0$ such that $t_0 \in \text{int}(I_0)$, $\beta \in W^{2,1}(I_0)$ and for almost every $t \in I_0$, for all $u \in [\beta(t) - \epsilon, \beta(t)]$ we have

$$\beta''(t) + f(t,u) \leq 0. \tag{3.27}$$

Remark 3.9. Every strict $W^{2,1}$-lower(upper) solution of problem (1.1) and (1.2) is a $W^{2,1}$-lower(upper) solution.

Now we are going to show that the solution curve of problem (1.1) and (1.2) cannot be tangent to upper or lower solutions from below or above.

Proposition 3.10. *Let $\alpha(t)$ and $\beta(t)$ be respective strict $W^{2,1}$-lower and upper solutions of problem (1.1) and (1.2) with $\alpha(t) \leq \beta(t)$ on $[0,1]$. If $u \in W^{2,1}(0,1)$ is a solution of problem (1.1) and (1.2) with $\alpha \leq u \leq \beta$ on $[0,1]$, then $\alpha(t) < u(t) < \beta(t)$, for any $t \in [0,1]$.*

Proof. As α is not a solution, u is not identical to α. Assume, the conclusion does not hold, then

$$t_0 := \inf\{t \in [0,1] \mid u(t) = \alpha(t)\} \tag{3.28}$$

exists. Hence, $u - \alpha$ has minimum at t_0, that is, $u(t_0) - \alpha(t_0) = 0$.

Case 1. Set $t_0 \in (0,1)$. Since $u - \alpha$ has minimum at t_0, we have $D_-\alpha(t_0) \geq D^+\alpha(t_0)$. According to the Definition 3.7, there exist I_0, $\epsilon_0 > 0$ and $t_1 \in I_0$ with $t_1 < t_0$ such that, for every $t \in (t_1, t_0)$, $u(t) \leq \alpha(t) + \epsilon_0$, $u'(t_1) - \alpha'(t_1) < 0$ and for a.e. $t \in (t_1, t_0)$

$$\alpha''(t) + f(t, u(t)) \geq 0. \tag{3.29}$$

Hence, we have the contradiction since

$$0 < (u' - \alpha')(t_0) - (u' - \alpha')(t_1) = -\int_{t_1}^{t_0} [f(t, u(t)) + \alpha''(t)] dt \leq 0. \tag{3.30}$$

Case 2. If $t_0 = 0$, by the definition of strict $W^{2,1}$-lower solution that $\alpha(0) < 0$, we then have

$$0 = u(0) - \alpha(0) > 0. \tag{3.31}$$

And we get a contradiction.

Case 3. If $t_0 = 1$, repeat the same arguments in Case 3 of the proof of Proposition 3.4. Therefore, we obtain $\alpha(t) < u(t)$ on $[0, 1]$. The inequality $u(t) < \beta(t)$ on $[0, 1]$ can be proved by the similar arguments as above.

$\square$

Theorem 3.11. *Let $\alpha(t)$ and $\beta(t)$ be strict $W^{2,1}$-lower and upper solutions of problem (1.1) and (1.2) such that $\alpha(t) < \beta(t)$ on $[0, 1]$ and let $f : E \to \mathbb{R}$ be a Carathéodory function, where*

$$E := \{(t, u) \in [0, 1] \times \mathbb{R} \mid \alpha(t) \leq u(t) \leq \beta(t), \ t \in [0, 1]\}. \tag{3.32}$$

Then, problem (1.1) and (1.2) has at least one solution $u \in W^{2,1}(0, 1)$ such that, for any $t \in [0, 1]$,

$$\alpha(t) < u(t) < \beta(t). \tag{3.33}$$

Proof. This is a consequence of Theorem 3.5 and Proposition 3.10 and hence, we omits this proof. $\square$

4. Singular Case

In this section we give a more general existence result than Theorem 3.11 by assuming the existence of $W^{2,1}$-lower and upper solutions. This makes us to deal with problem (1.1) and (1.2), where the function f is singular at the end point $t = 0$ and $t = 1$.

Theorem 4.1. *Let $\alpha(t)$ and $\beta(t)$ be $W^{2,1}$-lower and upper solutions of problem (1.1) and (1.2) such that $\alpha(t) \leq \beta(t)$ on $[0, 1]$ and let $f : (0, 1) \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(i) *for almost every $t \in (0, 1)$, $f(t, \cdot)$ is continuous on $\mathbb{R}$;*

(ii) *for any $u \in \mathbb{R}$, the function $f(\cdot, u)$ is measurable on $(0, 1)$;*

(iii) *there exists a function $h \in \mathcal{A}$ such that, for all $(t, u) \in E$,*

$$|f(t, u)| \leq h(t), \tag{4.1}$$

where

$$E := \{(t, u) \mid t \in (0, 1), \alpha(t) \leq u(t) \leq \beta(t)\} \subseteq (0, 1) \times \mathbb{R}. \tag{4.2}$$

Then problem (1.1) *and* (1.2) *has at least one solution* $u \in W^{2,\mathcal{A}}(0,1)$ *such that, for all* $t \in [0,1]$,

$$\alpha(t) \leq u(t) \leq \beta(t). \tag{4.3}$$

Proof. Consider the modified problem (3.3) and (1.2) with respect to the given $\alpha(t)$ and $\beta(t)$ and define $T : C[0,1] \rightarrow C[0,1]$ by (3.13). Note that by Lemma 2.2, T is well defined. Define

$$P := \{u \in C[0,1] \mid \|u\| \leq N\}, \tag{4.4}$$

where

$$N := \max_{t \in [0,1]} \int_0^1 |G(t,s)|[h(s) + M_1]ds < \infty, \tag{4.5}$$

and M_1 is defined by (3.17). The rest arguments are similar to the proof of Theorem 3.5. □

Remark 4.2. We have similar results of Theorems 3.5–4.1, respectively, for (1.1) equipped with

$$u(0) = A, \qquad u(1) = \delta u(\eta), \tag{4.6}$$

where $A \in \mathbb{R}$ is a constant and δ, η are given as (1.2).

Example 4.3. Consider the problem (4.7), for $0 < \alpha < 1, 0 < \beta_i < 2 - 2\alpha, i = 1,2$,

$$u''(t) + \frac{1}{t^{\beta_1}(1-t)^{\beta_2}}u(t)^\alpha + 1 = 0, \quad 0 < t < 1,$$

$$u(0) = 0, \qquad u\left(\frac{1}{2}\right) = u(1). \tag{4.7}$$

Clearly, 0 is a $W^{2,1}$-lower solution of (4.7) and

$$G(t,s) := 2t(1-s) - U(t,s) - 2V(t,s), \quad 0 \leq t, \ s \leq 1, \tag{4.8}$$

where

$$U(t,s) = \begin{cases} t - s, & s \leq t, \\ 0, & t \leq s, \end{cases}$$

$$V(t,s) = \begin{cases} t\left(\frac{1}{2} - s\right), & s \leq \frac{1}{2}, \\ 0, & \frac{1}{2} \leq s. \end{cases} \tag{4.9}$$

From Lemma 2.1, we have $\max_{0 \leq t \leq 1} \int_0^1 G(t, s)ds = 9/32$ and define $h_1(t) := 1/t^{\beta_1/(1-\alpha)}(1 - t)^{\beta_2/(1-\alpha)}$. Since, for $2 - \beta_i/(1 - \alpha) > 0$, $i = 1, 2$,

$$\int_0^1 t(1-t)h_1(t)dt = \int_0^1 t^{(2-\beta_1/(1-\alpha))-1}(1-t)^{(2-\beta_2/(1-\alpha))-1} < \infty, \tag{4.10}$$

that is, $h_1(t) \in \mathcal{A}$, we have, from Lemma 2.2, $\int_0^1 G(t, s)(1 - \alpha)h_1(s)ds \in W^{2,\mathcal{A}}$ and $\max_{0 \leq t \leq 1} \int_0^1 G(t, s)(1 - \alpha)h_1(s)ds$ exists. Let

$$B := \frac{1}{1 - (9/32)\alpha} \max_{0 \leq t \leq 1} \int_0^1 G(t, s)(1 - \alpha)h_1(s)ds + 1 \tag{4.11}$$

and, by Lemma 2.2 again, choose β such that

$$\beta''(t) + \alpha B + (1 - \alpha)h_1(t) + 1 = 0,$$
$$\beta(0) = 0, \qquad \beta(1) = \beta\left(\frac{1}{2}\right). \tag{4.12}$$

Note that according to the direct computation, we see that β is well-defined and is bounded by B. Next, let $f(t, u) := (1/t^{\beta_1}(1 - t)^{\beta_2})u^\alpha + 1$. By Young's inequality, it follows that

$$\beta''(t) + f(t, \beta(t)) \leq \beta''(t) + \alpha\beta(t) + (1 - \alpha)\frac{1}{t^{\beta_1/(1-\alpha)}(1 - t)^{\beta_2/(1-\alpha)}} + 1$$
$$= \beta''(t) + \alpha\beta(t) + (1 - \alpha)h_1(t) + 1. \tag{4.13}$$
$$\leq 0.$$

Hence, such $\beta(t)$ is a $W^{2,1}$-upper solution of (4.7) and $\beta(t) \geq 0$ on $[0, 1]$. Clearly, f satisfies (i), (ii) of Theorem 4.1. By using Young's inequality again, for $(t, u) \in E := \{(t, u) \mid t \in (0, 1), 0 \leq u(t) \leq \beta(t)\} \subseteq (0, 1) \times \mathbb{R}$., we have

$$f(t, u) \leq \alpha u + h_1(t)$$
$$\leq \alpha B + h_1(t) := h_2(t). \tag{4.14}$$

and $h_2(t) \in \mathcal{A}$. Therefore, f satisfies the assumption (iii) of Theorem 4.1. Consequently, we conclude that this problem has at least one solution $u \in W^{2,\mathcal{A}}(0, 1)$ such that, for all $t \in [0, 1]$,

$$0 \leq u(t) \leq \beta(t). \tag{4.15}$$

Notice that in Theorem 4.1, one can only deal with the case that f is singular at end points $t = 0$, $t = 1$. However, when f is singular at $u = 0$, there is no hope to obtain the solutions directly from Theorem 4.1. We will establish the following theorem to deal with this case by constructing upper and lower solutions to solve this problem.

Theorem 4.4. *Assume*

(H₁) *the function* $f : (0,1) \times \mathbb{R}^+ \to \mathbb{R}$ *is continuous;*

(H₂) *there exists* $k > \pi$ *and for any compact set* $K \subseteq (0,1)$, *there is* $\epsilon > 0$ *such that*

$$f(t,u) \geq k^2 u, \quad \forall t \in K, \ u \in (0,\epsilon]; \tag{4.16}$$

(H₃) *for some* $M > 0$ *and* $0 < \gamma < \sqrt{Q}$, *there is* $h \in \mathcal{A} \cap C(0,1)$ *such that*

$$f(t,u) \leq \gamma^2 u + h(t), \quad \forall t \in (0,1), \ u \in [M,\infty); \tag{4.17}$$

where Q^{-1} *is defined as in Lemma 2.1.*

(H₄) *for any compact set* $K \subseteq (0,\infty)$, *there is* $h_K \in \mathcal{A}$ *such that*

$$|f(t,u)| \leq h_K(t), \quad \forall t \in (0,1), \ u \in K. \tag{4.18}$$

Then problem (1.1) *and* (1.2) *with* $\delta = 1$ *has at least one solution*

$$u \in C([0,1], \mathbb{R}^+ \cup \{0\}) \cap C^2((0,1), \mathbb{R}^+). \tag{4.19}$$

Remark 4.5 (see [12, Remark 3.1]). Assumption (H₂) is equivalent to the assumption that there exists $k > \pi$ and a function $a_1 \in C_0^2([0,1], \mathbb{R}^+)$ such that:

(i) $a_1 > 0$ for all $t \in (0,1)$,

(ii) $f(t,u) \geq k^2 u$, for all $t \in (0,1)$, $0 < u \leq a_1(t)$,

(iii) $a_1''(t) > 0$, for all $t \in [0,1/3] \cup [2/3,1]$,

where

$$C_0^2([0,1], \mathbb{R}^+) := \left\{ u \in C^2([0,1], \mathbb{R}^+) \mid u(0) = u(1) = 0 \right\}. \tag{4.20}$$

Proof.

Step 1. Construction of lower solutions. Consider k_2 such that $\pi < k_2 < \min(k, 3\pi)$ and the function

$$\alpha_2(t) = A_2 \cos k_2 \left(t - \frac{1}{2} \right), \tag{4.21}$$

where A_2 is chosen small enough so that

$$f(t,u) \geq k^2 u, \quad \forall t \in \left(\frac{1}{2} - \frac{\pi}{2k_2}, \frac{1}{2} + \frac{\pi}{2k_2} \right), \ 0 < u \leq \alpha_2(t). \tag{4.22}$$

Next, we choose a_1 from the Remark 4.5, and let

$$\alpha_1(t) = A_1 a_1(t), \tag{4.23}$$

where $A_1 \in (0,1]$ is small enough so that for some points $t_1 \in (0,1/3)$, $t_2 \in (2/3,1)$, we have:

$$\alpha_1(t) \geq \alpha_2(t), \quad \forall t \in [0,t_1] \cup [t_2,1], \tag{4.24}$$

$$\alpha_2(t) \geq \alpha_1(t), \quad \forall t \in [t_1,t_2]. \tag{4.25}$$

Notice that by (4.24) and (4.25), for any $h : (0,1) \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$h(t,u) \geq f(t,u), \quad \text{for any } (t,u) \in (0,1) \times \mathbb{R}^+, \tag{4.26}$$

we have:

$$\alpha_1''(t) + h(t,\alpha_1(t)) \geq \alpha_1''(t) + k^2\alpha_1(t) > 0, \quad \text{for any } t \in [0,t_1] \cup [t_2,1], \tag{4.27}$$

$$\alpha_2''(t) + h(t,\alpha_2(t)) \geq -k_2^2\alpha_2(t) + k^2\alpha_2(t) > 0, \quad \text{for any } t \in [t_1,t_2]. \tag{4.28}$$

Step 2. Approximation problems. We define for each $n \in \mathbb{N}$, $n \geq 1$,

$$\eta_n(t) = \max\left\{ \frac{1}{2^{n+1}}, \min\left(t, 1 - \frac{1}{2^{n+1}}\right) \right\}, \quad t \in (0,1) \tag{4.29}$$

and set

$$\tilde{f}_n(t,u) = \max\{ f(\eta_n(t),u), f(t,u) \}. \tag{4.30}$$

We have that, for each index n, $\tilde{f}_n : (0,1) \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and

$$\tilde{f}_n(t,u) = f(t,u), \quad \text{for any } (t,u) \in K_n \times \mathbb{R}^+, \tag{4.31}$$

where

$$K_n = \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}} \right]. \tag{4.32}$$

Hence, the sequence of functions $\{\tilde{f}_n\}$ converges to f uniformly on any set $K \times \mathbb{R}^+$, where K is an arbitrary compact subset of $(0,1)$. Next we define

$$f_n(t,u) = \min\left\{ \tilde{f}_1(t,u), \ldots, \tilde{f}_n(t,u) \right\}. \tag{4.33}$$

Each of the functions f_i is a continuous function defined on $(0,1) \times \mathbb{R}^+$, moreover

$$f_1(t,u) \geq f_2(t,u) \geq \cdots \geq f_n(t,u) \geq f_{n+1}(t,u) \geq \cdots \geq f(t,u) \qquad (4.34)$$

and the sequence $\{f_n\}$ converges to f uniformly on the compact subsets of $(0,1) \times \mathbb{R}^+$ since

$$f_n(t,u) = f(t,u), \quad \forall t \in K_n, \ u \in \mathbb{R}^+. \qquad (4.35)$$

Define now a decreasing sequence $\{\epsilon_n\} \subseteq \mathbb{R}^+$ such that

$$\lim_{n \to \infty} \epsilon_n = 0,$$

$$f(t,u) \geq k^2 u, \quad \forall t \in K_n, \ u \in (0, \epsilon_n], \qquad (4.36)$$

and consider a sequence of the following approximation problems:

$$u''(t) + f_n(t, u(t)) = 0,$$

$$(BC)u(0) = \epsilon_n, \ u(1) = u(\eta), \qquad (P_n)$$

where $0 < \eta < 1$.

Step 3. A lower solution of (P_n). It is clear that for any $c \in (0, \epsilon_n]$,

$$\tilde{f}_n(t, c) \geq f(\eta_n(t), c) \geq k^2 c > 0. \qquad (4.37)$$

As the sequence $\{\epsilon_n\}$ is decreasing, we also have

$$f_n(t, \epsilon_n) = \min_{1 \leq k \leq n} \tilde{f}_k(t, \epsilon_n) \geq k^2 \epsilon_n > 0. \qquad (4.38)$$

Clearly, $\alpha_3(t) := \epsilon_n$ satisfies

$$\alpha_3''(t) + f_n(t, \alpha_3(t)) = f_n(t, \epsilon_n) > 0. \qquad (4.39)$$

It follows from (4.25) and (4.27) that $\alpha(t) := \max(\alpha_1(t), \alpha_2(t), \epsilon_n)$ is a lower solution of (P_n).

Step 4. Existence of a solution u_1 of (4.7) such that

$$\max(\alpha_1(t), \alpha_2(t), \epsilon_1) \leq u_1(t). \qquad (4.40)$$

From assumption (H_3), we can find $M \geq \max(\alpha_1(t), \alpha_2(t), \epsilon_1)$ and $h \in \mathcal{A}$ such that, for all $t \in (0,1), u \in [M, \infty)$,

$$f(t,u) \leq \gamma^2 u + h(t). \qquad (4.41)$$

Also, one has

$$f(\eta_1(t), u) \leq \gamma^2 u + h(\eta_1(t)) \leq \gamma^2 u + R, \tag{4.42}$$

where $R > 0$ is a suitable constant. Hence, we obtain, for such t and u,

$$f_1(t, u) = \max\{f(\eta_1(t), u), f(t, u)\} \leq \gamma^2 u + h(t) + R. \tag{4.43}$$

Let C be a constant such that

$$C > \frac{1}{1 - \gamma^2 Q^{-1}} \left\{ M + \max_{0 \leq t \leq 1} \int_0^1 G(t, s)(h(s) + R)ds \right\}. \tag{4.44}$$

Choose β such that

$$\beta''(t) + \gamma^2 C + h(t) + R = 0,$$
$$\beta(0) = M, \qquad \beta(1) = \beta(\eta), \tag{4.45}$$

that is,

$$\beta(t) = M + \int_0^1 G(t, s)\left(\gamma^2 C + h(s) + R\right)ds, \tag{4.46}$$

where $G(t, s)$ is defined by (2.1). Note that β is well-defined and $M \leq \beta(t) \leq C$ on $[0, 1]$ since $h \in \mathcal{A}$. It is easy to see that

$$\beta'' + f_1(t, \beta) \leq \beta'' + \gamma^2 \beta + h(t) + R$$
$$= \gamma^2(\beta - C) \tag{4.47}$$
$$\leq 0.$$

So by Remark 4.2, there is a solution u_1 of (4.7) such that

$$\max(\alpha_1(t), \alpha_2(t), \epsilon_1) \leq u_1(t) \leq \beta(t). \tag{4.48}$$

Step 5. The problem (P_n) has at least one solution u_n such that

$$\max(\alpha_1(t), \alpha_2(t), \epsilon_n) \leq u_n(t) \leq u_{n-1}(t). \tag{4.49}$$

Notice that u_{n-1} is an upper solution of (P_n), since

$$0 = u_{n-1}''(t) + f_{n-1}(t, u_{n-1}(t)) \geq u_{n-1}'' + f_n(t, u_{n-1}(t)),$$
$$u_{n-1}(0) = \epsilon_{n-1} \geq \epsilon_n. \tag{4.50}$$

Step 6. Existence of a solution. Consider the pointwise limit

$$\tilde{u}(t) = \lim_{n \to \infty} u_n(t), \quad \text{on } (0,1). \tag{4.51}$$

It is clear that, for any $n \geq 1$,

$$\max(\alpha_1(t), \alpha_2(t)) \leq \tilde{u}(t) \leq u_n(t), \quad \text{for any } t \in (0,1) \tag{4.52}$$

and therefore $\tilde{u}(t) > 0$ on $(0,1)$. Let $K \subseteq (0,1)$ be a compact interval. There is an index $n^* = n^*(K)$ such that $K \subseteq K_n$ for all $n \geq n^*$ and therefore for these $n \geq n^*$,

$$0 = u_n''(t) + f_n(t, u_n(t)) = u_n''(t) + f(t, u_n(t)), \quad \forall t \in K. \tag{4.53}$$

Moreover, we have

$$\sup\{|f(t,u)| \mid t \in K, \; \max(\alpha_1(t), \alpha_2(t)) \leq u \leq u_{n^*}(t)\} < \infty. \tag{4.54}$$

By Arzelá-Ascoli theorem it is standard to conclude that $\tilde{u}$ is a solution of problem (1.1) and (1.2) on the interval K. Since K is arbitrary, we find that $\tilde{u} \in C^2((0,1), \mathbb{R}^+)$ and for all $t \in (0,1)$,

$$\tilde{u}''(t) + f(t, \tilde{u}(t)) = 0. \tag{4.55}$$

Since

$$\tilde{u}(0) = \lim_{n \to \infty} \epsilon_n = 0, \tag{4.56}$$

it remains only to check the continuity of $\tilde{u}$ at $t = 0$. This can be deduced from the continuity of u_n and the fact that $u_n(0) = \epsilon_n \to 0$ as $n \to \infty$. $\qquad\square$

Example 4.6. Consider the following problem (P_2), for $\alpha > 0$, $0 < \beta_1, \beta_2 < 2$,

$$u''(t) + \frac{1}{t^{\beta_1}(1-t)^{\beta_2}(u(t))^\alpha} = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u\left(\frac{1}{2}\right) = u(1). \tag{4.57}$$

Let $f(t, u) = 1/t^{\beta_1}(1-t)^{\beta_2}u^\alpha$, where $(t, u) \in (0,1) \times \mathbb{R}^+$. Obviously, f satisfies (H_1) and (H_4). Moreover, for any given $k > \pi$ and for any compact set $K \subseteq (0,1)$, for $\epsilon > 0$ small enough, we have

$$f(t, u) \geq \frac{1}{u^\alpha} = u\frac{1}{u^{\alpha+1}} \geq k^2 u, \quad \forall t \in K, \; u \in (0, \epsilon], \tag{4.58}$$

Hence, (H_2) holds. Furthermore, for $M > 0$ large enough, $0 < \gamma < 4/\sqrt{3}$, we have, from Young's inequality by choosing $1 < p < \min\{1/\beta_1, 1/\beta_2\}$ and $1/p + 1/q = 1$,

$$f(t, u) \leq \frac{1}{p} t^{-\beta_1 p}(1-t)^{-\beta_2 p} + \frac{1}{q} u^{-\alpha q}$$

$$= h_1(t) + \frac{1}{q} u u^{-\alpha q - 1} \tag{4.59}$$

$$\leq h_1(t) + \gamma^2 u, \quad \forall t \in (0, 1), \ u \in [M, \infty),$$

where $h_1 := (1/p) t^{-\beta_1 p}(1-t)^{-\beta_2 p} \in \mathcal{A} \cap C(0, 1)$. Hence, (H_3) holds. By Theorem 4.4, (P_2) has at least one solution

$$u \in C([0, 1], \mathbb{R}^+ \cup \{0\}) \cap C^2((0, 1), \mathbb{R}^+). \tag{4.60}$$

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