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Research Article
Asymptotic Periodicity of a Higher-Order Difference Equation

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We give a complete picture regarding the asymptotic periodicity of positive solutions of the following difference equation: $x_n = f(x_{n-p_1},\dots,x_{n-p_k},x_{n-q_1},\dots,x_{n-q_m})$, $n \in \mathbb{N}_0$, where p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_m$ and $\gcd(p_1,\dots,p_k,q_1,\dots,q_m) = 1$, the function $f \in C[(0,\infty)^{k+m},(\alpha,\infty)]$, $\alpha > 0$, is increasing in the first k arguments and decreasing in other m arguments, there is a decreasing function $g \in C[(\alpha,\infty),(\alpha,\infty)]$ such that $g(g(x)) = x$, $x \in (\alpha,\infty)$, $x = f(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m})$, $x \in (\alpha,\infty)$, $\lim_{x\to\alpha+}g(x) = +\infty$, and $\lim_{x\to+\infty}g(x) = \alpha$. It is proved that if all p_i, $i \in \{1,\dots,k\}$, are even and all q_j, $j \in \{1,\dots,m\}$ are odd, every positive solution of the equation converges to (not necessarily prime) a periodic solution of period two, otherwise, every positive solution of the equation converges to a unique positive equilibrium.

1. Introduction

Recently, there is a huge interest in studying nonlinear difference equations; see, for example, [1–29] and the references therein.

In [26], we proved the following theorem.

Theorem A. *Consider the following difference equation:*

$$x_n = 1 + \frac{\sum_{i=1}^{k}\alpha_i x_{n-p_i}}{\sum_{j=1}^{m}\beta_j x_{n-q_j}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where α_i, $i \in \{1,\dots,k\}$, and β_j, $j \in \{1,\dots,m\}$, are positive numbers such that $\sum_{i=1}^{k}\alpha_i = \sum_{j=1}^{m}\beta_j = 1$, and p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$. Assume that

$$G := \gcd\left(p_1,\dots,p_k,q_1,\dots,q_m\right) = 1. \tag{1.2}$$

Then if all p_i, $i \in \{1,\dots,k\}$, are even and all q_j, $j \in \{1,\dots,m\}$, are odd, every positive solution of (1.1) converges to a periodic solution of period two. Otherwise, every positive solution of (1.1) converges to a unique positive equilibrium.

On the other hand, by the main result in [15], in [18], we proved the following result.

Theorem B. *Consider the difference equation*

$$x_{n+1} = F\left(x_n,x_{n-2},\dots,x_{n-2k}\right), \tag{1.3}$$

where $k \in \mathbb{N}$ is fixed. If

(a) $F \in C[(0,+\infty)^{k+1},(0,+\infty)]$ *is nonincreasing in each of its arguments,*
(b) $F(z_1,z_2,\dots,z_{k+1})$ *is strictly decreasing in the first argument z_1,*
(c) $g(g(x)) = x$ *for all $x \in (0,+\infty)$, where $g(x) = F(x,x,\dots,x)$,*

then every positive solution of (1.3) converges to (not necessarily prime) a period-two solution.

For closely related results to Theorem B, see [5, 7, 14, 16, 19] and the references therein.

These two theorems motivated us to investigate the behavior of positive solutions of the following difference equation:

$$x_n = f\left(x_{n-p_1},\dots,x_{n-p_k},x_{n-q_1},\dots,x_{n-q_m}\right), \quad n \in \mathbb{N}_0, \tag{1.4}$$

where p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$, and the function $f \in C[(0,\infty)^{k+m},(\alpha,\infty)]$, $\alpha > 0$, satisfies the following conditions:

(a) f is increasing in first k arguments and decreasing in other m arguments;
(b) there is a decreasing function $g \in C[(\alpha,\infty),(\alpha,\infty)]$ such that $g(g(x)) = x$, $x \in (\alpha,\infty)$;
(c)

$$x = f\left(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m}\right), \quad x \in (\alpha,\infty); \tag{1.5}$$

(d)

$$\lim_{x\to\alpha+} g(x) = +\infty, \qquad \lim_{x\to+\infty} g(x) = \alpha. \tag{1.6}$$

Note that if x is sufficiently close to α, then from (d) it follows that $x < g(x)$. From this and by (a) and (c), we have that

$$x = f\left(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m}\right) < f\left(\underbrace{x,\dots,x}_{k+m}\right). \tag{1.7}$$

On the other hand, if x is sufficiently large, from (d), it follows that $g(x) < x$. This, along with (a) and (c), yields

$$x = f\left(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m}\right) > f\left(\underbrace{x,\dots,x}_{k+m}\right). \tag{1.8}$$

Hence the equation $x = f(x,\dots,x)$ has a solution x^* on the interval (α,∞). In view of (c), it must be

$$x^* = f(x^*,\dots,x^*) = f\left(\underbrace{x^*,\dots,x^*}_{k},\underbrace{g(x^*),\dots,g(x^*)}_{m}\right). \tag{1.9}$$

This, and (a), imply that $g(x^*) = x^*$, which, along with (b), shows that x^* is a unique solution of the equation $g(x) = x$ on the interval (α,∞), and consequently, it is a unique solution of the equation $x = f(x,\dots,x)$ on (α,∞).

Here, we give a complete picture regarding the asymptotic stability of positive solutions of (1.4).

We may assume that

$$G := \gcd(p_1,\dots,p_k,q_1,\dots,q_m) = 1, \tag{1.10}$$

otherwise, (1.4) can be separated into the following G independent difference equations

$$x_l^{(t)} = f\left(x_{l-p_1/G}^{(t)},\dots,x_{l-p_k/G}^{(t)},x_{l-q_1/G}^{(t)},\dots,x_{l-q_m/G}^{(t)}\right), \quad l \in \mathbb{N}_0, \tag{1.11}$$

where $x_l^{(t)} = x_{Gl+t}$ and $t \in \{0,1,\dots,G-1\}$.

Remark 1.1. Note that by the definition of G, it follows that at least one of the numbers p_i/G, $i \in \{1,\dots,k\}$ and q_j/G, $j \in \{1,\dots,m\}$ is odd. This fact will be used in the proof of the main result of this paper, in Theorem 2.4.

Remark 1.2. Note also that some of p_i and q_j can be equal.

We also need the following result by Karakostas [10] (see also [11]).

Theorem C. *Let J be an interval of real numbers, $f \in C[J^l,J]$, and let $(x_n)_{n=-l}^{\infty}$ be a bounded solution of the difference equation*

$$x_n = f(x_{n-1},\dots,x_{n-l}), \quad n \in \mathbb{N}_0, \tag{1.12}$$

with $I = \liminf_{n\to\infty} x_n$, $S = \limsup_{n\to\infty} x_n$, *and with* $I, S \in J$. *Then there exist two solutions* $(I_n)_{n=-\infty}^{\infty}$ *and* $(S_n)_{n=-\infty}^{\infty}$ *of the difference equation*

$$x_n = f(x_{n-1}, \ldots, x_{n-l}), \tag{1.13}$$

which satisfy the equation for all $n \in \mathbb{Z}$, *with* $I_0 = I$, $S_0 = S$, $I_n, S_n \in [I, S]$ *for all* $n \in \mathbb{Z}$, *such that for every* $N \in \mathbb{Z}$, I_N *and* S_N *are limit points of* $(x_n)_{n=-l}^{\infty}$. *Furthermore, for every* $m \le -l$, *there exist two subsequences* (x_{r_n}) *and* (x_{l_n}) *of the solution* $(x_n)_{n=-l}^{\infty}$ *such that the following are true:*

$$\lim_{n\to\infty} x_{r_n+N} = I_N, \quad \lim_{n\to\infty} x_{l_n+N} = S_N, \quad \text{for every } N \ge m. \tag{1.14}$$

The solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of (1.13) are called full-limiting solutions of (1.13) associated with the solution $(x_n)_{n=-l}^{\infty}$ of (1.12).

2. Main results

The first result in this section concerns the boundedness character of positive solutions of (1.4). Some other closely related results can be found, for example, in [2, 3, 8, 17, 20–24, 26, 27].

Theorem 2.1. *Every positive solution of (1.4) is bounded.*

Proof. Assume that $(x_n)_{n=-l}^{\infty}$ is a positive solution of (1.4). Then since $f : (0,\infty)^{k+m} \to (\alpha, \infty)$, we have that $x_n > \alpha$ for $n \ge 0$. From this and in view of condition (d), we have that there is a positive number l greater than α such that $l \le x_i \le g(l)$ for $i \in \{0, 1, \ldots, s-1\}$, where $s = \max\{p_k, q_m\}$. Employing condition (c) and (1.4), we obtain

$$\begin{aligned} l = f(l, \ldots, l, g(l), \ldots, g(l)) &\le f(x_{s-p_1}, \ldots, x_{s-p_k}, x_{s-q_1}, \ldots, x_{s-q_m}) = x_s, \\ x_s = f(x_{s-p_1}, \ldots, x_{s-p_k}, x_{s-q_1}, \ldots, x_{s-q_m}) &\le f(g(l), \ldots, g(l), l, \ldots, l) = g(l). \end{aligned} \tag{2.1}$$

By the induction, we obtain that $x_n \in [l, g(l)]$ for every $n \in \mathbb{N}_0$, finishing the proof of the theorem. □

Theorem 2.2. *Assume that* $(x_n)_{n=-l}^{\infty}$ *is a positive solution of (1.4) and let* $\liminf_{n\to\infty} x_n = I$ *and* $\limsup_{n\to\infty} x_n = S$. *Then* $I = g(S)$ *and* $S = g(I)$.

Proof. First, note that in view of Theorem 2.1, every positive solution (x_n) of (1.4) is bounded, which implies that there are finite $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$, moreover, we have that $\alpha < I$. By taking the limit inferior and limit superior in (1.4) and using condition (c), we obtain, respectively,

$$f(I, \ldots, I, S, \ldots, S) \le I = f(I, \ldots, I, g(I), \ldots, g(I)), \tag{2.2}$$

$$f(S, \ldots, S, g(S), \ldots, g(S)) = S \le f(S, \ldots, S, I, \ldots, I). \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$g(I) \le S, \qquad I \le g(S), \tag{2.4}$$

which, in view of condition (b), implies that

$$g(S) \le g(g(I)) = I, \qquad S = g(g(S)) \le g(I). \tag{2.5}$$

Hence

$$I = g(S), \qquad S = g(I), \tag{2.6}$$

as desired. □

Remark 2.3. Note that if all p_i are even and all q_j are odd, then for every $I > \alpha$, the sequence

$$(\dots, I, g(I), I, g(I), \dots) = (\dots, I, S, I, S, \dots) \tag{2.7}$$

is a period two solution of (1.4).

Before we formulate and prove the main result of this paper, we need the following notation. Let

$$\mathcal{P} = \{p_i \mid i = 1, \dots, k\}, \qquad \mathcal{Q} = \{q_j \mid j = 1, \dots, m\}. \tag{2.8}$$

Theorem 2.4. *Consider (1.4), where the function f satisfies conditions (a)–(d). Assume that*

$$G := \gcd(p_1, \dots, p_k, q_1, \dots, q_m) = 1. \tag{2.9}$$

Then if all p_i, $i \in \{1, \dots, k\}$, are even and all q_j, $j \in \{1, \dots, m\}$, are odd, every positive solution of (1.4) converges to (not necessarily prime) a periodic solution of period two. Otherwise, every positive solution of (1.4) converges to a unique positive equilibrium.

Proof. Let $(L_{-i})_{i \in \mathbb{Z}}$ be a full-limiting sequence of a solution $(x_n)_{n=-l}^{\infty}$ of (1.4) such that $L_0 = S$. Since $(L_{-i})_{i \in \mathbb{Z}}$ is a solution of (1.4) belonging to the interval $[I, S]$, by employing Theorems 2.1 and 2.2 and condition (c), we obtain

$$\begin{aligned} S = L_0 &= f(L_{-p_1}, \dots, L_{-p_k}, L_{-q_1}, \dots, L_{-q_m}) \\ &\le f(S, \dots, S, I, \dots, I) = f(S, \dots, S, g(S), \dots, g(S)) = S. \end{aligned} \tag{2.10}$$

From (2.10), it follows that $L_{-p_i} = S$ for every $i \in \{1, \dots, k\}$ and $L_{-q_j} = I$ for every $j \in \{1, \dots, m\}$.

If we assume further that $\mathcal{P} \cap \mathcal{Q} \neq \varnothing$, then we obtain $I = S$, from which the result follows in this case.

Now we assume that $\mathcal{P} \cap \mathcal{Q} = \varnothing$ and there is $p_{i_0} \in \mathcal{P}$, which is odd. Let $p_{i_0} = 2s + 1$ and let q_{j_0} be an arbitrary element of $\mathcal{Q}$. Then (1.4) can be written in the form

$$x_n = f(\dots, x_{n-(2s+1)}, \dots, x_{n-q_{j_0}}, \dots). \tag{2.11}$$

Let $(L_{-i})_{i \in \mathbb{Z}}$ be a full-limiting sequence of a solution (x_n) of (1.4) such that $L_0 = S = \limsup_{n \to \infty} x_n$. From

$$S = L_0 = f(\dots, L_{-(2s+1)}, \dots, L_{-q_{j_0}}, \dots), \tag{2.12}$$

similar to (2.10), we obtain

$$L_{-(2s+1)} = S, \qquad L_{-q_{j_0}} = I. \tag{2.13}$$

From (2.13), and since $(L_{-i})_{i\in\mathbb{Z}}$ is a solution of (2.11), it follows that

$$L_{-2(2s+1)} = S, \qquad L_{-2q_{j_0}} = S. \tag{2.14}$$

Indeed, since

$$\begin{aligned} S = L_{-(2s+1)} &= f\left(\dots,L_{-2(2s+1)},\dots,L_{-q_{j_0}-(2s+1)},\dots\right) \\ &\leq f(S,\dots,S,I,\dots,I) = f\left(S,\dots,S,g(S),\dots,g(S)\right) = S, \end{aligned} \tag{2.15}$$

we obtain the first equality in (2.14). On the other hand, from

$$\begin{aligned} I = L_{-q_{j_0}} &= f\left(\dots,L_{-q_{j_0}-(2s+1)},\dots,L_{-2q_{j_0}},\dots\right) \\ &\geq f(I,\dots,I,S,\dots,S) = I, \end{aligned} \tag{2.16}$$

the second equality in (2.14) follows.

By induction, we obtain

$$L_{-(2s+1)i} = S, \quad i \in \mathbb{N}, \tag{2.17}$$

$$L_{-q_{j_0}j} = \begin{cases} I, & j \text{ odd}, \\ S, & j \text{ even}. \end{cases} \tag{2.18}$$

If we take $i = q_{j_0}$ in (2.17) and $j = 2s+1$ in (2.18), we obtain $I = L_{-(2s+1)q_{j_0}} = S$, as desired.

Now assume that all $p_i \in P$ are even, and $\mathfrak{Q}$ has odd as well as even elements. Then (1.4) can be written in the form

$$x_n = f\left(x_{n-p_1},\dots,x_{n-p_k},\dots,x_{n-q_{j_0}},\dots,x_{n-q_{j_1}},\dots\right), \tag{2.19}$$

where $q_{j_0} = 2s$ and $q_{j_1} = 2t+1$.

Condition $G = 1$ implies that for each sufficiently large n, for example, $n \geq n_0$, there are nonnegative numbers $d_i \in \mathbb{N}_0$, $i \in \{1,\dots,k+m\}$, such that

$$\sum_{i=1}^{k} p_i d_i + \sum_{j=1}^{m} q_j d_{k+j} = n, \tag{2.20}$$

see, for example, [13]. From condition $G = 1$, by using (2.19), (2.20), and employing the procedure described above for getting formulae (2.17) and (2.18), we obtain that the subsequence $(L_{-i})_{i\geq n_0}$ of the full-limiting sequence $(L_i)_{i\in\mathbb{Z}}$ with $L_0 = S$ takes only values I and S.

If we replace n in (2.19) by $-n_0 - l$, $l \in \{0,1,\dots,p_1 - 1\}$, we obtain that $L_{-n_0-l} = L_{-n_0-l-p_1 i}$ for every $i \in \mathbb{N}$ and each $l \in \{0,1,\dots,p_1 - 1\}$, that is, $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period p_1. Similarly, it can be proven that $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with periods $p_2,\dots,p_k$. The periodicity of $(L_{-i})_{i\in\mathbb{N}}$ with periods $2q_1,\dots,2q_m$ can be proved similar to (2.13), (2.14), and by using induction.

Since all $p_i \in \mathcal{P}$ are even and $G = 1$, we have that

$$\begin{aligned} 2 &\le \gcd\left(p_1, p_2,\dots, p_k, 2q_1,\dots,2q_m\right) \\ &= 2\gcd\left(\frac{p_1}{2},\frac{p_2}{2},\dots,\frac{p_k}{2},q_1,\dots,q_m\right) \le 2G = 2, \end{aligned} \tag{2.21}$$

that is,

$$\gcd\left(p_1,p_2,\dots,p_k,2q_1,\dots,2q_m\right) = 2. \tag{2.22}$$

Hence the sequence $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period two. Since $(L_i)_{i\in\mathbb{Z}}$ is a solution of (1.4), we obtain that $(L_i)_{i\in\mathbb{Z}}$ is also periodic with period two. From this, since $L_0 = S$ and by Theorem 2.2, we have that

$$L_{2i} = S, \quad L_{2i-1} = I = g(S), \quad i \in \mathbb{Z}. \tag{2.23}$$

From (2.19), (2.23), and condition (c), we have that

$$f\Big(\underbrace{S,\dots,S}_{k},I,\dots,I\Big) = S = L_0 = f\Big(\underbrace{S,\dots,S}_{k},\dots,S,\dots,I,\dots\Big). \tag{2.24}$$

This and condition (a) imply that $S = I$.

If $\mathcal{P}$ contains only even elements while $\mathcal{Q}$ contains only odd elements, then from condition (c), we see that (1.4) has infinite prime two periodic solutions of the form $x,g(x),x,g(x),\dots$. Similar to (2.22), it can be proven that, in this case, the full-limiting sequence $(L_i)_{i\in\mathbb{Z}}$, $L_0 = S$ is periodic with period two and that

$$L_{2i} = S, \quad L_{2i-1} = I = g(S), \quad i \in \mathbb{Z}. \tag{2.25}$$

From (2.25) and condition (d), we have that for every $\varepsilon \in (0,S)$, there is a $k_0 \in \mathbb{Z}$ and $j \in \{1,2,\dots,[s/2]+1\}$ such that

$$S-\varepsilon < x_{k_0+2j}, \qquad x_{k_0+2j-1} < g(S-\varepsilon), \tag{2.26}$$

where $s = \max\{p_k,q_m\}$.

From (2.26), (1.4), and by conditions (b) and (c), we have that

$$\begin{aligned} x_{k_0+2[s/2]+3} &< f\left(g(S-\varepsilon),\dots,g(S-\varepsilon),S-\varepsilon,\dots,S-\varepsilon\right) = g(S-\varepsilon), \\ x_{k_0+2[s/2]+4} &> f\left(S-\varepsilon,\dots,S-\varepsilon,g(S-\varepsilon),\dots,g(S-\varepsilon)\right) = S-\varepsilon. \end{aligned} \tag{2.27}$$

By induction, we obtain

$$x_{k_0+2i+1} < g(S-\varepsilon), \qquad x_{k_0+2i} > S-\varepsilon, \tag{2.28}$$

for every $i \in \mathbb{N}$.

From (2.28) and the fact that $\lim_{\varepsilon\to 0} g(S-\varepsilon) = g(S) = I$, it follows that $\lim_{n\to\infty} x_{2n} = S$ and $\lim_{n\to\infty} x_{2n-1} = I$, or $\lim_{n\to\infty} x_{2n} = I$ and $\lim_{n\to\infty} x_{2n-1} = S$, finishing the proof of the theorem. □

Remark 2.5. If, in Theorem 2.4, all p_i, $i \in \{1,\dots,k\}$, are even and all q_j, $j \in \{1,\dots,m\}$, are odd, then the two periodic solutions to which the other solutions converge can be essentially different from each other in the sense that one of them cannot be transformed into another one by means of cyclic permutations.

Remark 2.6. Note that Theorem 2.4 extends Theorem A as well as Theorem B (for the case when all arguments of the function F are decreasing).

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