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Research Article

On the Asymptotic Behavior of a Difference Equation with Maximum

Fangkuan Sun

College of Computer Science, Chongqing University, Chongqing 400044, China

Correspondence should be addressed to Fangkuan Sun, franksun85@163.com

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We study the asymptotic behavior of positive solutions to the difference equation $x_n = \max\{A/x_{n-1}^{\alpha}, B/x_{n-2}^{\beta}\}$, $n = 0, 1, \ldots$, where $0 < \alpha, \beta < 1$, $A, B > 0$. We prove that every positive solution to this equation converges to $x^* = \max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$.

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1. Introduction

Recently, there has been a considerable interest in studying, the so-called, max-type difference equations, see for example, [1–21] and the references cited therein. The max-type operators arise naturally in certain models in automatic control theory (see [9, 11]). The investigation of the difference equation

$$x_n = \max\left\{\frac{A_1}{x_{n-1}}, \frac{A_2}{x_{n-2}}, \ldots, \frac{A_p}{x_{n-p}}\right\}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $p \in \mathbb{N}$, A_i, $i = 1, \ldots, p$, are real numbers such that at least one of them is different from zero and the initial values $x_{-1}, \ldots, x_{-p}$ are different from zero, was proposed in [6]. Some results about (1.1) and its generalizations can be found in [1, 3–5, 7, 8, 10, 12, 17–19] (see also the references therein). The study of max-type equations whose some terms contain nonconstant numerators was initiated by Stević, see for example, [2, 14–16]. For some closely related papers, see also [20, 21].

Motivated by the aforementioned papers and by computer simulations, in this paper we study the asymptotic behavior of positive solutions to the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}^{\beta}}\right\}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $0 < \alpha, \beta < 1$, $A, B > 0$. We prove that every positive solution of this equation converges to $x^* = \max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$.

2. Main results

In this section, we will prove the following result concerning (1.2).

Theorem 2.1. *Let (x_n) be a positive solution to (1.2).*
Then

$$x_n \longrightarrow \max\left\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\right\} \quad \text{as } n \longrightarrow \infty. \tag{2.1}$$

In order to establish Theorem 2.1, we need the following lemma and its corollary which can be found in [13].

Lemma 2.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers which satisfies the inequality*

$$a_{n+k} \le q \ \max\left\{a_{n+k-1}, a_{n+k-2}, \dots, a_n\right\}, \quad \text{for } n \in \mathbb{N}, \tag{2.2}$$

where $q > 0$ and $k \in \mathbb{N}$ are fixed. Then there exist $L \in \mathbb{R}_+$ such that

$$a_{km+r} \le Lq^m \quad \forall m \in \mathbb{N}_0,\ 1 \le r \le k. \tag{2.3}$$

Corollary 2.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers as in Lemma 2.2. Then there exists $M > 0$ such that*

$$a_n \le M\left(\sqrt[k]{q}\right)^n, \quad n \in \mathbb{N}. \tag{2.4}$$

Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We proceed by distinguishing two possible cases.

Case 1 ($A^{1/(\alpha+1)} \ge B^{1/(\beta+1)}$). We prove $x_n \to A^{1/(\alpha+1)}$ as $n \to \infty$.
Set $x_n = y_n A^{1/(\alpha+1)}$, then (1.2) becomes

$$y_n = \max\left\{\frac{1}{y_{n-1}^{\alpha}}, \frac{C}{y_{n-2}^{\beta}}\right\}, \quad n = 0, 1, \dots, \tag{2.5}$$

where $C = B/A^{(\alpha+1)/(\beta+1)}$. Since $A^{1/(\alpha+1)} \ge B^{1/(\beta+1)}$, we have $C \le 1$. To prove $x_n \to A^{1/(\alpha+1)}$ as $n \to \infty$, it suffices to prove $y_n \to 1$ as $n \to \infty$.
We proceed by two cases: $C = 1$ and $0 < C < 1$.

Case $C = 1$. In this case (2.5) is reduced to

$$y_n = \max\left\{\frac{1}{y_{n-1}^{\alpha}}, \frac{1}{y_{n-2}^{\beta}}\right\}, \quad n = 0, 1, \dots, \tag{2.6}$$

where $0 < \alpha$, $\beta < 1$. Choose a number D so that $0 < D < 1$. Let $y_n = D^{z_n}$, $n \geq -2$. Then, (z_n) is a solution to the difference equation

$$z_n = \min\{-\alpha z_{n-1}, -\beta z_{n-2}\}, \quad n = 0, 1, \ldots. \tag{2.7}$$

To prove $y_n \to 1$ as $n \to \infty$, it suffices to prove $z_n \to 0$ as $n \to \infty$.

It can be easily proved that there is a positive integer N such that for all $n \geq 0$,

$$z_{3n+N} \geq 0, \qquad z_{3n+N+1} \leq 0, \qquad z_{3n+N+2} \leq 0. \tag{2.8}$$

By simple computation, we get that, for all $n \geq 0$,

$$z_{3n+N+2} = \min\{-\alpha z_{3n+N+1}, -\beta z_{3n+N}\} = -\beta z_{3n+N}, \tag{2.9}$$

$$0 \leq z_{3n+N+3} = \min\{-\alpha z_{3n+N+2}, -\beta z_{3n+N+1}\} = \min\{\alpha\beta z_{3n+N}, -\beta z_{3n+N+1}\} \leq \alpha\beta z_{3n+N}, \tag{2.10}$$

$$z_{3n+N+4} = \min\{-\alpha z_{3n+N+3}, -\beta z_{3n+N+2}\} = -\alpha z_{3n+N+3}. \tag{2.11}$$

Since $0 < \alpha\beta < 1$, (2.10) implies $z_{3n+N} \to 0$ as $n \to \infty$. From (2.9) and (2.11), it follows that $z_{3n+N+1} \to 0$, $z_{3n+N+2} \to 0$ as $n \to \infty$. This implies $z_n \to 0$.

Case $0 < C < 1$. Let $y_n = C^{z_n}$, then (z_n) is a solution to the difference equation

$$z_n = \min\{-\alpha z_{n-1}, 1 - \beta z_{n-2}\}, \quad n = 0, 1, \ldots. \tag{2.12}$$

To prove $y_n \to 1$ as $n \to \infty$, it suffices to prove $z_n \to 0$ as $n \to \infty$. If $z_{-1} = 0$, $z_{-2} = 0$, then we have $z_n = 0$ for all $n \geq -2$. Next, we assume either $z_{-1} \neq 0$ or $z_{-2} \neq 0$. Then the following four claims are obviously true.

Claim 1. If $z_{n-1} \geq 0$ and $z_{n-2} \geq 0$ for some n, then

$$|z_n| \leq \max\{\alpha|z_{n-1}|, \beta|z_{n-2}| - 1\}. \tag{2.13}$$

Claim 2. If $z_{n-1} \leq 0$ and $z_{n-2} \leq 0$ for some n, then $|z_n| \leq \alpha|z_{n-1}|$.

Claim 3. If $z_{n-1} \geq 0$ and $z_{n-2} \leq 0$ for some n, then $|z_n| = \alpha|z_{n-1}|$.

Claim 4. If $z_{n-1} \leq 0$ and $z_{n-2} \geq 0$ for some n, then

$$|z_n| \leq \max\{\alpha|z_{n-1}|, \beta|z_{n-2}| - 1\}. \tag{2.14}$$

In general, we have

$$|z_n| \leq \max\{\alpha|z_{n-1}|, \beta|z_{n-2}| - 1\} \leq \max\{\alpha|z_{n-1}|, \beta|z_{n-2}|\} \leq \gamma \max\{|z_{n-1}|, |z_{n-2}|\}, \tag{2.15}$$

where $0 < \gamma = \max\{\alpha, \beta\} < 1$. From (2.15) and Corollary 2.3, there exists $M > 0$ such that

$$|z_n| \leq M(\sqrt{\gamma})^n. \tag{2.16}$$

This implies $z_n \to 0$ as $n \to \infty$.

Case 2 ($A^{1/(\alpha+1)} < B^{1/(\beta+1)}$). We prove $x_n \to B^{1/(\beta+1)}$ as $n \to \infty$.

Similar to the proof of Case 1, we set $x_n = y_n B^{1/(\beta+1)}$, then (1.2) becomes

$$y_n = \max\left\{\frac{C}{y_{n-1}^{\alpha}}, \frac{1}{y_{n-2}^{\beta}}\right\}, \quad n = 0, 1, \dots, \tag{2.17}$$

where $C = A/B^{(\alpha+1)/(\beta+1)} < 1$. To prove $x_n \to B^{1/(\beta+1)}$ as $n \to \infty$, it suffices to prove $y_n \to 1$ as $n \to \infty$. Let $y_n = C^{z_n}$, then (z_n) is a solution to the difference equation

$$z_n = \min\{1 - \alpha z_{n-1}, -\beta z_{n-2}\}, \quad n = 0, 1, \dots. \tag{2.18}$$

To prove $y_n \to 1$ as $n \to \infty$, it suffices to prove $z_n \to 0$ as $n \to \infty$. If $z_{-1} = 0$, $z_{-2} = 0$, then we have $z_n = 0$ for all $n \geq -2$. Next, we assume either $z_{-1} \neq 0$ or $z_{-2} \neq 0$, then the following four claims are obviously true.

Claim 1. If $z_{n-1} \geq 0$ and $z_{n-2} \geq 0$ for some n, then

$$|z_n| \leq \max\{\alpha|z_{n-1}| - 1, \beta|z_{n-2}|\}. \tag{2.19}$$

Claim 2. If $z_{n-1} \leq 0$ and $z_{n-2} \leq 0$ for some n, then $|z_n| \leq \beta|z_{n-2}|$.

Claim 3. If $z_{n-1} \geq 0$ and $z_{n-2} \leq 0$ for some n, then

$$|z_n| \leq \max\{\alpha|z_{n-1}| - 1, \beta|z_{n-2}|\}. \tag{2.20}$$

Claim 4. If $z_{n-1} \leq 0$ and $z_{n-2} \geq 0$ for some n, then $|z_n| = \beta|z_{n-2}|$.

In general, we have

$$|z_n| \leq \max\{\alpha|z_{n-1}| - 1, \beta|z_{n-2}|\} \leq \max\{\alpha|z_{n-1}|, \beta|z_{n-2}|\} \leq \gamma \max\{|z_{n-1}|, |z_{n-2}|\}, \tag{2.21}$$

where $0 < \gamma = \max\{\alpha, \beta\} < 1$. Then the rest of the proof is similar to the proof of Case 1 and will be omitted. The proof is complete. □

Theorem 2.4. *Every solution to the difference equation* $x_n = A/x_{n-m}^{\alpha}$, $0 < \alpha < 1$, $A > 0$ *converges to* $x^* = A^{1/(\alpha+1)}$.

Proof. Let $x_n = y_n A^{1/(\alpha+1)}$, then the equation becomes

$$y_n = \frac{1}{y_{n-m}^{\alpha}} = y_{n-2m}^{\alpha^2} = y_{n-4m}^{\alpha^4} = \cdots = y_{n-2[n/2m]m}^{\alpha^{2[n/2m]}}. \tag{2.22}$$

From this and the condition $0 < \alpha < 1$, it follows that $y_n \to 1$ as $n \to \infty$ which implies $x_n \to A^{1/(\alpha+1)}$ as $n \to \infty$. □

3. Conclusions and remarks

This paper examines the asymptotic behavior of positive solutions to the difference equation (1.2) with $0 < \alpha,\ \beta < 1,\ A, B > 0$. The method used in this work may provide insight into the asymptotic behavior of positive solutions to the generic difference equation

$$x_n = \max\left\{\frac{A_1}{x_{n-1}^{\alpha_1}}, \frac{A_2}{x_{n-2}^{\alpha_2}}, \dots, \frac{A_p}{x_{n-p}^{\alpha_p}}\right\}, \quad n = 0, 1, \dots, \tag{3.1}$$

where $0 < \alpha_i < 1,\ A_i > 0,\ i = 1, \dots, p$. We close this work by proposing the following conjecture.

Conjecture 3.1. *Assume that* (x_n) *is a positive solution to* (3.1). *Then* $x_n \to \max_{1\le i\le p}\{A_i^{1/(\alpha_i+1)}\}$ *as* $n \to \infty$.

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