NONLINEAR VOLTERRA DIFFERENCE EQUATIONS IN SPACE l^p

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We consider a class of vector nonlinear discrete-time Volterra equations in space l^p and derive estimates for the norms of solutions. These estimates give us explicit stability conditions, which allow us to avoid finding Lyapunov functionals.

1. Introduction and statement of the main result

Volterra difference equations arise in the mathematical modeling of some real phenomena, and also in numerical schemes for solving differential and integral equations (cf. [7, 8] and the references therein).

One of the basic methods in the theory of stability and boundedness of Volterra difference equations is the direct Lyapunov method (see [1, 3, 4] and the references therein). But finding the Lyapunov functionals for Volterra difference equations is a difficult mathematical problem.

In this paper, we derive estimates for the c_0- and l^p-norms of solutions for a class of vector Volterra difference equations. These estimates give us explicit stability conditions. To establish the solution estimates, we will interpret the Volterra equations with matrix kernels as operator equations in appropriate spaces. Such an approach for Volterra difference equations has been used by Kolmanovskii and Myshkis [7], Kolmanovskii et al. [8], Kwapisz [9], Medina [10, 11], and Gil' and Medina [6]. Under some restriction, our results generalize the main results from [6, 8, 11].

Let $\mathbb{C}^n$ be an n-dimensional complex Euclidean space with the Euclidean norm $\|\cdot\|_{\mathbb{C}^n}$. For a positive $r \le \infty$, put

$$\omega_r = \{h \in \mathbb{C}^n : \|\cdot\|_{\mathbb{C}^n} \le r\}. \tag{1.1}$$

As usual, $c_0 = c_0(\mathbb{C}^n)$ is the Banach space of sequences of vectors from $\mathbb{C}^n$ equipped with the norm

$$\|h\|_{c_0} = \sup_k \|h_k\|_{\mathbb{C}^n} \quad (h = (h_k)_{k=1}^{\infty} \in c_0,\ h_k \in \mathbb{C}^n,\ k = 1,2,\dots) \tag{1.2}$$

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and $l^p = l^p(\mathbb{C}^n)$ $(1 < p < \infty)$ is the Banach space of sequences of vectors from $\mathbb{C}^n$ equipped with the norm

$$\|h\|_{l^p} = \left[\sum_{k=1}^{\infty} ||h_k||_{\mathbb{C}^n}^p\right]^{1/p}, \quad (h = (h_k)_{k=1}^{\infty} \in l^p,\ h_k \in \mathbb{C}^n,\ k = 1,2,\dots). \tag{1.3}$$

Let

$$a_{jk}(h_1,\dots,h_{j-1}) \quad (h_1,\dots,h_{j-1} \in \omega_r;\ k < j,\ j = 1,2,\dots) \tag{1.4}$$

be $n \times n$ matrices dependent on $j-1$ arguments. Consider the equation

$$x_j = G_j(x_1,\dots,x_{j-1}) + \sum_{k=1}^{j-1} a_{jk}(x_1,\dots,x_{j-1})x_k \quad (j = 1,2,\dots), \tag{1.5}$$

where the mappings $G_j : \mathbb{C}^{(j-1)n} \to \mathbb{C}^n$ have the properties

$$f_j := \sup_{h_1,\dots,h_{j-1} \in \omega_r} ||G_j(h_1,\dots,h_{-1})||_{\mathbb{C}^n} < \infty \quad (j = 1,2,\dots). \tag{1.6}$$

Moreover, $G_1 \in \mathbb{C}^n$ is given and

$$f := \{f_1, f_2,\dots\} \in l^p(\mathbb{C}). \tag{1.7}$$

In addition, it is assumed that

$$v_{jk} = \sup_{h_1,\dots,h_{j-1}} ||a_{jk}||_{\mathbb{C}^n} < \infty \quad (k < j, j = 1,2,\dots), \tag{1.8}$$

$$N_p(V) := \left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j-1} v_{jk}^{p'}\right)^{p/p'}\right]^{1/p} < \infty \tag{1.9}$$

with

$$\frac{1}{p'} + \frac{1}{p} = 1. \tag{1.10}$$

To formulate the result, denote

$$m_p(V) = \sum_{k=0}^{\infty} \frac{N_p^k(V)}{\sqrt[p]{k!}},$$
$$Q_p(V) = \sup_{j=1,2,\ldots} \left[\sum_{k=1}^{j-1} v_{jk}^{p'} \right]^{1/p'}. \tag{1.11}$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let conditions (1.7) and (1.9) hold. Then a solution $x = (x_1, x_2, \ldots)$ of (1.5) satisfies the inequalities*

$$\|x\|_{l^p} \le m_p(V)\|f\|_{l^p},$$
$$\|x\|_{c_0} \le \|f\|_{c_0} + Q_p(V)m_p(V)\|f\|_{l^p}, \tag{1.12}$$

provided

$$\|f\|_{c_0} + m_p(V)Q_p(V)\|f\|_{l^p} < r. \tag{1.13}$$

Note that due to the Hölder inequality,

$$m_p(V) = \sum_{k=0}^{\infty} \frac{a^k N_p^k(V)}{a^k \sqrt[p]{k!}} \le \left[\sum_{k=0}^{\infty} a^{kp'} \right]^{1/p'} \left[\sum_{k=0}^{\infty} \frac{N_p^{pk}(V)}{a^{kp} k!} \right]^{1/p} \tag{1.14}$$

for any positive $a < 1$. So

$$m_p(V) \le (1 - a^{p'})^{-1/p'} \exp\left[\frac{N_p^p(V)}{pa^p} \right]. \tag{1.15}$$

In particular, taking

$$a = \sqrt[p]{\frac{1}{p}}, \tag{1.16}$$

we have

$$m_p(V) \le b_p \exp[N_p^p(V)], \tag{1.17}$$

where

$$b_p = \left(1 - \frac{1}{p^{p'/p}}\right)^{-1/p'}. \tag{1.18}$$

2. Proof of Theorem 1.1

First, assume that $r = \infty$. Then conditions (1.7) and (1.9) imply

$$\|x_j\|_{\mathbb{C}^n} \le f_j + \sum_{k=1}^{j-1} v_{jk}\|x_k\|_{\mathbb{C}^n} \quad (j = 1,2,\dots). \tag{2.1}$$

Define on $l^p = l^p(\mathbb{R})$ the operator V by

$$[Vh]_j = \sum_{k=1}^{j-1} h_k. \tag{2.2}$$

Here, $[h]_j$ means the jth coordinate of the element $h \in l^p(\mathbb{R})$. The operator V is a quasinilpotent one. So, due to the well-known lemma from the book by Dalec'kiĭ and Kreĭn (see [2, Lemma 3.2.1]) (the comparison principle),

$$\|x_j\|_{\mathbb{C}^n} \le y_j, \tag{2.3}$$

where y_j is a solution of the equation

$$y_j = f_j + \sum_{k=1}^{j-1} v_{jk} y_k \quad (j = 1,2,\dots). \tag{2.4}$$

Rewrite this equation as

$$y = f + Vy. \tag{2.5}$$

Lemma 2.1. *Let conditions (1.7) and (1.9) hold. Then a solution y of (2.5) satisfies the inequality*

$$\|y\|_{l^p} \le m_p(V)\|f\|_{l^p}. \tag{2.6}$$

Proof. Rewrite (2.5) as

$$y = (I - V)^{-1} f. \tag{2.7}$$

By [5, Lemma 4.3],

$$\|V^k\|_{l^p} \le \frac{N_p^k(V)}{\sqrt[p]{k!}}. \tag{2.8}$$

Since

$$(I - V)^{-1} = \sum_{k=0}^{\infty} V^k, \tag{2.9}$$

we have

$$||(I-V)^{-1}||_{l^p} \leq m_p(V), \tag{2.10}$$

concluding the proof. □

LEMMA 2.2. *Let conditions (1.7) and (1.9) hold. Then a solution y of (2.5) satisfies the inequality*

$$\|y\|_{c_0} \leq \|f\|_{c_0} + m_p(V)\|f\|_{l^p}. \tag{2.11}$$

Proof. From (2.5) it follows that

$$\|y\|_{c_0} \leq \|f\|_{c_0} + \|Vy\|_{c_0}. \tag{2.12}$$

But due to Hölder's inequality

$$\|Vy\|_{c_0} \leq \sup_{j=1,2,\ldots} \left[\sum_{k=1}^{j-1} v_{jk}^{p'}\right]^{1/p'} \|y\|_{l^p} = Q_p(V)\|y\|_{l^p}, \tag{2.13}$$

now (2.12) and Lemma 2.1 yield

$$\|y\|_{c_0} \leq \|f\|_{c_0} + Q_p(V)\|y\|_{l^p} \leq \|f\|_{c_0} + Q_p(V)m_p(V)\|f\|_{l^p} \tag{2.14}$$

as claimed. □

Proof of Theorem 1.1. If $r = \infty$, then the required result follows from Lemmas 2.1 and 2.2. Let now $r < \infty$. By a simple application of the Urysohn lemma and Lemma 2.2, we get the required result. □

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