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Research Article

On the Recursive Sequence $x_n = 1 + \sum_{i=1}^{k}\alpha_i x_{n-p_i}/\sum_{j=1}^{m}\beta_j x_{n-q_j}$

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We give a complete picture regarding the behavior of positive solutions of the following important difference equation: $x_n = 1 + \sum_{i=1}^{k}\alpha_i x_{n-p_i}/\sum_{j=1}^{m}\beta_j x_{n-q_j}$, $n \in \mathbb{N}_0$, where α_i, $i \in \{1,\dots,k\}$, and β_j, $j \in \{1,\dots,m\}$, are positive numbers such that $\sum_{i=1}^{k}\alpha_i = \sum_{j=1}^{m}\beta_j = 1$, and p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$. The case when $\gcd(p_1,\dots,p_k,q_1,\dots,q_m) = 1$ is the most important. For the case we prove that if all p_i, $i \in \{1,\dots,k\}$, are even and all q_j, $j \in \{1,\dots,m\}$, are odd, then every positive solution of this equation converges to a periodic solution of period two, otherwise, every positive solution of the equation converges to a unique positive equilibrium.

1. Introduction and preliminaries

In [1], we studied the behavior of positive solutions of the recursive equation

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

with $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0,\infty)$ and $k,m \in \{1,2,3,4,\dots\}$, where $s = \max\{k,m\}$. We proved that if 2^i is the highest power of 2 which divides m, then if $2^{i+1} \nmid k$, y_n tends to 2, exponentially, and otherwise every solution tends to a period t solution, with $t = 2\gcd(k,m)$. The method we used in [1] is a little bit complicated and its idea essentially stems from the theory of nonexpansive metrics. Since the above result is formulated in number theoretic language, we expect that the result is a particular case of a more general result, which motivates us to investigate the following somewhat natural generalization of (1.1):

$$x_n = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i}}{\sum_{j=1}^{m} \beta_j x_{n-q_j}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where α_i, $i \in \{1,\dots,k\}$, and β_j, $j \in \{1,\dots,m\}$, are positive numbers such that $\sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{m} \beta_j = 1$, and p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$.

Here, we give a complete picture regarding the asymptotic behavior of positive solutions of (1.2). For closely related results, see, for example, [1–16] and the references therein.

In the proof of the main result of this paper, we need the following result by Karakostas (see [8, 9]).

Theorem 1.1. *Let J be some interval of real numbers, let $f \in C[J^2, J]$, and let $(x_n)_{n=-1}^{\infty}$ be a bounded solution of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \tag{1.3}$$

with $I = \liminf_{n\to\infty} I_n$, $S = \limsup_{n\to\infty} x_n$ and with $I, S \in J$. Then there exist two solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}) \tag{1.4}$$

which satisfy the equation for all $n \in \mathbb{Z}$, with $I_0 = I$, $S_0 = S$, $I_n, S_n \in [I, S]$ for all $n \in \mathbb{Z}$ and such that for every $N \in \mathbb{Z}$, I_N and S_N are limit points of $(x_n)_{n=-1}^{\infty}$. Furthermore, for every $m \le -1$, there exist two subsequences (x_{r_n}) and (x_{l_n}) of the solution $(x_n)_{n=-1}^{\infty}$ such that the following are true:

$$\lim_{n\to\infty} x_{r_n+N} = I_N, \quad \lim_{n\to\infty} x_{l_n+N} = S_N \quad \text{for every } N \ge m. \tag{1.5}$$

The solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of (1.4) are called full limiting solutions of (1.4) associated with the solution $(x_n)_{n=-1}^{\infty}$ of (1.3).

2. Main results

First, we study the boundedness character of positive solutions of (1.2). For closely related results, see, for example, [4, 6, 12–14].

Theorem 2.1. *Every positive solution of (1.2) is bounded.*

Proof. Assume that (x_n) is a positive solution of (1.2). Note that $x_n > 1$ for $n \ge 0$. Hence, it is possible to choose positive numbers l and L greater than one such that $lL = L + l$ and $l \le x_i \le L$ for $i \in \{0, 1, \dots, s-1\}$, where $s = \max\{p_k, q_m\}$. Employing (1.2), we obtain

$$l = 1 + \frac{l}{L} \le x_s = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{s-p_i}}{\sum_{j=1}^{m} \beta_j x_{s-q_j}} \le 1 + \frac{L}{l} = L. \tag{2.1}$$

By the induction, we obtain that $x_n \in [l,L]$ for every $n \in \mathbb{N}_0$, finishing the proof of the theorem. □

We are now in a position to formulate and prove the main result of this paper.

Theorem 2.2. *Consider (1.2). Assume that*

$$G := \gcd(p_1,\ldots,p_k,q_1,\ldots,q_m) = 1. \tag{2.2}$$

Then if all p_i, $i \in \{1,\ldots,k\}$, are even and all q_j, $j \in \{1,\ldots,m\}$, are odd, every positive solution of (1.2) converges to a periodic solution of period two. Otherwise, every positive solution of (1.2) converges to a unique positive equilibrium.

Proof. Let

$$\mathcal{P} = \{p_i \mid i = 1,\ldots,k\}, \qquad \mathcal{Q} = \{q_j \mid j = 1,\ldots,m\}. \tag{2.3}$$

Assume first that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$. In view of Theorem 2.1, every positive solution (x_n) of (1.2) is bounded which implies that there are finite $\liminf_{n\to\infty} x_n = I$ and $\limsup_{n\to\infty} x_n = S$. Letting $n \to \infty$ in (1.2), we obtain

$$1 + \frac{I}{S} \le I \le S \le 1 + \frac{S}{I}, \tag{2.4}$$

from which it follows that

$$SI = I + S. \tag{2.5}$$

Let $(L_{-i})_{i\in\mathbb{Z}}$ be a full limiting sequence of a solution (x_n) of (1.2), such that $L_0 = S$. Since $(L_{-i})_{i\in\mathbb{Z}}$ is a solution of (1.2) belonging to the interval $[I,S]$, we have that

$$S = L_0 = 1 + \frac{\sum_{i=1}^{k} \alpha_i L_{-p_i}}{\sum_{j=1}^{m} \beta_j L_{-q_j}} \le 1 + \frac{S}{I} = S. \tag{2.6}$$

From (2.6), it follows that $L_{-p_i} = S$ for every $i \in \{1,\ldots,k\}$ and $L_{-q_j} = I$ for every $j \in \{1,\ldots,m\}$. Employing assumption $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$, we obtain $I = S$, from which the result follows in this case.

Now we assume that $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Further, assume that there is $p_{i_0} \in \mathcal{P}$ which is odd. Let $p_{i_0} = 2s+1$ and let q_{j_0} be an arbitrary element of $\mathcal{Q}$. Then, (1.2) can be written in the form

$$x_n = 1 + \frac{\alpha_{i_0} x_{n-(2s+1)} + \sum_{i=1, i\neq i_0}^{k} \alpha_i x_{n-p_i}}{\beta_{j_0} x_{n-q_{j_0}} + \sum_{j=1, j\neq j_0}^{m} \beta_j x_{n-q_j}}. \tag{2.7}$$

Let $(L_{-i})_{i\in\mathbb{Z}}$ be a full limiting sequence of a solution (x_n) of (1.2), such that $L_0 = S = \limsup_{n\to\infty} x_n$. From

$$S = L_0 = 1 + \frac{\alpha_{i_0} L_{-(2s+1)} + \sum_{i=1, i\neq i_0}^{k} \alpha_i L_{-p_i}}{\beta_{j_0} L_{-q_{j_0}} + \sum_{j=1, j\neq j_0}^{m} \beta_j L_{-q_j}}, \tag{2.8}$$

similar to (2.6), we obtain

$$L_{-(2s+1)} = S, \qquad L_{-q_{j_0}} = I. \tag{2.9}$$

From (2.9) and since $(L_{-i})_{i\in\mathbb{Z}}$ is a solution of (2.7), it follows that

$$L_{-2(2s+1)} = S, \qquad L_{-2q_{j_0}} = S. \tag{2.10}$$

Indeed, since

$$S = L_{-(2s+1)} = 1 + \frac{\alpha_{i_0}L_{-2(2s+1)} + \sum_{i=1,i\neq i_0}^{k}\alpha_i L_{-p_i-(2s+1)}}{\beta_{j_0}L_{-q_{j_0}-(2s+1)} + \sum_{j=1,j\neq j_0}^{m}\beta_j L_{-q_j-(2s+1)}} \leq 1 + \frac{S}{I} = S, \tag{2.11}$$

we obtain the first equality in (2.10). On the other hand, from

$$I = L_{-q_{j_0}} = 1 + \frac{\alpha_{i_0}L_{-q_{j_0}-(2s+1)} + \sum_{i=1,i\neq i_0}^{k}\alpha_i L_{-q_{j_0}-p_i}}{\beta_{j_0}L_{-2q_{j_0}} + \sum_{j=1,j\neq j_0}^{m}\beta_j L_{-q_{j_0}-q_j}} \geq 1 + \frac{I}{S} = I, \tag{2.12}$$

the second equality in (2.10) follows.

By induction we obtain

$$L_{-(2s+1)i} = S, \quad i \in \mathbb{N}, \tag{2.13}$$

$$L_{-q_{j_0}j} = \begin{cases} I, & j \text{ odd}, \\ S, & j \text{ even}. \end{cases} \tag{2.14}$$

If we take $i = q_{j_0}$ in (2.13) and $j = 2s+1$ in (2.14), we obtain $I = L_{-(2s+1)q_{j_0}} = S$, as desired.

Now, assume that all $p_i \in \mathcal{P}$ are even, and $\mathcal{Q}$ has odd as well as even elements. Then, (1.2) can be written in the form

$$x_n = 1 + \frac{\sum_{i=1}^{k}\alpha_i x_{n-p_i}}{\beta_{j_0}x_{n-q_{j_0}} + \beta_{j_1}x_{n-q_{j_1}} + \sum_{j=1,j\neq j_0,j_1}^{m}\beta_j x_{n-q_j}}, \tag{2.15}$$

where $q_{j_0} = 2s$ and $q_{j_1} = 2t+1$.

From a result in number theory [11], we know that the condition $G = 1$ implies that for each sufficiently large n, say, $n \geq n_0$, there are nonnegative numbers $d_i \in \mathbb{N}_0$, $i \in \{1,\dots,k+m\}$, such that

$$\sum_{i=1}^{k} p_i d_i + \sum_{j=1}^{m} q_j d_{k+j} = n. \tag{2.16}$$

From condition $G = 1$, by using (2.15) and (2.16), and employing the procedure described above for getting formulae (2.13) and (2.14), we obtain that the subsequence $(L_{-i})_{i\geq n_0}$ of the full limiting sequence $(L_i)_{i\in\mathbb{Z}}$ with $L_0 = S$ takes values I and S.

Now we prove that the sequence $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with periods $p_1, p_2, \dots, p_k$ and also with periods $2q_1, \dots, 2q_m$. Indeed, if we replace n in (2.15) by $-n_0 - l$, $l \in \{0, 1, \dots, p_1 - 1\}$, we obtain that $L_{-n_0-l} = L_{-n_0-l-p_1 i}$ for every $i \in \mathbb{N}$ and each $l \in \{0, 1, \dots, p_1 - 1\}$, that is, $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period p_1. Similarly it can be proven that $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with periods $p_2, \dots, p_k$. The periodicity with periods $2q_1, \dots, 2q_m$ can be proven similar to (2.9) and (2.10) and by using induction.

Since all $p_i \in \mathcal{P}$ are even and $G = 1$, we have that

$$2 \le \gcd\left(p_1, p_2, \dots, p_k, 2q_1, \dots, 2q_m\right) = 2\gcd\left(\frac{p_1}{2}, \frac{p_2}{2}, \dots, \frac{p_k}{2}, q_1, \dots, q_m\right) \le 2G = 2, \tag{2.17}$$

that is,

$$\gcd\left(p_1, p_2, \dots, p_k, 2q_1, \dots, 2q_m\right) = 2. \tag{2.18}$$

Hence, the sequence $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period two. Since $(L_i)_{i\in\mathbb{Z}}$ is a solution of (1.2), we obtain that $(L_i)_{i\in\mathbb{Z}}$ is also periodic with period two.

Assume now that

$$\dots, x, y, x, y, x, y, \dots, \tag{2.19}$$

is a two-periodic solution of (2.15). Then we have

$$x = 1 + \frac{x}{cx + (1-c)y}, \qquad y = 1 + \frac{y}{cy + (1-c)x}, \tag{2.20}$$

for some $c \in (0, 1)$. Hence,

$$(c-1)xy = cx^2 - (c+1)x - (1-c)y = cy^2 - (c+1)y - (1-c)x, \tag{2.21}$$

from which it follows that $c(x - y)(x + y - 2) = 0$. If $x + y = 2$ and $x \ne y$, then we have that x and y are different positive solutions of the equation

$$x = 1 + \frac{x}{cx + (1-c)(2-x)}, \tag{2.22}$$

which implies that $(2c - 1)(x - 1)^2 = 1$. Hence, if $c \le 1/2$, then this equation does not have real roots. If $c > 1/2$, then $x = 1 \pm (1/(2c - 1))^{1/2}$ are solutions. However, since $c \in (1/2, 1)$, the number $1 - (1/(2c - 1))^{1/2}$ is negative. Therefore, it follows that $x = y$ as desired.

Assume now that the set $\mathcal{P}$ contains only even elements while $\mathcal{Q}$ contains only odd elements. Then, it is easy to see that (1.2) in this case has infinite prime two-periodic solutions of the form $x, y, x, y, \dots$, such that $xy = x + y$. Similar to (2.18), it can be proven that, in this case, the full limiting sequence $(L_i)_{i\in\mathbb{Z}}$, $L_0 = S$ is periodic with period two and that

$$L_{2i} = S, \quad L_{2i-1} = I, \quad i \in \mathbb{Z}. \tag{2.23}$$

Assume that $\varepsilon, \delta \in (0, S)$ are such that

$$(S-\varepsilon)(I+\delta) = (S-\varepsilon)+(I+\delta). \tag{2.24}$$

Then, for such chosen ε and δ, there is a $k_0 \in \mathbb{Z}$ such that

$$x_{k_0+2j} > S-\varepsilon, \qquad x_{k_0+2j-1} < I+\delta, \tag{2.25}$$

for $j \in \{1,2,\dots,[s/2]+1\}$, where $s = \max\{p_k, q_m\}$.

From (1.2) and (2.25), we have that

$$\begin{aligned} x_{k_0+2[s/2]+3} &< 1+\frac{I+\delta}{S-\varepsilon} = I+\delta, \\ x_{k_0+2[s/2]+4} &> 1+\frac{S-\varepsilon}{I+\delta} = S-\varepsilon. \end{aligned} \tag{2.26}$$

By induction, we obtain

$$x_{k_0+2i+1} < I+\delta, \qquad x_{k_0+2i} > S-\varepsilon, \tag{2.27}$$

for every $i \in \mathbb{N}$. From (2.27) and the fact that $\varepsilon \to 0$ implies $\delta \to 0$, it follows that $\lim_{n\to\infty} x_{2n} = S$ and $\lim_{n\to\infty} x_{2n-1} = I$, or $\lim_{n\to\infty} x_{2n} = I$ and $\lim_{n\to\infty} x_{2n-1} = S$, finishing the proof of the theorem. □

Remark 2.3. Note that the case when all p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are even is excluded from the consideration in Theorem 2.1 since we assume that $G = 1$. However, this case is reduced to the cases considered in Theorem 2.1. Indeed, let 2^s be the highest power of 2 which divides G, then (1.2) can be separated into 2^s different equations of the form

$$x_n^{(t)} = 1+\frac{\sum_{i=1}^{k}\alpha_i x_{n-p_i/2^s}^{(t)}}{\sum_{j=1}^{m}\beta_j x_{n-q_j/2^s}^{(t)}}, \quad n \in \mathbb{N}_0, \tag{2.28}$$

where $t \in \{0,1,\dots,2^s-1\}$. Note that by the definition of 2^s, it follows that at least one of the numbers $p_i/2^s$, $i \in \{1,\dots,k\}$, and $q_j/2^s$, $j \in \{1,\dots,m\}$, is odd. Hence, Theorem 2.1 can be applied to the equations in (2.28).

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