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Research Article

Perron-Type Criterion for Linear Difference Equations with Distributed Delay

Jehad O. Alzabut and Thabet Abdeljawad

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It is shown that if a linear difference equation with distributed delay of the form $\Delta x(n) = \sum_{k=-d}^{0}\Delta_k\zeta(n+1,k-1)x(n+k-1)$, $n \geq 1$, satisfies a Perron condition then its trivial solution is uniformly asymptotically stable.

1. Introduction and preliminaries

The characteristics of many real dynamical models are described by delay differential equations, see the list of recent papers [1–6]. However, delay differential equations are hard to manage analytically and thus many articles have examined the models by using the corresponding delay difference equations. In practice, one can easily formulate a discrete model directly by experiments or observations. For simulation purposes, nevertheless, it is important that a discrete analog faithfully inherits the characteristics of the continuous time parent system. In the recent years, the stability of solutions of linear delay difference equations has been extensively studied in the literature. Many authors have addressed this problem by using various methods and applying different techniques, see for instance the papers [7–13] and references therein.

It is well known in the theory of ordinary differential equations (see, e.g., [14, page 120]) that if for every continuous function $f(t)$ bounded on $[0,\infty)$, the solution of the equation

$$x'(t) = A(t)x(t) + f(t), \quad t \geq 0, \tag{1.1}$$

satisfying $x(0) = 0$ is bounded on $[0,\infty)$, then the trivial solution of the corresponding homogeneous equation

$$x'(t) = A(t)x(t), \quad t \ge 0, \tag{1.2}$$

is uniformly asymptotically stable. This result is referred as Perron's criterion [15]. Later, Perron's criterion has been extended by Halanay [14, page 371] to differential equations with distributed delay. Indeed, it was shown that if for every continuous function $f(t)$ bounded on $[0,\infty)$, the solution of the equation

$$x'(t) = \int_{-\tau}^{0} \mathrm{d}_s \eta(t,s)x(t+s) + f(t), \quad t \ge 0, \tag{1.3}$$

satisfying $x(t) = 0$ for $t \in [-\tau, 0]$ is bounded on $[0,\infty)$, then the trivial solution of the equation

$$x'(t) = \int_{-\tau}^{0} \mathrm{d}_s \eta(t,s)x(t+s), \quad t \ge 0, \tag{1.4}$$

is uniformly asymptotically stable. Recently, the above result has been carried out for linear impulsive delay differential equations [16] and for linear impulsive differential equations with distributed delay [17]. See also the papers [18–21] for related works. The paper [22] deserves more attention as it discusses a closed result.

For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points. But unless we can explicitly solve both equations, it is impossible to satisfy this requirement. Most of the time, it is desirable that a difference equation, when derived from a differential equation, preserves the dynamical features of the corresponding continuous time model. If such discrete models can be derived from continuous time delay models, then the discrete time models can be used without any loss of functional similarities of continuous models. In this paper, we will consider a discrete time analog of the continuous-time Perron criterion. We will show that a discrete time analog will preserve the stability condition of its continuous time counterpart. Our approach is based on the technique followed in [14] and totally different from the one used in [22].

There are several ways of deriving discrete time version of dynamical systems corresponding to continuous time formulations. One of the ways is based on appropriate modifications of the models. For this technique, differential equations with piecewise constant arguments prove helpful; see [23] for more information. Let us assume that the average growth rate in (1.4) changes at regular intervals of time, then we can incorporate this aspect in (1.4) and obtain the following modified equation:

$$x'(t) = \int_{-\tau}^{0} \mathrm{d}_s \eta([t],s)x([t+s]), \quad t \ge 0, \tag{1.5}$$

where $[t]$ denotes the integer part of t for $t \geq 0$. This equation occupies a position midway between differential and difference equations. Integrating (1.5) on any interval of the form $[n, n+1)$, $n = 0, 1, 2, \ldots$, we obtain

$$x(t) - x(n) = \int_{-\tau}^{0} \mathrm{d}_s \eta(n,s) x(n+s)(t-n). \tag{1.6}$$

Letting $t \to n+1$, we have

$$\Delta x(n) = \int_{-\tau}^{0} \mathrm{d}_s \eta(n,s) x(n+s), \tag{1.7}$$

where $\Delta x(n) := x(n+1) - x(n)$. Let $\mathbb{N}$ and $\mathbb{Z}$ denote, as usual, the sets of natural and integer numbers, respectively. Let $\zeta : \mathbb{N} \times \mathbb{Z} \to \mathbb{R}^{m\times m}$ be a kernel function satisfying the following conditions:

(i) $\zeta(n,k)$ is normalized so that $\zeta(n,s) = 0$ for $s \geq -1$ and for $s \leq -d+1$, where $d > 3$ is a positive integer;

(ii) there exists a positive real number γ such that $\sup_{t\geq 0} \sum_{s=-d}^{0} \|\Delta_s \zeta(t,s)\| \leq \gamma$.

It turns out that the discrete analog of (1.7) should have the form

$$\Delta x(n) = \sum_{k=-d}^{0} \Delta_k \zeta(n,k) x(n+k), \quad n \geq 1, \tag{1.8}$$

where by $\Delta_k \zeta(n,k)$ we mean the difference $\zeta(n,k+1) - \zeta(n,k)$.

For the sake of convenience, however, we will consider an equation of the form

$$\Delta x(n) = \sum_{k=-d}^{0} \Delta_k \zeta(n+1,k-1) x(n+k-1), \quad n \geq 1. \tag{1.9}$$

For any $a, b \in \mathbb{N}$, define $\mathbb{N}(a) = \{a, a+1, \ldots\}$ and $\mathbb{N}(a,b) = \{a, a+1, \ldots, b\}$, where $a \leq b$. By a solution of (1.9), we mean a sequence $x(n)$ of elements in $\mathbb{R}^m$ which is defined for all $n \in \mathbb{N}(n_0 - d + 1)$ and satisfies (1.9) for $n \in \mathbb{N}(n_0)$ for some $n_0 \in \mathbb{N}$. It is easy to see that for any given $n_0 \in \mathbb{N}$ and initial conditions of the form

$$x(n) = \phi(n), \quad n \in \mathbb{N}(n_0 - d + 1, n_0 + 1), \tag{1.10}$$

Equation (1.9) has a unique solution $x(n)$ which is defined for $n \in \mathbb{N}(n_0 - d + 1)$ and satisfies the initial conditions (1.10). To emphasize the dependence of the solution on the initial point n_0 and the initial functions ϕ, we may use the notation $x(n) = x(n; n_0, \phi)$.

Consider the nonhomogeneous equation

$$\Delta x(n) = \sum_{k=-d}^{0} \Delta_k \zeta(n+1,k-1) x(n+k-1) + f(n), \quad n \geq 1, \tag{1.11}$$

where f is a sequence with values in $\mathbb{R}^m$. Perron's condition for (1.9) is formulated as follows.

Definition 1.1. Equation (1.9) is said to verify Perron's condition if for every bounded sequence $f(n)$ for $n \geq 1$, the solution of (1.11) with $\phi(n) = 0$ for $n \in \mathbb{N}(-d+1,1)$ is bounded for $n \geq 1$.

The main result of this paper is stated in the following theorem.

THEOREM 1.2. *If (1.9) verifies Perron's condition, then its trivial solution is uniformly asymptotically stable.*

The proof is provided in the last section.

2. Background lemmas

This section is devoted to certain auxiliary assertions that will be needed in the proof of Theorem 1.2. Lemma 2.1 which introduces the main result of this section is needed to define an adjoint equation of (1.9). Lemmas 2.3 and 2.6 give representations of solutions of the considered equations. Lemmas 2.9 and 2.10 are concerned with the boundedness of fundamental functions of (1.9).

We will construct the adjoint equation of (1.9) which resembles the one constructed by Halanay in [14, page 365]. It turns out that the function has the form

$$\langle x(n), y(n)\rangle = x^T(n)y(n) + \sum_{s=n-d}^{n} x^T(s-1)\Delta_s \sum_{\alpha=n+1}^{s+d-1} \zeta^T(\alpha, s-\alpha)y(\alpha), \tag{2.1}$$

where by "T" we mean the transposition.

Consider the equation

$$\Delta y(n) = -\Delta_n \sum_{k=-d}^{0} \zeta^T(n-k, k+1)y(n-k), \tag{2.2}$$

where $\Delta_n \zeta(n,k) := \zeta(n+1,k) - \zeta(n,k)$. We claim that (2.2) is the adjoint equation of (1.9). The following lemma generalizes a fundamental result to linear difference equations with distributed delay.

LEMMA 2.1. *Let $x(n)$ be any solution of (1.9) and let $y(n)$ be any solution of (2.2), then*

$$\langle x(n), y(n)\rangle = \text{constant}, \tag{2.3}$$

where $\langle\, ,\rangle$ is defined by (2.1).

Proof. Clearly, it suffices to show that $\Delta\langle x(n), y(n)\rangle = 0$. It follows that

$$\Delta\langle x(n), y(n)\rangle = x^T(n)\Delta y(n) + \Delta x^T(n)y(n+1) + \Delta_n \sum_{s=n-d}^{n} g(s,n), \tag{2.4}$$

where

$$g(s,n) = x^T(s-1)\Delta_s \sum_{\alpha=n+1}^{s+d-1} \zeta^T(\alpha, s-\alpha)y(\alpha). \tag{2.5}$$

It is easy to see that

$$\Delta_n \sum_{s=n-d}^{n} g(s,n) = g(n+1,n+1) - g(n-d,n) + \sum_{s=n-d+1}^{n} \Delta_n g(s,n). \tag{2.6}$$

Therefore (2.4) becomes

$$\begin{aligned}\Delta\langle x(n),y(n)\rangle &= x^T(n)\Delta y(n) + \Delta x^T(n)y(n+1) + g(n+1,n+1)\\ &\quad - g(n-d,n) + \sum_{s=n-d+1}^{n} \Delta_n g(s,n).\end{aligned} \tag{2.7}$$

Thus

$$\begin{aligned}\Delta\langle x(n),y(n)\rangle &\overset{\text{by (2.2)}}{=} x^T(n)\left[-\Delta_n \sum_{k=-d}^{0} \zeta^T(n-k,k+1)y(n-k)\right]\\ &\overset{\text{by (1.9)}}{+} \left[\sum_{k=-d}^{0} x^T(n+k-1)\Delta_k \zeta^T(n+1,k-1)\right] y(n+1)\\ &\overset{\text{by (2.5)}}{+} x^T(n) \sum_{\alpha=n+3}^{n+d} \Delta_n \zeta^T(\alpha,n+1-\alpha)y(\alpha)\\ &\overset{\text{by (2.5)}}{-} x^T(n-d-1)\Delta_{n-d} \sum_{\alpha=n+1}^{n-1} \zeta^T(\alpha,n-d-\alpha)y(\alpha)\\ &\overset{\text{by (2.5)}}{-} \sum_{s=n-d+1}^{n} x^T(s-1)\Delta_s \zeta^T(n+1,s-n-1)y(n+1).\end{aligned} \tag{2.8}$$

By changing the indices and using the properties of ζ, we see that the above equation is equal to zero. The proof is finished. □

In virtue of Lemma 2.1, we may say that (2.2) is an adjoint of (1.9). It is easy to verify also that the adjoint of (2.2) is (1.9), that is, they are mutually adjoint of each other.

Definition 2.2. A matrix solution $X(n,\alpha)$ of (1.9) satisfying $X(\alpha,\alpha) = I$, (I is an identity matrix), and $X(n,\alpha) = 0$ for $n < \alpha$ is called a fundamental function of (1.9).

Lemma 2.3. *Let $X(n,\alpha)$ be a fundamental function of (1.9) and $n_0 \in \mathbb{N}$. If $x(n)$ is a solution of (1.11)), then*

$$x(n) = X(n,n_0)x(n_0) + \sum_{s=n_0-d}^{n_0} \Delta_s \sum_{\alpha=n_0+1}^{s+d-1} X(n,\alpha)\zeta(\alpha,s-\alpha)x(s-1) + \sum_{k=n_0}^{n-1} X(n,k+1)f(k). \tag{2.9}$$

Proof. A direct substitution of (2.9) in (1.11) leads to the desired result. Indeed,

$$\begin{aligned}\Delta x(n) = \Delta X(n,n_0)x(n_0) + \Delta_n \sum_{s=n_0-d}^{n_0} \Delta_s \sum_{\alpha=n_0+1}^{s+d-1} X(n,\alpha)\zeta(\alpha,s-\alpha)x(s-1) \\ + \Delta_n \sum_{k=n_0}^{n-1} X(n,k+1)f(k),\end{aligned} \tag{2.10}$$

or

$$\begin{aligned}\Delta x(n) &\overset{\text{by (1.9)}}{=} \sum_{k=-d}^{0} \Delta_k\zeta(n+1,k-1)X(n+k-1,n_0)x(n_0) \\ &\quad + \sum_{s=n_0-d}^{n_0} \left[\Delta_s \sum_{\alpha=n_0+1}^{s+d-1} \left\{ \sum_{k=-d}^{0} \Delta_k\zeta(n+1,k-1)X(n+k-1,\alpha) \right\}\zeta(\alpha,s-\alpha)\right] x(s-1) \\ &\quad + f(n) + \sum_{k=n_0}^{n-1} \left\{ \sum_{k=-d}^{0} \Delta_k\zeta(n+1,k-1)X(n+k-1,k+1) \right\} f(k) \\ &= f(n) + \sum_{k=-d}^{0} \Delta_k\zeta(n+1,k-1)x(n+k-1).\end{aligned} \tag{2.11}$$

□

Corollary 2.4. *Let $X(n,\alpha)$ be a fundamental function of (1.9) and $n_0 \in \mathbb{N}$. If $x(n)$ is a solution of (1.9), then*

$$x(n) = X(n,n_0)x(n_0) + \sum_{s=n_0-d}^{n_0} \Delta_s \sum_{\alpha=n_0+1}^{s+d-1} X(n,\alpha)\zeta(\alpha,s-\alpha)x(s-1). \tag{2.12}$$

Definition 2.5. A matrix solution $Y(n,\alpha)$ of (2.2) satisfying $Y(\alpha,\alpha) = I$ and $Y(n,\alpha) = 0$ for $n > \alpha$ is called a fundamental function of (2.2).

Lemma 2.6. *Let $Y(n,\alpha)$ be a fundamental function of (2.2) and $n_0 \in \mathbb{N}$. If $y(n)$ is a solution of (1.10), then*

$$y(n) = Y(n,n_0)y(n_0) + \sum_{s=n_0-d}^{n_0} Y(n,s-1)\Delta_s \sum_{\alpha=n_0+1}^{s+d-1} \zeta^T(\alpha,s-\alpha)y(\alpha). \tag{2.13}$$

For more details on the derivation of solutions representations for similar difference equations, see the papers [24, 25].

Corollary 2.7. *Let $X(n,n_0)$ be a fundamental function of (1.9) and let $Y(n,n_0)$ be a fundamental function of (2.2). Then*

$$X(n,n_0) = Y^T(n_0,n). \tag{2.14}$$

Proof. By following the same arguments used by Halanay in [14, page 364], (2.3) can be written as follows:

$$\langle x(n), y(n)\rangle = \langle x(n_0), y(n_0)\rangle, \quad \text{for any } n_0 \in \mathbb{N}. \tag{2.15}$$

Further

$$\begin{aligned} X^T(n,n)Y(n,n_0) &+ \sum_{s=n-d}^{n} X^T(s-1,n)\Delta_s \sum_{\alpha=n+1}^{s+d-1} \zeta^T(\alpha,s-\alpha)Y(\alpha,n_0) \\ &\overset{\text{by (2.1)}}{=} X^T(n_0,n)Y(n_0,n_0) + \sum_{s=n-d}^{n} X^T(s-1,n)\Delta_s \sum_{\alpha=n+1}^{s+d-1} \zeta^T(\alpha,s-\alpha)Y(\alpha,n_0). \end{aligned} \tag{2.16}$$

Upon using the properties of the fundamental functions $X(n,n_0)$ and $Y(n,n_0)$, identity (2.14) is obtained. □

Remark 2.8. Formulas (2.12) and (2.13) can be derived from function (2.1). Indeed, replacing X by x or Y by y in (2.16), using (2.14) and employing the properties of X and Y, we obtain the desired results.

Let D denote the space of bounded sequences $f = \{f(n)\}$ with values in $\mathbb{R}^m$ and equipped with the norm $\|f\|_\infty = \sup_n \|f(n)\|$. Clearly, D is a Banach space.

LEMMA 2.9. *If (1.9) verifies Perron's condition, then there exists a constant C such that*

$$\sum_{\alpha=0}^{n-1} ||X(n,\alpha+1)|| < C, \quad \textit{for} \ \geq 1. \tag{2.17}$$

Proof. From formula (2.12), the solution $x(n)$ satisfying (1.10) with $\phi(n) = 0$, $n \in \mathbb{N}(-d+1,1)$, has the form

$$x(n) = \sum_{\alpha=0}^{n-1} X(n,\alpha+1)f(\alpha). \tag{2.18}$$

For each $n \in \mathbb{N}$, define a sequence of linear operators $U_n : D \to \mathbb{R}^m$ by

$$U_n(f) = \sum_{\alpha=0}^{n-1} X(n,\alpha+1)f(\alpha). \tag{2.19}$$

By using the estimate

$$||U_n(f)|| \leq \sum_{\alpha=0}^{n-1} ||X(n,\alpha+1)|| \|f\|_\infty, \tag{2.20}$$

it follows that the operators U_n are bounded. In virtue of Perron's condition, we deduce that for each $f \in D$, we can find $c_f > 0$ such that $\sup_n \|U_n(f)\| \leq c_f$. Hence, by using the

Banach-Steinhaus theorem, there exists a constant $T > 0$ such that

$$\sup_n \|U_n(f)\| \leq T\|f\|_\infty, \quad \forall f \in D. \tag{2.21}$$

For fixed $n \geq 1$, let x_{rk}, $(1 \leq r,k \leq m)$, be the elements of the matrix $X(n,\alpha+1)$ where $0 \leq \alpha \leq n-1$. Let e_p denote the canonical basis having the unity at the pth place and zero otherwise. Let f_α^r be the element of D with its α-component the vector V_r of $\mathbb{R}^m$ and zeros otherwise, where $V_r = \sum_{k=1}^m \operatorname{sign} x_{rk} e_k$. The vector $X f_\alpha^r(\alpha)$ will have its rth component equal to $\sum_{k=1}^m |x_{rk}|$.

From (2.21), we can write

$$\left\|\sum_{\alpha=0}^{n-1} X(n,\alpha+1) f_\alpha^r(\alpha)\right\| \leq M_2, \tag{2.22}$$

where $M_2 = T\sup_r \|V_r\|$. Hence

$$\sum_{\alpha=0}^{n-1}\sum_{k=1}^{m} \left| x_{rk}(n,\alpha+1)\right| \leq M_2. \tag{2.23}$$

Since this relation is true for every r, we take the summation $\sum_{r=1}^m$ of both sides to deduce that there exists C such that

$$\sum_{\alpha=0}^{n-1} \|X(n,\alpha+1)\| < C \quad \text{for } n \geq 1, \tag{2.24}$$

which completes the proof. □

Lemma 2.10. *If (1.9) verifies Perron's condition, then there exists a constant M such that*

$$\|X(n,n_0)\| < M, \quad \textit{for } n \geq n_0 \geq 1. \tag{2.25}$$

Proof. Taking into account that $Y^T(\alpha,t)$ satisfies (2.2), we take $\sum_{\alpha=n_0}^{n-1}$ of both sides

$$\sum_{\alpha=n_0}^{n-1} \Delta_\alpha Y^T(\alpha,n) = -\sum_{\alpha=n_0}^{n-1} \Delta_\alpha \sum_{k=-d}^{0} Y^T(\alpha-k,n)\zeta(\alpha-k,k+1). \tag{2.26}$$

It follows that

$$X(n,n_0) \overset{\text{by}(16)}{=} I + \sum_{\alpha=n_0}^{n-1} \Delta_\alpha \sum_{\alpha=-d}^{0} X(n,\alpha-k)\zeta(\alpha-k,k+1). \tag{2.27}$$

By setting $\alpha - k = r$ and then changing the order of summations, we obtain

$$X(n,n_0) = I + \sum_{r=n_0}^{n_0+d}\sum_{\alpha=n_0}^{r} X(n,r)\Delta_\alpha\zeta(r,\alpha-r+1) + \sum_{r=n_0+d+1}^{n}\sum_{\alpha=r-d}^{r} X(n,r)\Delta_\alpha\zeta(r,\alpha-r+1), \tag{2.28}$$

where that $X(n,r)=0$ for $r>n$ is used. Taking the norm of both sides, we have

$$||X(n,n_0)|| \\ \leq 1+\sum_{r=n_0}^{n_0+d}\sum_{\alpha=n_0}^{r}||X(n,r)\Delta_\alpha\zeta(r,\alpha-r+1)||+\sum_{r=n_0+d+1}^{n-1}\sum_{\alpha=r-d}^{r}||X(n,r)\Delta_\alpha\zeta(r,\alpha-r+1)||. \tag{2.29}$$

By using the properties of ζ, it follows that

$$||X(n,n_0)||\leq 1+\gamma\sum_{r=n_0}^{n-1}||X(n,r)||. \tag{2.30}$$

In view of Lemma 2.9, we end up with the desired result. □

We are now in a position to give the proof of Theorem 1.2.

3. Proof of Theorem 1.2

From formula (2.12), the solution of (1.9) has the form

$$x(n;n_0,\phi)=X(n,n_0)x(n_0)+\sum_{s=n_0-d}^{n_0}\Delta_s\sum_{\alpha=n_0+1}^{s+d-1}X(n,\alpha)\zeta(\alpha,s-\alpha)x(s-1). \tag{3.1}$$

Changing the order of summations, we get

$$x(n;n_0,\phi)=X(n,n_0)x(n_0)+\sum_{s=n_0+1}^{n_0+d-1}X(n,\alpha)\sum_{s=\alpha-d+1}^{n_0}\Delta_s\zeta(\alpha,s-\alpha)x(s-1). \tag{3.2}$$

In virtue of Lemma 2.10, we obtain

$$||x(n;n_0,\phi)||\leq M_1\|\phi\|_0, \tag{3.3}$$

where

$$M_1=M[1+\gamma d],\qquad \|\phi\|_0=\sup_{n\in\mathbb{N}(n_0-d+2,n_0)}||x(n)||. \tag{3.4}$$

Thus, the trivial solution is uniformly stable.

It remains to prove that

$$\lim_{n\to\infty}x(n;n_0,\phi)=0 \tag{3.5}$$

uniformly with respect to n_0 and ϕ. For our purpose, let $m_0\geq n_0$, then the solution has the form

$$x(n;n_0,\phi)=X(n,m_0)x(m_0;n_0,\phi)+\sum_{\alpha=m_0+1}^{m_0+d-1}X(n,\alpha)\sum_{s=\alpha-d+1}^{m_0}\Delta_s\zeta(\alpha,s-\alpha)x(s-1;n_0,\phi). \tag{3.6}$$

Taking the summation $\sum_{m_0=n_0}^{n}$ of both sides, we have

$$(n-n_0)x(n;n_0,\phi) = \sum_{m_0=n_0}^{n} X(n,m_0)x(m_0;n_0,\phi) + \sum_{m_0=n_0}^{n}\sum_{\alpha=m_0+1}^{m_0+d-1} X(n,\alpha)z(m_0,\alpha), \tag{3.7}$$

where

$$z(m_0,\alpha) = \sum_{s=\alpha-d+1}^{m_0} \Delta_s\zeta(\alpha,s-\alpha)x(s-1,n_0,\phi). \tag{3.8}$$

Interchanging the order of summations, we obtain

$$\begin{aligned}(n-n_0)x(n;n_0,\phi) = {} & \sum_{m_0=n_0}^{n} X(n,m_0)x(m_0;n_0,\phi) \\ & + \sum_{\alpha=n_0+1}^{n_0+d-1}\sum_{m_0=n_0}^{\alpha-1} X(n,\alpha)z(m_0,\alpha) \\ & + \sum_{\alpha=n_0+d}^{n+1}\sum_{m_0=\alpha-d+1}^{\alpha-1} X(n,\alpha)z(m_0,\alpha),\end{aligned} \tag{3.9}$$

where that $X(n,\alpha)=0$ for $\alpha>n$ is used. Taking the norm of both sides, we get

$$\begin{aligned}&(n-n_0)\|x(n;n_0,\phi)\| \\ &\quad\le M_1\|\phi\|_0\sum_{m_0=n_0}^{n}\|X(n,m_0)\| + d\gamma M_1\|\phi\|_0\sum_{\alpha=n_0+d}^{n+1}\|X(n,\alpha)\| + d^2\gamma MM_1\|\phi\|_0.\end{aligned} \tag{3.10}$$

It follows that

$$\|x(n;n_0,\phi)\| \le \frac{M_2}{(n-n_0)}\|\phi\|_0, \tag{3.11}$$

where

$$M_2 = M_1[C + d\gamma C + Md^2\gamma]. \tag{3.12}$$

Clearly, (3.5) follows from (3.11). The proof is finished.

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Jehad O. Alzabut: Department of Mathematics, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia
Current address: Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey
Email address: jehad@cankaya.edu.tr

Thabet Abdeljawad: Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey
Email address: thabet@cankaya.edu.tr