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Research Article

Triple Positive Solutions for Third-Order m-Point Boundary Value Problems on Time Scales

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We study the following third-order m-point boundary value problems on time scales $(\varphi(u^{\Delta\nabla}))^{\nabla} + a(t)f(u(t)) = 0$, $t \in [0,T]_{\mathbf{T}}$, $u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, $u^{\Delta}(T) = 0$, $\varphi(u^{\Delta\nabla}(0)) = \sum_{i=1}^{m-2} c_i \varphi(u^{\Delta\nabla}(\xi_i))$, where $\varphi : R \to R$ is an increasing homeomorphism and homomorphism and $\varphi(0) = 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$. We obtain the existence of three positive solutions by using fixed-point theorem in cones. The conclusions in this paper essentially extend and improve the known results.

1. Introduction

The theory of time scales was initiated by Hilger [1] as a mean of unifying and extending theories from differential and difference equations. The study of time scales has led to several important applications in the study of insect population models, neural networks, heat transfer, and epidemic models; see, for example [2–6]. Recently, the boundary value problems with p-Laplacian operator have also been discussed extensively in the literature, for example, see [7–15].

A time scale $\mathbf{T}$ is a nonempty closed subset of R. We make the blanket assumption that 0, T are points in $\mathbf{T}$. By an interval $(0,T)$, we always mean the intersection of the real interval $(0,T)$ with the given time scale; that is, $(0,T) \cap \mathbf{T}$.

In [16], Anderson considered the following third-order nonlinear boundary value problem (BVP):

$$\begin{gathered} x'''(t) = f(t, x(t)), \quad t_1 \le t \le t_3, \\ x(t_1) = x'(t_2) = 0, \qquad \gamma x(t_3) + \delta x''(t_3) = 0. \end{gathered} \tag{1.1}$$

He used the Krasnoselskii and Leggett-Williams fixed-point theorems to prove the existence of solutions to the nonlinear boundary value problem.

In [9, 10], He considered the existence of positive solutions of the p-Laplacian dynamic equations on time scales

$$\left(\phi_p\left(u^{\Delta}\right)\right)^{\nabla} + a(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbf{T}}, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) - B_0\left(u^{\Delta}(\eta)\right) = 0, \qquad u^{\Delta}(T) = 0, \tag{1.3}$$

or

$$u^{\Delta}(0) = 0, \qquad u(T) - B_1\left(u^{\Delta}(\eta)\right) = 0, \tag{1.4}$$

where $\eta \in (0, \rho(T))$. He obtained the existence of at least double and triple positive solutions of the problems by using a new double fixed point theorem and triple fixed point theorem, respectively.

In [15], Zhou and Ma firstly studied the existence and iteration of positive solutions for the following third-order generalized right-focal boundary value problem with p-Laplacian operator

$$\begin{gathered} (\phi_p(u''))'(t) = q(t)f(t,u(t)), \quad 0 \le t \le 1, \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i). \end{gathered} \tag{1.5}$$

They established a corresponding iterative scheme for the problem by using the monotone iterative technique.

However, to the best of our knowledge, little work has been done on the existence of positive solutions for the increasing homeomorphism and positive homomorphism operator on time scales. So the goal of the present paper is to improve and generate p-Laplacian operator and establish some criteria for the existence of multiple positive solutions for the following third-order m-point boundary value problems on time scales

$$\begin{gathered} \left(\varphi\left(u^{\Delta\nabla}\right)\right)^{\nabla} + a(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbf{T}}, \\ u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \qquad u^{\Delta}(T) = 0, \qquad \varphi\left(u^{\Delta\nabla}(0)\right) = \sum_{i=1}^{m-2} c_i \varphi\left(u^{\Delta\nabla}(\xi_i)\right), \end{gathered} \tag{1.6}$$

where $\varphi : R \to R$ is an increasing homeomorphism and homomorphism and $\varphi(0) = 0$, and b_i, c_i, a, f satisfy

(H_1) $b_i, c_i \in [0, +\infty)$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, $0 < \sum_{i=1}^{m-2} b_i < 1$, $0 < \sum_{i=1}^{m-2} c_i < 1$;

(H_2) $f : [0, +\infty) \rightarrow R^+$ is continuous, $a \in C_{\mathrm{ld}}([0, T]_{\mathbf{T}}, R^+)$ and there exits $t_0 \in [0, T)_{\mathbf{T}}$ such that $a(t_0) > 0$, where $R^+ = [0, +\infty)$.

A projection $\varphi : R \rightarrow R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:

(i) if $x \leq y$, then $\varphi(x) \leq \varphi(y)$, $\forall x, y \in R$;

(ii) φ is continuous bijection and its inverse mapping is also continuous;

(iii) $\varphi(xy) = \varphi(x)\varphi(y)$, $\forall x, y \in R$.

2. Preliminaries and Lemmas

For convenience, we list the following definitions which can be found in [1–5].

Definition 2.1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\begin{aligned} \sigma(t) &= \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T}, \\ \rho(r) &= \sup\{\tau \in \mathbf{T}\tau < r\} \in \mathbf{T}, \end{aligned} \tag{2.1}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \rightarrow R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|, \tag{2.2}$$

for all $s \in U$.

For $f : \mathbf{T} \rightarrow R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t, denoted by $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \leq \epsilon \left| \rho(t) - s \right|, \tag{2.3}$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e., ld-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$.

Definition 2.4. If $\phi^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = \phi(b) - \phi(a). \tag{2.4}$$

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a). \tag{2.5}$$

Definition 2.5. Let E be a real Banach space over R. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $u \in P$, $a \geq 0$ implies $au \in P$;

(ii) $u, -u \in P$ implies $u = 0$.

Definition 2.6. Given a cone P in a real Banach space E, a functional $\psi : P \to P$ is said to be increasing on P, provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

Definition 2.7. Given a cone P in a real Banach space E, we define for each $a > 0$ the set

$$P_a = \{x \in P \mid \|x\| < a\}. \tag{2.6}$$

Definition 2.8. A map α is called nonnegative continuous concave functional on a cone P of a real Banach space X if $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(\lambda x + (1 - \lambda)y) \geq \lambda\alpha(x) + (1 - \lambda)\alpha(y) \tag{2.7}$$

for all $x, y \in P$ and $\lambda \in [0, 1]$. Similarly we say that the map β is called nonnegative continuous concave functional on a cone P of a real Banach space X if $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(\lambda x + (1 - \lambda)y) \leq \lambda\beta(x) + (1 - \lambda)\beta(y) \tag{2.8}$$

for all $x, y \in P$ and $\lambda \in [0, 1]$.

Let γ, θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P. For nonnegative real numbers a, b, k, and c we define the following convex set:

$$\begin{gathered}
P(\gamma, c) = \{x \in P : \gamma(x) < c\}, \\
P(\alpha, b; \gamma, c) = \{x \in P : b \leq \alpha(x),\ \gamma(x) \leq c\}, \\
R(\psi, a; \gamma, c) = \{x \in P : a \leq \psi(x),\ \gamma(x) \leq c\}, \\
P(\alpha, b; \theta, k; \gamma, c) = \{x \in P : b \leq \alpha(x),\ \theta(x) \leq k,\ \gamma(x) \leq c\}.
\end{gathered} \tag{2.9}$$

Theorem 2.9 ([17]). *Let P be a cone in a real Banach space X. Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and c, $\alpha(x) \leq \psi(x)$ and $\|x\| \leq M\gamma(x)$ for all $x \in \overline{P(\gamma, c)}$. Suppose that $\Phi : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is a completely continuous operator and there exist nonnegative numbers a, b, and k with $0 < a < b$ such that*

(i) $\{x \in P(\alpha, b; \theta, k; \gamma, c) : \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(\Phi(x)) > b$ *for* $x \in P(\alpha, b; \theta, k; \gamma, c)$;

(ii) $\alpha(\Phi(x)) > b$ *for* $x \in P(\alpha, b; \gamma, c)$ *with* $\theta(\Phi(x)) > k$;

(iii) $0 \overline{\in} R(\psi, a; \gamma, c)$ *and* $\psi(\Phi(x)) < a$ *for* $x \in R(\psi, a; \gamma, c)$ *with* $\psi(x) = a$.

Then Φ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ satisfying

$$\begin{gathered} \gamma(x_i) \leq c, \quad i = 1, 2, 3, \\ b < \alpha(x_1), \quad \alpha(x_2) < b, \qquad a < \psi(x_2), \quad \psi(x_3) < a. \end{gathered} \tag{2.10}$$

Theorem 2.10 ([18]). *Let A be a bounded closed convex subset of a Banach space E. Assume that A_1, A_2 are disjoint closed convex subsets of A and U_1, U_2 are nonempty open subsets of A with $U_1 \subset A_1$ and $U_2 \subset A_2$. Suppose that $\Phi : A \to A$ is completely continuous and the following conditions hold:*

(i) $\Phi(A_1) \subset A_1$, $\Phi(A_2) \subset A_2$;

(ii) Φ *has no fixed points in* $(A_1 \setminus U_1) \cup (A_2 \setminus U_2)$.

Then Φ has at least three points x_1, x_2, x_3 such that $x_1 \in U_1$, $x_2 \in U_2$, and $x_3 \in A \setminus (A_1 \cup A_2)$.

Lemma 2.11. *If condition (H_1) holds, then for $h \in C_{\mathrm{ld}}[0, T]_{\mathbf{T}}$, the boundary value problem (BVP)*

$$\begin{gathered} u^{\Delta\nabla} + h(t) = 0, \quad t \in (0, T), \\ u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \qquad u^{\Delta}(T) = 0 \end{gathered} \tag{2.11}$$

has the unique solution

$$u(t) = \int_0^t (T - s) h(s) \nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T - s) h(s) \nabla s}{1 - \sum_{i=1}^{m-2} b_i}. \tag{2.12}$$

Proof. By caculating, we can easily get (2.12). So we omit it. □

Lemma 2.12. *If condition (H_1) holds, then for $h \in C_{\mathrm{ld}}[0, T]_{\mathbf{T}}$, the boundary value problem (BVP)*

$$\begin{gathered} \left(\varphi\left(u^{\Delta\nabla}\right)\right)^{\nabla} + h(t) = 0, \quad t \in (0, T), \\ u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \qquad u^{\Delta}(T) = 0, \qquad \varphi\left(u^{\Delta\nabla}(0)\right) = \sum_{i=1}^{m-2} c_i \varphi\left(u^{\Delta\nabla}(\xi_i)\right) \end{gathered} \tag{2.13}$$

has the unique solution

$$u(t)=\int_0^t (T-s)\varphi^{-1}\left(\int_0^s h(r)\nabla r+C\right)\nabla s+\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(T-s)\varphi^{-1}\left(\int_0^s h(r)\nabla r+C\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i}, \quad (2.14)$$

where $C=\sum_{i=1}^{m-2} c_i\int_0^{\xi_i} h(r)\nabla r/(1-\sum_{i=1}^{m-2} c_i)$, $\varphi^{-1}(s)$ *is the inverse function to* $\varphi(s)$.

Proof. Integrating both sides of equation in (2.13) on $[0,t]$, we have

$$\varphi\left(u^{\Delta\nabla}(t)\right)=\varphi\left(u^{\Delta\nabla}(0)\right)-\int_0^t h(r)\nabla r. \quad (2.15)$$

So,

$$\varphi\left(u^{\Delta\nabla}(\xi_i)\right)=\varphi\left(u^{\Delta\nabla}(0)\right)-\int_0^{\xi_i} h(r)\nabla r. \quad (2.16)$$

By boundary value condition $\varphi(u^{\Delta\nabla}(0))=\sum_{i=1}^{m-2} c_i\varphi(u^{\Delta\nabla}(\xi_i))$, we have

$$\varphi\left(u^{\Delta\nabla}(0)\right)=-\frac{\sum_{i=1}^{m-2} c_i\int_0^{\xi_i} h(r)\nabla r}{1-\sum_{i=1}^{m-2} c_i}. \quad (2.17)$$

By (2.15) and (2.17) we know

$$u^{\Delta\nabla}(t)=-\varphi^{-1}\left(\frac{\sum_{i=1}^{m-2} c_i\int_0^{\xi_i} h(r)\nabla r}{1-\sum_{i=1}^{m-2} c_i}+\int_0^t h(r)\nabla r\right). \quad (2.18)$$

This together with Lemma 2.11 implies that

$$u(t)=\int_0^t (T-s)\varphi^{-1}\left(\int_0^s h(r)\nabla r+C\right)\nabla s+\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(T-s)\varphi^{-1}\left(\int_0^s h(r)\nabla r+C\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i}, \quad (2.19)$$

where $C=\sum_{i=1}^{m-2} c_i\int_0^{\xi_i} h(r)\nabla r/(1-\sum_{i=1}^{m-2} c_i)$. The proof is complete. □

Lemma 2.13. *Let condition* (H_1) *hold. If* $h\in C_{\mathrm{ld}}[0,T]_{\mathbf{T}}$ *and* $h(t)\geq 0$, *then the unique solution* $u(t)$ *of* (2.13) *satisfies*

$$u(t)\geq 0, \quad t\in[0,T]_{\mathbf{T}}. \quad (2.20)$$

Proof. By $u^{\Delta\nabla}(t)=-\varphi^{-1}((\sum_{i=1}^{m-2} c_i\int_0^{\xi_i} h(r)\nabla r/(1-\sum_{i=1}^{m-2} c_i))+\int_0^t h(r)\nabla r)\leq 0$, we can know that the graph of $u(t)$ is concave down on $(0,T)_{\mathbf{T}}$ and $u^{\Delta}(t)$ is nonincreasing on $[0,T]_{\mathbf{T}}$. This together with the assumption that the boundary condition is $u^{\Delta}(T) = 0$ implies that $u^{\Delta}(t) \geq 0$ for $t \in [0,T]_{\mathbf{T}}$. This implies that

$$\|u\| = u(T), \qquad \min_{t\in[0,T]_{\mathbf{T}}} u(t) = u(0). \tag{2.21}$$

So we only prove $u(0) \geq 0$. By condition (H_1) we have

$$\begin{aligned} u(0) &= \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s h(r)\nabla r + C\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\ &\geq 0. \end{aligned} \tag{2.22}$$

The proof is completed. □

3. Triple Positive Solutions

In this section, some existence results of positive solutions to BVP (1.6) are established by imposing some conditions on f and defining a suitable Banach space and a cone.

Let $E = C_{\mathrm{ld}}[0,T]_{\mathbf{T}}$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0,T]_{\mathbf{T}}$, and $\|u\| = \max_{t\in[0,T]_{\mathbf{T}}}|u(t)|$ is defined as usual by maximum norm. Clearly, it follows that $(E, \|u\|)$ is a Banach space.

We define a cone by

$$P = \left\{u : u \in E,\ u(t) \text{ is concave, nondecreasing, and nonnegative on } [0,T]_{\mathbf{T}},\ u^{\Delta}(T) = 0\right\}. \tag{3.1}$$

Let

$$\eta = \max\left\{t \in \mathbf{T} : t \geq \frac{T}{2}\right\}, \tag{3.2}$$

and fix $l \in \mathbf{T}$ such that

$$0 < \eta < l < T, \tag{3.3}$$

and define the nonnegative continuous convex functionals γ and θ, the nonnegative continuous concave functional α, and the nonnegative continuous functional ψ on the cone P by

$$\begin{gathered} \gamma(u) = \theta(u) = \max_{t\in[0,l]_{\mathbf{T}}} u(t) = u(l), \\ \alpha(u) = \min_{t\in[\eta,T]_{\mathbf{T}}} u(t) = u(\eta), \qquad \psi(u) = \max_{t\in[0,\eta]_{\mathbf{T}}} = u(\eta). \end{gathered} \tag{3.4}$$

For notational convenience, denote

$$\widehat{C} = \frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)\nabla r}{1 - \sum_{i=1}^{m-2} c_i},$$

$$m_\eta = \int_0^\eta (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s, \quad m = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i},$$

$$M_l = \int_0^l (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i},$$

$$M_\eta = \int_0^\eta (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i},$$

$$M = \int_0^T (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i}. \tag{3.5}$$

Lemma 3.1 ([9]). *If $u \in P$, then*

(1) $u(t) \geq (t/T)\|u\|$ *for all* $t \in [0,T]_{\mathbf{T}}$;

(2) $u(s)/s \geq u(t)/t$ *for* $t, s \in [0,T]_{\mathbf{T}}$ *with* $s \leq t$.

Define an operator $\Phi : P \to E$ *by*

$$(\Phi u)(t) = \int_0^t (T-s)\varphi^{-1}\left(\int_0^s a(r)f(u(r))\nabla r + \widetilde{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)f(u(r))\nabla r + \widetilde{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i}, \tag{3.6}$$

where $\widetilde{C} = \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)f(u(r))\nabla r/(1-\sum_{i=1}^{m-2} c_i)$. *Then, u is a solution of boundary value problem* (1.6) *if and only if u is a fixed point of operator Φ. Obviously, for $u \in P$ one has $(\Phi u)(t) \geq 0$ for $t \in [0,T]_{\mathbf{T}}$. In addition, $(\Phi u)^{\Delta\nabla}(t) \leq 0$ for $t \in [0,T]_{\mathbf{T}}$ and $(\Phi u)^{\Delta}(T) = 0$, this implies $\Phi P \subset P$. With standard argument one may show that $\Phi : P \to P$ is completely continuous.*

Theorem 3.2. *Suppose conditions (H_1) and (H_2) hold, and there exist positive numbers $a < (\eta/T)b < b < (l/T)c$, $M_l b < mc$ such that*

(B_1) $f(u) \leq \varphi(c/M_l)$, $u \in [0, Tc/l]$;

(B_2) $f(u) > \varphi(b/m_\eta)$, $u \in [b, T^2 b/l^2]$;

(B_3) $f(u) < \varphi(a/M_\eta)$, $u \in [0, Ta/\eta]$.

Then, the BVP (1.6) *has at least three positive solutions* $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ *satisfying*

$$\begin{gathered} \gamma(u_i) \le c, \quad i = 1,2,3, \\ b < \alpha(u_1), \quad \alpha(u_2) < b, \qquad a < \psi(u_2), \quad \psi(u_3) < a. \end{gathered} \tag{3.7}$$

Proof. Based on Lemma 3.1, it is clear that for $u \in P$ and $\lambda \in [0,1]$, there are $\alpha(u) = \psi(u)$, $\psi(\lambda u) = \lambda\psi(u)$ and $\|u\| \le (T/l)u(l) = (T/l)\gamma(u)$. Furthermore, $\psi(0) = 0 < a$ and therefore $0 \overline{\in} R(\psi, a; \gamma, c)$.

Take $u \in \overline{P(\gamma, c)}$, then $0 \le u \le \|u\| \le (T/l)\gamma(u) \le (T/l)c$. By means of (B_1) one derives

$$\begin{aligned} \gamma(\Phi u) &= \Phi u(l) \\ &= \int_0^l (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\le \frac{c}{M_l}\left(\int_0^l (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}\right) \\ &= c. \end{aligned} \tag{3.8}$$

Thus $\Phi : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$.

Set $u \equiv Tb/l$ and $k = Tb/l$, it follows that

$$\alpha(u) = u(\eta) = \frac{Tb}{l} > b, \qquad \theta(u) = u(l) = \frac{Tb}{l}, \qquad \gamma(u) = \frac{Tb}{l} < c, \tag{3.9}$$

which means $\{u \in P(\alpha, b; \theta, Tb/l; \gamma, c) : \alpha(u) > b\} \neq \emptyset$.

For $u \in P(\alpha, b; \theta, Tb/l; \gamma, c)$, we have $b \le u(t) \le T^2 b/l^2$ for $t \in [\eta, T]_{\mathbf{T}}$. By condition (B_2) we have

$$\begin{aligned} \alpha(\Phi u) &= \Phi u(\eta) \\ &= \int_0^\eta (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s \\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\ge \frac{b}{m_\eta}\int_0^\eta (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s \\ &= b. \end{aligned} \tag{3.10}$$

So, (i) of Theorem 2.9 is fulfilled.

If $u \in P(\alpha, b; \gamma, c)$ and $\theta(\Phi u) > c$, then due to (2) of Lemma 3.1

$$\alpha(\Phi u) = \Phi u(\eta) \geq \frac{\eta}{l}(\Phi u)(l) = \frac{\eta}{l}\theta(\Phi u) > \frac{\eta c}{l} > \frac{T\eta b}{l^2} > b. \tag{3.11}$$

Therefore, (ii) of Theorem 2.9 is fulfilled.

Take $u \in R(\psi, a; \gamma, c)$ and $\psi(u) = a$, then $0 \leq u \leq \|u\| \leq (T/\eta)u(\eta) = (T/\eta)\psi(u) = Ta/\eta$, it then follows from (B_3) that

$$\begin{aligned}
\psi(\Phi u) &= \Phi u(\eta) \\
&= \int_0^\eta (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&\leq \frac{a}{M_\eta}\left(\int_0^\eta (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i}\right) \\
&= a.
\end{aligned} \tag{3.12}$$

As a result, all the conditions of Theorem 2.9 are verified. This completes the proof. □

Theorem 3.3. *Suppose that conditions (H_1) and (H_2) hold. Let $0 < a < b < c$, $Mb < mc$ and assume that the following conditions are satisfied:*

(C_1) $f(u) < \varphi(a/M)$, $u \in [0, a]$;

(C_2) *there exists a number $d > c$ such that $f(u) < \varphi(d/M)$, $u \in [0, d]$;*

(C_3) $\varphi(b/m) < f(u) < \varphi(c/M)$, $u \in [b, c]$.

Then, the BVP (1.6) *has at least three positive solutions u_1, u_2, and u_3 such that*

$$\begin{gathered}
b < u_1(t) < c, \qquad \|u_2\| < a, \\
\|u_3\| > a.
\end{gathered} \tag{3.13}$$

Where for real number b, $\phi_b : [0, T]_{\mathbf{T}} \to [0, +\infty)$ is continuous, $\phi_b(t) = b$, for $t \in [0, T]_{\mathbf{T}}$.

Proof. We first show that $A(\overline{P}_a) \subseteq P_a \subset \overline{P}_a$ if condition (C_1) holds. If $u \in \overline{P}_a$, then $0 \leq u \leq \|u\| \leq a$, which implies $f(u) < \varphi(a/M)$. We have

$$\begin{aligned}
\|\Phi u\| &= (\Phi u)(T) \\
&\leq \int_0^T (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&< \frac{a}{M}\left(\int_0^T (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i}\right) \\
&= a.
\end{aligned} \tag{3.14}$$

This implies that $\Phi(\overline{P}_a) \subseteq P_a \subset \overline{P}_a$.

Next, condition (C_2) indicates that there exists $d > c$ such that $\Phi(\overline{P}_d) \subset \overline{P}_d$. Now we let

$$A = \overline{P}_d, \qquad A_1 = [\varphi_b, \varphi_c], \qquad U_1 = \operatorname{int}(A_1), \qquad A_2 = \overline{P}_a, \qquad U_2 = P_a, \tag{3.15}$$

where $\operatorname{int}(A_1)$ is the interior of A_1. Then we have $\Phi(A) \subset A$, $\Phi(A_2) \subset A_2$. Moreover, $\Phi(\overline{P}_a) \subseteq P_a \subset \overline{P}_a$ means $\Phi(A_2) \subseteq U_2 \subset A_2$. Thus Φ has no fixed point in $(A_2 \setminus U_2)$.

To show $\Phi(A_1) \subset A_1$ and Φ has no fixed point in $(A_1 \setminus U_1)$, set $u \in A_1$, following the definition of φ_b, we can know $b \leq u(t) \leq c$, for $t \in [0,T]_{\mathbf{T}}$. Condition (C_3) then gives rise to $\varphi(b/m) < f(u) < \varphi(c/M)$, which in turn produces

$$\begin{aligned}
(\Phi u)(t) &\geq (\Phi u)(0) \\
&= \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r) f(u(r))\nabla r + \widetilde{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&> \frac{b}{m}\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&= b,
\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
(\Phi u)(t) \le (\Phi u)(T)2 \\
&\le \int_0^T (T-s)\varphi^{-1}\left(\int_0^s a(r)f(u(r))\nabla r + \widetilde{C}\right)\nabla s \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(T-s)\varphi^{-1}\left(\int_0^s a(r)f(u(r))\nabla r + \widetilde{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&< \frac{c}{M}\left(\int_0^T (T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(T-s)\varphi^{-1}\left(\int_0^s a(r)\nabla r + \widehat{C}\right)\nabla s}{1-\sum_{i=1}^{m-2} b_i}\right) \\
&= c.
\end{aligned}
\tag{3.17}
$$

Combining the above two inequalities one achieves $\phi_b(t) = b < (\Phi u)(t) < c = \phi_c(t)$, for $t \in [0,T]_{\mathbf{T}}$. That is, $\Phi u \in U_1$. So $\Phi(A_1) \subseteq U_1 \subset A_1$ and Φ has no fixed point in $(A_1 \setminus U_1)$. Therefore, all conditions of Theorem 2.10 are fulfilled, and the BVP (1.6) has at least three positive solutions u_1, u_2, and u_3 such that

$$
b < u_1(t) < c, \qquad \|u_2\| < a, \qquad \|u_3\| > a. \tag{3.18}
$$

□

4. Some Examples

In the section, we present some simple examples to explain our results. We only study the case $\mathbf{T} = R$, $(0,T)_{\mathbf{T}} = (0,1)$.

Example 4.1. Consider the following third-order three-point boundary value problem:

$$
\begin{gathered}
(\varphi(u''))' + a(t)f(u) = 0, \quad 0 < t < 1, \\
u(0) = \frac{1}{3}u\left(\frac{1}{2}\right), \qquad u'(1) = 0, \qquad \varphi(u''(0)) = \frac{1}{4}\varphi\left(u''\left(\frac{1}{2}\right)\right),
\end{gathered}
\tag{4.1}
$$

where $\varphi(x) = x$, $a(t) \equiv 1$, $b_1 = 1/3$, $c_1 = 1/4$, $\xi_1 = 1/2$.

We choose $\eta = 1/2$, by computing we can know $m_\eta = 11/24$, $M_l = 351/256$, $M_\eta = 33/48$. Let $a = 100$, $b = 245$, $c = 770$, $l = 7/8$, then $a < \eta b < b < lc$. Obviously, $M_l b < mc$. We define a nonlinearity f as follows:

$$
f(u) = \begin{cases}
140, & u \in [0,200], \\
140 + \dfrac{410}{45}(u-200), & u \in [200,245], \\
550, & u \in [245,320], \\
550 + \dfrac{5}{560}(u-320), & u \in [320,+\infty).
\end{cases}
\tag{4.2}
$$

Then, by the definition of f, we have

(i) $f(u) \le \varphi(c/M_l) \approx 557.2$, $u \in [0,880]$;

(ii) $f(u) > \varphi(b/m_\eta) \approx 534.5$, $u \in [245,320]$;

(iii) $f(u) < \varphi(a/M_\eta) \approx 145.4$, $u \in [0,200]$.

By Theorem 3.2, BVP (4.1) has at least three positive solutions.

Example 4.2. Consider the following third-order three-point boundary value problem:

$$\begin{gathered} (\varphi(u''))' + a(t)f(u) = 0, \quad 0 < t < 1, \\ u(0) = \frac{1}{3}u\left(\frac{1}{2}\right), \qquad u'(1) = 0, \qquad \varphi(u''(0)) = \frac{1}{4}\varphi\left(u''\left(\frac{1}{2}\right)\right), \end{gathered} \tag{4.3}$$

where $\varphi(x) = x$, $a(t) \equiv 1$, $b_1 = 1/3$, $c_1 = 1/4$, $\xi_1 = 1/2$, $\eta = 1/2$.

By computing, we can know $m = 11/48$, $M = 83/48$. Let $a = 7$, $b = 12$, $c = 336$, $l = 7/8$, then $a < b < c$. Obviously, $Mb < mc$. We define a nonlinearity f as follows:

$$f(u) = \begin{cases} 3, & u \in [0,7], \\ 3 + \dfrac{97}{25}(u-7)^2, & u \in [7,12], \\ 100, & u \in [12,336], \\ 100 + \dfrac{1100}{1764}(u-336), & u \in [336,+\infty). \end{cases} \tag{4.4}$$

Then, by the definition of f, we have

(i) $f(u) < \varphi(a/M) \approx 4.2$, $u \in [0,7]$;

(ii) and there exists $d = 2100 > c$ such that $f(u) \le \varphi(d/M) \approx 1214.4$, $u \in [0,2100]$;

(iii) $\varphi(b/m) \approx 52.4 < f(u) < \varphi(c/M) \approx 194.3$, $u \in [12,336]$.

By Theorem 3.3, BVP (4.3) has at least three positive solutions.

Remark 4.3. Consider following nonlinear m-point boundary value problem:

$$\begin{gathered} \left(\varphi\left(u^{\Delta\nabla}\right)\right)^{\nabla} + a(t)f(u(t)) = 0, \quad 0 < t < T, \\ u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u^{\Delta}(T) = 0, \qquad \varphi\left(u^{\Delta\nabla}(0)\right) = \sum_{i=1}^{m-2} c_i \varphi\left(u^{\Delta\nabla}(\xi_i)\right), \end{gathered} \tag{4.5}$$

where

$$\varphi(u)=\begin{cases}u^3, & u\le 0,\\ u^2, & u>0,\end{cases} \tag{4.6}$$

f and a satisfy the conditions (H_1) and (H_2). It is clear that $\varphi : R \to R$ is an increasing homeomorphism and homomorphism and $\varphi(0)=0$. Because p-Laplacian operators are odd, they do not apply to our example. Hence we generalize boundary value problem with p-Laplacian operator, and the results [8–11, 13–15] do not apply to the example.

Remark 4.4. In a similar way, we can get the corresponding results for the following boundary value problem:

$$\begin{gathered}\left(\varphi\left(u^{\Delta\nabla}\right)\right)^{\nabla}+a(t)f(u(t))=0,\quad t\in(0,T)_{\mathbf{T}},\\ u(T)=\sum_{i=1}^{m-2}a_iu(\xi_i),\qquad u^{\Delta}(0)=0,\qquad \varphi\left(u^{\Delta\nabla}(0)\right)=\sum_{i=1}^{m-2}c_i\varphi\left(u^{\Delta\nabla}(\xi_i)\right).\end{gathered} \tag{4.7}$$

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