Research Article

A Sufficient Condition for Asymptotic Stability of Discrete-Time Interval System with Delay

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The asymptotic stability of discrete-time interval system with delay is discussed. A new sufficient condition for preserving the asymptotic stability of the system is presented by means of the inequality techniques. By mathematical analysis, the stability criterion is less conservative than that in previous result. Finally, one example is given to demonstrate the applicability of the present scheme.

1. Introduction

The stability analysis of interval system is very useful for the robustness analysis of nominally stable system subject to model perturbations. Therefore, there has been considerable interest in the stability analysis of interval system in literature ([1–15], and references therein). In general, those approaches can be classified into two categories: the first is the polynomial and the second is the matrix approach. However, due to information transmission between elements or systems, data computation, natural property of system elements, and so forth, time delays also inherently exist in controlled systems and therefore must be integrated into system models. The stability analysis for interval systems with delays becomes more complicated. In [6], a sufficient condition for the stability of discrete-time systems is given in terms of pulse-response sequence matrix. In [11], based on the Gersgorin theorem, the stability testing problem for continuous and discrete systems including a time delay is discussed.

The objective of this paper is to deal with the asymptotic stability of a discrete-time interval system with delay. Based on the inequality techniques [16], a new sufficient condition for preserving the asymptotic stability of the system is presented. By mathematical analysis, the stability criterion is less conservative than that in previous result. An example is given to compare the proposed method with one reported.

2. System description and notations

Consider the discrete-time interval system with delay described by

$$x(k+1) = A_I x(k) + B_I x(k-p), \quad k = 0, 1, 2, \ldots,$$
$$x(k) = \varphi(k), \quad k = -p, \ldots, 0, \tag{2.1}$$

where delay p is a positive integer, $x(k) = \mathrm{col}\{x_1(k), x_2(k), \ldots x_n(k)\} \in R^n$, A_I, B_I are the interval matrices described as $A_I \triangleq [\underline{A}, \overline{A}] = \{A = (a_{ij}) : a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]\}$, $B_I \triangleq [\underline{B}, \overline{B}] = \{B = (b_{ij}) : b_{ij} \in [\underline{b}_{ij}, \overline{b}_{ij}]\}$, $i, j = 1, 2, \ldots, n$, $\varphi(k), k = -p, \ldots, 0$, are bounded.

In the sequel, the following notations will be used: $R^n(R^n_+)$: the space of n-dimensional (nonnegative) real column vectors; $R^{n \times m}(R^{n \times m}_+)$: the set of $n \times m$ (nonnegative) real matrices; $\rho(A)$: the spectral radius of matrix $A \in R^{n \times n}$; $A \geq B(A > B)$: each pair of corresponding elements of A and B satisfies the inequality "$\geq (>)$," where $A, B \in R^{n \times m}$ or $A, B \in R^n$; $W_\rho(A)$: for $A \in R^{n \times n}_+$, $W_\rho(A) \triangleq \{z \in R^n_+ \mid Az = \rho(A)z\}$; $[\cdot]^+$: the vector (or matrix) obtained by replacing each entry of $[\cdot]$ by its absolute value; $Z^{[a,b]}$: $Z^{[a,b]} \equiv \{a, a+1, \ldots, b\}$, where a, b are nonnegative integers; if $a > b$, we define $Z^{[a,b]} = \varnothing$, where $\varnothing$ is the empty set; if $b = \infty$, we write $Z^{[a,b]}$ as $Z^{[a]}$; $\mathcal{A}_r(\mathcal{B}_r)$: the set of matrices obtained by exchanging corresponding r column (s) of $\underline{A}$ ($\underline{B}$) and $\overline{A}$ ($\overline{B}$), so there are $2 \times \binom{n}{r}$ matrices in each $\mathcal{A}_r(\mathcal{B}_r)$, where $r \in Z^{[0,n]}$, $\binom{n}{r} = n!/r!(n-r)!$ and $n!$ denotes the factorial of n.

3. Main result

In order to prove our main result, we first need the following technical lemmas.

Lemma 3.1 (see [17, Theorem 8.3.1]). *If $A \in R^{n \times n}_+$, then there is a nonnegative vector $z \geq 0$, $z \neq 0$, such that $Az = \rho(A)z$.*

So it is clear that $W_\rho(A)$ is not empty by Lemma 3.1.

Lemma 3.2. *Let $P, Q \in R^{n \times n}_+$, $u(k) \in R^n_+$ satisfy that*

$$u(k+1) \leq Pu(k) + Qu(k-p), \quad k \in Z^{[0]}. \tag{3.1}$$

If

$$\rho(P+Q) < 1, \tag{3.2}$$

then there exists a constant $\lambda > 0$ such that

$$u(k) \leq ze^{-\lambda k}, \quad k \in Z^{[0]}, \tag{3.3}$$

for some $z = (z_1, z_2, \ldots, z_n)^T \in W_\rho(P + Qe^{\lambda p})$.

Proof. Since $\rho(P+Q) < 1$, using continuity, there must be a sufficiently small constant $\lambda > 0$ such that

$$e^{\lambda(p+1)}\rho(P + Qe^{\lambda p}) \leq 1. \tag{3.4}$$

Let

$$
\begin{aligned}
y(k) &= u(k)e^{\lambda k}, \quad k \in Z^{[0]}, \\
y(k) &= u(k), \quad k \in Z^{[-p,0]},
\end{aligned}
\tag{3.5}
$$

so we have

$$
u(k) = y(k)e^{-\lambda k} \le y(k)e^{-\lambda(k-p)}, \quad k \in Z^{[0]}.
\tag{3.6}
$$

By (3.1), we have

$$
y(k+1) = u(k+1)e^{\lambda(k+1)} \le \big(Pu(k) + Qu(k-p)\big)e^{\lambda(k+1)}, \quad k \in Z^{[0]}.
\tag{3.7}
$$

Since $P, Q \in R_+^{n \times n}$, we derive that

$$
y(k+1) \le \big(Py(k) + Qy(k-p)e^{\lambda p}\big)e^{\lambda(p+1)}, \quad k \in Z^{[0]}.
\tag{3.8}
$$

We next show that for any $k \in Z^{[0]}$,

$$
y(k) \le z.
\tag{3.9}
$$

If this is not true, then there must be a positive constant $l > 0$ and some integer m such that

$$
y_m(l+1) > z_m, \qquad y(k) \le z \quad \text{for } k \in Z^{[-p,l]}.
\tag{3.10}
$$

By using (3.4) and (3.8), we obtain that

$$
y(l+1) \le e^{\lambda(p+1)}\big(P + Qe^{\lambda p}\big)z = e^{\lambda(p+1)}\rho\big(P + Qe^{\lambda p}\big)z \le z
\tag{3.11}
$$

which contradicts the first inequality of (3.10). Thus (3.9) holds for all $k \in Z^{[0]}$. Therefore, we have

$$
u(k) \le ze^{-\lambda k}, \quad k \in Z^{[0]},
\tag{3.12}
$$

and the proof is completed. □

Theorem 3.3. *For any $C \in \mathcal{A} \triangleq \bigcup_{r=0}^{n}\mathcal{A}_r$, $D \in \mathcal{B} \triangleq \bigcup_{r=0}^{n}\mathcal{B}_r$, if the inequality*

$$
\rho\big([C]^+ + [D]^+\big) < 1
\tag{3.13}
$$

holds, then the discrete-time interval system (2.1) is asymptotically stable.

Proof. Let $\Gamma_k = \{i \mid x_i(k) < 0\} \equiv \{i_1, i_2, \ldots, i_m\}$, $\Lambda_k = \{j \mid x_j(k-p) < 0\} \equiv \{j_1, j_2, \ldots, j_l\}$, where m, l satisfying $0 \le m \le n$, $0 \le l \le n$, $m + l = n$ are integers and $m = 0$ or $l = 0$ is equivalent to that Γ_k or Λ_k is empty, respectively. Obviously, $\bigcup_{k=1}^{\infty}\Gamma_k$ and $\bigcup_{k=1}^{\infty}\Lambda_k$ are finite sets.

By the definitions of $\mathcal{A}$ and $\mathcal{B}$, we can obtain matrices $A_{1k}, A_{2k} \in \mathcal{A}$ by exchanging the corresponding i_1th, i_2th, ..., i_mth columns of $\underline{A}$ and $\overline{A}$ (if $m = 0$, then $A_{1k} = \underline{A}$, $A_{2k} = \overline{A}$), and

matrices $B_{1k}, B_{2k} \in \mathcal{B}$ by exchanging the corresponding j_1th, j_2th, ..., j_lth columns of $\underline{B}$ and $\overline{B}$ (if $l = 0$, then $B_{1k} = \underline{B}, B_{2k} = \overline{B}$) such that the following inequalities hold:

$$
\begin{aligned}
A_{1k}x(k) \leq Ax(k) \leq A_{2k}x(k), \quad A \in A_I, \\
B_{1k}x(k-p) \leq Bx(k-p) \leq B_{2k}x(k-p), \quad B \in B_I.
\end{aligned}
\tag{3.14}
$$

So together with (2.1), we have

$$
A_{1k}x(k) + B_{1k}x(k-p) \leq x(k+1) \leq A_{2k}x(k) + B_{2k}x(k-p).
\tag{3.15}
$$

From the above, we see that A_{1k}, A_{2k} and B_{1k}, B_{2k} depend only on the position of the negative components of $x(k)$ and $x(k-p)$, respectively.

Then, from (3.15), we derive

$$
\begin{aligned}
\left[x(k+1)\right]^+ &\leq \max\left\{ \left[A_{1k}x(k) + B_{1k}x(k-p)\right]^+, \left[A_{2k}x(k) + B_{2k}x(k-p)\right]^+\right\} \\
&\leq \max\left\{ \left[A_{1k}\right]^+\left[x(k)\right]^+ + \left[B_{1k}\right]^+\left[x(k-p)\right]^+, \left[A_{2k}\right]^+\left[x(k)\right]^+ + \left[B_{2k}\right]^+\left[x(k-p)\right]^+\right\}.
\end{aligned}
\tag{3.16}
$$

So we have

$$
\left[x(k+1)\right]^+ \leq \left[A_{1k}\right]^+\left[x(k)\right]^+ + \left[B_{1k}\right]^+\left[x(k-p)\right]^+
\tag{3.17}
$$

or

$$
\left[x(k+1)\right]^+ \leq \left[A_{2k}\right]^+\left[x(k)\right]^+ + \left[B_{2k}\right]^+\left[x(k-p)\right]^+.
\tag{3.18}
$$

Since $A_{1k}, A_{2k} \in \mathcal{A}$, $B_{1k}, B_{2k} \in \mathcal{B}$, by the definitions of $\mathcal{A}$ and $\mathcal{B}$ again, (3.17) and (3.18), for any $k \in Z^{[0]}$, we can find corresponding matrices $A_i \in \mathcal{A}, B_j \in \mathcal{B}$, $i, j \in \{1, 2, \ldots, n\}$, such that

$$
\left[x(k+1)\right]^+ \leq \left[A_i\right]^+\left[x(k)\right]^+ + \left[B_j\right]^+\left[x(k-p)\right]^+.
\tag{3.19}
$$

In view of condition (3.13), we obtain that

$$
\rho\left(\left[A_i\right]^+ + \left[B_j\right]^+\right) < 1, \quad i, j \in \{1, 2, \ldots, n\}.
\tag{3.20}
$$

Thus, by Lemma 3.2 and (3.19), (3.20), for any $k \in Z^{[0]}$, there exist constants $\lambda_{ij} > 0$ and some $z_{ij} \in W_\rho(\left[A_i\right]^+ + \left[B_j\right]^+ e^{\lambda_{ij}p})$, $i, j \in \{1, 2, \ldots, n\}$, such that

$$
x(k) \leq z_{ij}e^{-\lambda_{ij}k}.
\tag{3.21}
$$

Set $\lambda = \min_{1 \leq i, j \leq n}\{\lambda_{ij}\}$, $z_{ij} = \{z_{ij}^{(1)}, z_{ij}^{(2)}, \ldots, z_{ij}^{(n)}\}^T$, $z^{(h)} = \max_{1 \leq i, j \leq n}\{z_{ij}^{(h)}\}$, $h = 1, 2, \ldots, n$, $z = \{z^{(1)}, z^{(2)}, \ldots, z^{(n)}\}^T$, obviously, λ and z are independent of any choice of k, so by (3.21), we derive that

$$
x(k) \leq ze^{-\lambda k}, \quad k \in Z^{[0]},
\tag{3.22}
$$

which implies that the conclusion of the theorem holds. $\square$

Remark 3.4. By the meanings of $\mathscr{A}_r$ and $\mathscr{B}_r$, we know that $\mathscr{A}_r = \mathscr{A}_{n-r}$, $\mathscr{B}_r = \mathscr{B}_{n-r}$, $r \in Z^{[0,n]}$. So there are $(\sum_{r=0}^{n} 2 \times \binom{n}{r})/2 = 2^n$ matrices in the sets $\mathscr{A}$ and $\mathscr{B}$, respectively. Furthermore, if number n is even, that is, $n = 2k$, $k = 1, 2, \ldots$, then the equality $A_r = A_{n-r}$ $(B_r = B_{n-r})$ for $r = k$ is transformed to be $A_k = A_k$ $(B_k = B_k)$ and then $\mathscr{A}_k$ $(\mathscr{B}_k)$ contains only $\binom{2k}{k}$ different matrices. Therefore, condition (3.13) can be verified easily and quickly by computer software (such as MATLAB).

Corollary 3.5 (see [11, Theorem IV]). *The discrete-time interval system* (2.1) *is asymptotically stable if the following condition is satisfied:*

$$\rho(K + F) < 1, \tag{3.23}$$

where matrices K and F are defined as $K = (k_{ij})$, $k_{ij} = \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$, $F = (f_{ij})$, $f_{ij} = \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$, $i, j = 1, 2, \ldots, n$.

Proof. Clearly, for any $C \in \mathscr{A} = \bigcup_{i=1}^{n} \mathscr{A}_i$, $D \in \mathscr{B} = \bigcup_{i=1}^{n} \mathscr{B}_i$, the inequality

$$[C]^+ + [D]^+ \le K + F \tag{3.24}$$

holds, then from [18] (i.e., for $A, B \in R_+^{n \times n}$, if $A \le B$, then $\rho(A) \le \rho(B)$) and associated with (3.23), we have

$$\rho([C]^+ + [D]^+) \le \rho(K + F) < 1. \tag{3.25}$$

Therefore, system (2.1) is asymptotically stable in terms of **Theorem 3.3.** □

4. Illustrative examples

Example 4.1. Consider the discrete-time interval system (2.1) with delay and

$$\underline{A} = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{1}{4} \\ 0 & \dfrac{1}{4} \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} -\dfrac{1}{3} & 0 \\ \dfrac{1}{4} & \dfrac{1}{2} \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} -\dfrac{1}{4} & -\dfrac{1}{6} \\ -\dfrac{1}{4} & -\dfrac{1}{8} \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} -\dfrac{1}{5} & 0 \\ 0 & \dfrac{1}{10} \end{pmatrix}. \tag{4.1}$$

For this case, $\mathscr{A}_1 = \{A_1 = \begin{pmatrix} -1/3 & -1/4 \\ 1/4 & 1/4 \end{pmatrix}, A_2 = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\}$,

$$\mathscr{A}_2 = \left\{ A_3 = \begin{pmatrix} -\dfrac{1}{3} & 0 \\ \dfrac{1}{4} & \dfrac{1}{2} \end{pmatrix}, A_4 = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{1}{4} \\ 0 & \dfrac{1}{4} \end{pmatrix} \right\},$$

$$\mathscr{B}_1 = \left\{ B_1 = \begin{pmatrix} -\dfrac{1}{5} & -\dfrac{1}{6} \\ 0 & -\dfrac{1}{8} \end{pmatrix}, B_2 = \begin{pmatrix} -\dfrac{1}{4} & 0 \\ -\dfrac{1}{4} & \dfrac{1}{10} \end{pmatrix} \right\}, \tag{4.2}$$

$$\mathscr{B}_2 = \left\{ B_3 = \begin{pmatrix} -\dfrac{1}{5} & 0 \\ 0 & \dfrac{1}{10} \end{pmatrix}, B_4 = \begin{pmatrix} -\dfrac{1}{4} & -\dfrac{1}{6} \\ -\dfrac{1}{4} & -\dfrac{1}{8} \end{pmatrix} \right\}.$$
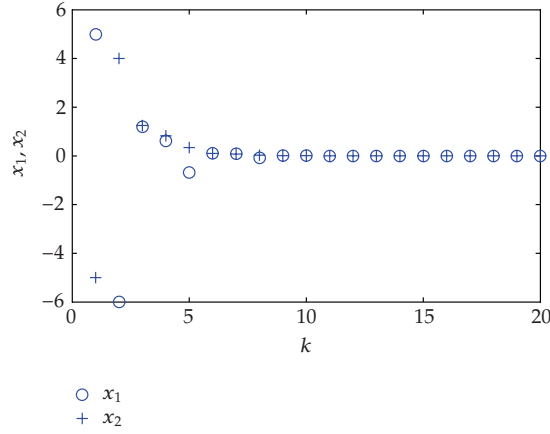
Figure 1: Stability for discrete-time interval system.

By simple calculation, we have $\rho([A_i]^+ + [B_j]^+) \leq \rho([A_1]^+ + [B_4]^+)$ for $i, j = 1, 2, 3, 4$, and

$$[A_1]^+ + [B_4]^+ = \begin{pmatrix} \dfrac{1}{3} & \dfrac{1}{4} \\ \dfrac{1}{4} & \dfrac{1}{4} \end{pmatrix} + \begin{pmatrix} \dfrac{1}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{1}{8} \end{pmatrix} = \begin{pmatrix} \dfrac{7}{12} & \dfrac{5}{12} \\ \dfrac{1}{2} & \dfrac{3}{8} \end{pmatrix}. \tag{4.3}$$

So we have

$$\rho\big([A_i]^+ + [B_j]^+\big) \leq \rho\big([A_1]^+ + [B_4]^+\big) = 0.9473 < 1, \quad i, j = 1, 2, 3, 4. \tag{4.4}$$

Therefore, the system (2.1) is asymptotically stable by means of Theorem 3.3.
In what follows, the simulation result is illustrated in Figure 1.

Remark 4.2. If [11, Theorem IV] is applied to Example 4.1, we obtain

$$K = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{4} \\ \dfrac{1}{4} & \dfrac{1}{2} \end{pmatrix}, \qquad F = \begin{pmatrix} \dfrac{1}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{1}{8} \end{pmatrix}, \tag{4.5}$$

where K, F are defined by Corollary 3.5, that is, [11, Theorem IV]. Then

$$\rho(K + F) = 1.1482 > 1, \tag{4.6}$$

that is, [11, Theorem IV] cannot be applied. So the sufficient condition (3.13) proposed in this paper is less conservative than condition (3.23) proposed by [11].

5. Conclusion

In this paper, we have investigated the asymptotic stability of discrete-time interval system with delay. A new sufficient condition for preserving the asymptotic stability of the system is

developed. By mathematical analysis, the presented criterion is to be less conservative than that proposed by [11]. So, the result of this paper indeed allows us to have more freedom for checking the stability of the discrete-time interval systems with delay. From the proposed example, it is easily seen that the criterion presented in this paper for the stability of the discrete-time interval system with delay is very helpful. We believe that the present scheme is applicable to robust control design.

Acknowledgments

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