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Research Article

On the Difference Equation $x_{n+1} = \alpha + x_{n-m}/x_n^k$

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We investigate the global behaviour of the difference equation of higher order $x_{n+1} = \alpha + x_{n-m}/x_n^k$, $n = 0, 1, \ldots$, where the parameters $\alpha, k \in (0, \infty)$ and the initial values $x_{-m}, x_{-(m-1)}, \ldots, x_{-2}, x_{-1}$, and x_0 are arbitrary positive real numbers.

1. Introduction

Although difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the global behavior of their solutions. See, for example, [1–12] and the relevant references cited therein. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in various scientific branches, such as in ecology, economy, physics, technics, sociology, and biology. Hamza and Morsy in [5] investigated the global behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values x_{-1} and x_0 are arbitrary positive real numbers.

Equation (1.1) was investigated when $k = 1$ where $\alpha \in (0, \infty)$ (see [1, 3]). There are some other examples of the research regarding (1.1) (e.g., [4, 8]).

Yalçinkaya in [11] investigated the global behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-2}}{x_n^k}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values are arbitrary positive real numbers.

Also, in [12], we investigated the global behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-3}}{x_n^k}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values are arbitrary positive real numbers.

In this paper, we consider the following difference equation of higher order

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where the parameters $\alpha, k \in (0, \infty)$ and the initial values are arbitrary positive real numbers.

Here, we review some results which will be useful in our investigation of the behavior of (1.4) solutions (cf. [10]).

Definition 1.1. Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function where k is a nonnegative integer. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.5}$$

with the initial values $x_{-k}, \ldots, x_0 \in I$. A point $\overline{x}$ is called an *equilibrium point* of (1.5) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{1.6}$$

Definition 1.2. Let $\overline{x}$ be an equilibrium point of (1.5).

(a) The equilibrium $\overline{x}$ is called *locally stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0, \ldots, x_{-k} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-k} - \overline{x}| < \delta$, then

$$|x_n - \overline{x}| < \varepsilon, \quad \forall n \geq -k. \tag{1.7}$$

(b) The equilibrium $\overline{x}$ is called *locally asymptotically stable* if it is locally stable and if there exists $\gamma > 0$ such that if $x_0, \ldots, x_{-k} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-k} - \overline{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{1.8}$$

(c) The equilibrium $\overline{x}$ is called *global attractor* if for every $x_0, \ldots, x_{-k} \in I$,

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{1.9}$$

(d) The equilibrium $\overline{x}$ is called *globally asymptotically stable* if it is locally stable and is a global attractor.

(e) The equilibrium $\overline{x}$ is called *unstable* if is not stable.

Definition 1.3. Let $a_i = (\partial f/\partial u_i)(\overline{x}, \dots, \overline{x})$ for each $i = 0, 1, \dots, k$ denote the partial derivatives of $f(u_0, u_1, \dots, u_k)$ evaluated at an equilibrium $\overline{x}$ of (1.5). Then

$$z_{n+1} = a_0 z_n + a_1 z_{n-1} + \cdots + a_k z_{n-k}, \quad n = 0, 1, \dots \tag{1.10}$$

is called the *linearized equation* of (1.5) about the equilibrium point $\overline{x}$.

Theorem 1.4 (Clark's theorem). *Consider the difference equation* (1.10). *Then*

$$\sum_{i=0}^{k} |a_i| < 1 \tag{1.11}$$

is a sufficient condition for the locally asymptotically stability of (1.5).

Definition 1.5. The sequence $\{x_n\}$ is said to be *periodic* with period p if $x_{n+p} = x_n$ for $n = 0, 1, \dots$ (cf. [2]).

2. Main results

In this section, we investigate the global behavior, the boundedness, and some periodicity of (1.4).

A point $\overline{x} \in \mathbb{R}$ is an equilibrium point of (1.4) if and only if it is a zero for the function

$$g(x) = x - x^{1-k} - \alpha, \tag{2.1}$$

that is,

$$\overline{x} - \overline{x}^{1-k} - \alpha = 0. \tag{2.2}$$

Lemma 2.1. *Equation* (1.4) *has a unique equilibrium point* $\overline{x} > 1$.

Proof

Case 1. Assume that $k = 1$, then (1.4) has a unique equilibrium point $\overline{x} = \alpha + 1 > 1$.

Case 2. Assume that $0 < k < 1$. The function g defined by (2.1) is decreasing on $[0, (1-k)^{1/k}]$ and increasing on $[(1-k)^{1/k}, \infty)$. Since $g(1) = -\alpha$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique zero $\overline{x} > 1$.

Case 3. Assume that $1 < k$. Since g is increasing on $[0, \infty)$, $g(1) = -\alpha$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique zero $\overline{x} > 1$.

Therefore, the proof is complete. □

Theorem 2.2. *Assume that* $\overline{x}$ *is the equilibrium point of* (1.4). *If* $k(k+1)^{(1-k)/k} < \alpha$, *then* $\overline{x}$ *is locally asymptotically stable.*

Proof. From (1.5) and (1.10), we see that

$$f(u_0, u_1, \dots, u_m) = \alpha + u_0^{-k} u_m, \tag{2.3}$$

then

$$a_0 = \frac{-k}{\overline{x}^k}, \qquad a_i = 0 \quad \forall i \in \{1, 2, \dots, m-1\}, \quad a_m = \frac{1}{\overline{x}^k}. \tag{2.4}$$

By using Clark's theorem, we get that $\overline{x}$ is locally asymptotically stable if and only if $\overline{x}^k > k+1$.

Let $k(k+1)^{(1-k)/k} < \alpha$, a simple calculations shows that

$$g\left((k+1)^{1/k}\right) = k(k+1)^{(1-k)/k} - \alpha < 0, \tag{2.5}$$

where g is defined by (2.1). Then, since $\lim_{x\to\infty} g(x) = \infty$, $\overline{x} > (k+1)^{1/k}$ and $\overline{x}^k > k+1$. Therefore, the proof is complete. □

Lemma 2.3. *If $\alpha \neq 1$, then every solution of* (1.4) *is bounded.*

Proof. We get that

$$\alpha < x_{n+1} < \alpha + \beta x_{n-m}, \quad \forall n = 1, 2, \dots, \tag{2.6}$$

where $\beta = 1/\alpha^k$.

By induction we obtain

$$\alpha < x_{(m+1)n+p} < \alpha \frac{1-\beta^n}{1-\beta} + \beta^n x_p, \quad \forall p \in \{-(m-1), -(m-2), \dots, -1, 0, 1\}. \tag{2.7}$$

Also, we see that if $\alpha > 1$,

$$\alpha < x_{(m+1)n+p} < \alpha \frac{1}{1-\beta} + x_p, \quad \forall p \in \{-(m-1), (-m-2), \dots, -1, 0, 1\}. \tag{2.8}$$

Therefore, the proof is complete. □

Theorem 2.4. *Assume that $\overline{x}$ is the equilibrium point of* (1.4). *If $\alpha > k^{1/k} \geq 1$, then $\overline{x}$ is globally asymptotically stable.*

Proof. We must show that the equilibrium point $\overline{x}$ of (1.4) is both locally asymptotically stable and $\lim_{x\to\infty} x_n = \overline{x}$.

Firstly, since $k \geq 1$, then $k \geq k(k+1)^{(1-k)/k}$ and since $\alpha > k^{1/k}$, we get $\alpha > k(k+1)^{(1-k)/k}$. By Theorem 2.2, $\overline{x}$ is locally asymptotically stable.

Let $\{x_n\}_{n=-m}^{\infty}$ be a solution of (1.4). By Lemma 2.3, $\{x_n\}_{n=-m}^{\infty}$ is bounded.

Let us introduce

$$\Lambda_1 = \lim_{n\to\infty} \inf x_n, \qquad \Lambda_2 = \lim_{n\to\infty} \sup x_n. \tag{2.9}$$

Then, for all $\varepsilon \in (0, \Lambda_1)$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we get

$$\Lambda_1 - \varepsilon \leq x_n \leq \Lambda_2 + \varepsilon. \tag{2.10}$$

This implies that

$$\alpha + \frac{\Lambda_1 - \varepsilon}{(\Lambda_2 + \varepsilon)^k} \leq x_{n+1} \leq \alpha + \frac{\Lambda_2 + \varepsilon}{(\Lambda_1 - \varepsilon)^k} \quad \text{for } n \geq n_0 + 1. \tag{2.11}$$

Then, we obtain

$$\alpha + \frac{\Lambda_1 - \varepsilon}{(\Lambda_2 + \varepsilon)^k} \leq \Lambda_1 \leq \Lambda_2 \leq \alpha + \frac{\Lambda_2 + \varepsilon}{(\Lambda_1 - \varepsilon)^k}, \tag{2.12}$$

and from the above inequality

$$\alpha + \frac{\Lambda_1}{{\Lambda_2}^k} \leq \Lambda_1 \leq \Lambda_2 \leq \alpha + \frac{\Lambda_2}{{\Lambda_1}^k}, \tag{2.13}$$

which implies that

$$(\alpha {\Lambda_2}^k {\Lambda_1}^{k-1} + {\Lambda_1}^k) \leq \Lambda_1^k \Lambda_2^k \leq (\alpha {\Lambda_2}^{k-1} {\Lambda_1}^k + {\Lambda_2}^k). \tag{2.14}$$

Consequently, we obtain

$$\alpha {\Lambda_2}^{k-1} {\Lambda_1}^{k-1} (\Lambda_2 - \Lambda_1) \leq (\Lambda_2^k - \Lambda_1^k). \tag{2.15}$$

Suppose that $\Lambda_1 \neq \Lambda_2$, we get that

$$\alpha {\Lambda_2}^{k-1} {\Lambda_1}^{k-1} \leq \frac{\Lambda_2^k - \Lambda_1^k}{\Lambda_2 - \Lambda_1}. \tag{2.16}$$

There exists $\gamma \in (\Lambda_1, \Lambda_2)$ such that

$$\frac{\Lambda_2^k - \Lambda_1^k}{\Lambda_2 - \Lambda_1} = k\gamma^{k-1} \leq k\Lambda_2^{k-1}. \tag{2.17}$$

This implies that $\alpha^k \le k$, which is a contradiction. Hence, $\Lambda_1 = \Lambda_2 = \overline{x}$. So, we have shown that

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{2.18}$$

Therefore, the proof is complete. □

Theorem 2.5. *Suppose that m is odd, then let* $\{x_n\}_{n=-m}^{\infty}$ *be a positive solution of* (1.4) *which consists of at least two semicycles. Then* $\{x_n\}_{n=-m}^{\infty}$ *is oscillatory and, except possibly for the first semicycle, every semicycle is of length one.*

Proof. Assume that $x_{n-2a} < \overline{x} \le x_{n-(2a+1)}$, $\forall a \in \{0,1,2,\dots,(m-1)/2\}$ for some $n \ge 0$, then

$$\begin{aligned} x_{n+1} &> \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}, \\ x_{n+2} &< \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}, \\ x_{n+3} &> \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}, \\ &\vdots \\ x_{n+m+1} &< \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}. \end{aligned} \tag{2.19}$$

Second, consider $x_{n-(2a+1)} < \overline{x} < x_{n-2a}$, $\forall a \in \{0,1,2,\dots,(m-1)/2\}$, then

$$\begin{aligned} x_{n+1} &< \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}, \\ x_{n+2} &> \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}, \\ x_{n+3} &< \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}, \\ &\vdots \\ x_{n+m+1} &> \alpha + \frac{\overline{x}}{\overline{x^k}} = \overline{x}, \end{aligned} \tag{2.20}$$

which ends the proof. □

Theorem 2.6. *Equation* (1.4) *has a period* $(m+1)$ *solution (not necessary prime)* $\{x_n\}_{n=-m}^{\infty}$ *if and only if* $(x_{-m}, x_{-m+1}, \dots, x_{-2}, x_{-1}, x_0)$ *is a solution of the system*

$$a_t = \alpha + \frac{a_t}{a_{t-1}^k}, \quad \forall t \in \{1,2,\dots,m+1\}. \tag{2.21}$$

Moreover, if at least one of the initial values of (1.4) *is different from the others, then* $\{x_n\}_{n=-m}^{\infty}$ *has a prime period* $(m+1)$ *solution.*

Proof. First, assume that $\{x_n\}_{n=-m}^{\infty}$ is a prime period $(m+1)$ solution of (1.4), then

$$x_{-m} = x_1 = \alpha + \frac{x_{-m}}{x_0^k}, \tag{2.22}$$

and for all $t \in \{2, 3, 4, \ldots, m+1\}$,

$$x_{t-(m+1)} = x_t = \alpha + \frac{x_{t-(1+m)}}{x_{t-1}^k} = \alpha + \frac{x_t}{x_{t-1}^k}. \tag{2.23}$$

Then, $(x_{-m}, x_{-m+1}, \ldots, x_{-2}, x_{-1}, x_0)$ is a solution of the system (2.21).

Second, assume that $(x_{-m}, x_{-m+1}, \ldots, x_{-2}, x_{-1}, x_0)$ is a solution of the system (2.21), then

$$\begin{aligned}
x_{-m} &= \alpha + \frac{x_{-m}}{x_0^k} = x_1, \\
x_{-(m-1)} &= \alpha + \frac{x_{-(m-1)}}{x_{-m}^k} = \alpha + \frac{x_{-(m-1)}}{x_1^k} = x_2, \\
x_{-(m-2)} &= \alpha + \frac{x_{-(m-2)}}{x_{-(m-1)}^k} = \alpha + \frac{x_{-(m-2)}}{x_2^k} = x_3, \\
&\vdots \\
x_{-1} &= \alpha + \frac{x_{-1}}{x_{-2}^k} = \alpha + \frac{x_{-1}}{x_{m-1}^k} = x_m, \\
x_0 &= \alpha + \frac{x_0}{x_{-1}^k} = \alpha + \frac{x_0}{x_m^k} = x_{m+1}.
\end{aligned} \tag{2.24}$$

By induction we see that

$$x_{n+m+1} = x_n \quad \forall n \geq -m. \tag{2.25}$$

In the case where at least one of the initial values of (1.4) is different from the others, clearly $\{x_n\}_{n=-m}^{\infty}$ is a prime period $(m+1)$ solution. □

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