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Research Article

Permanence and Global Attractivity of Discrete Predator-Prey System with Hassell-Varley Type Functional Response

Runxin Wu and Lin Li

Department of Mathematics and Physics, Fujian University of Technology, Fuzhou, Fujian 350108, China

Correspondence should be addressed to Runxin Wu, runxinwu@163.com

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By constructing a suitable Lyapunov function and using the comparison theorem of difference equation, sufficient conditions which ensure the permanence and global attractivity of the discrete predator-prey system with Hassell-Varley type functional response are obtained. Example together with its numerical simulation shows that the main results are verifiable.

1. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. The most popular predator-prey model is the one with Holling type II functional response [2]:

$$\begin{aligned} \frac{dx}{dt} &= ax\left(1-\frac{x}{k}\right)-\frac{cxy}{m+x}, \\ \frac{dy}{dt} &= y\left(-d+\frac{fx}{m+x}\right), \\ x(0) &> 0, \qquad y(0) > 0, \end{aligned} \tag{1.1}$$

where x, y denote the density of prey and predator species at time t, respectively. The constants a, k, c, m, f, d are all positive constants that stand for prey intrinsic growth rate, carrying capacity of prey species, capturing rate, half saturation constant, maximal predator growth rate, predator death rate, respectively.

Standard Lotka-Volterra type models, on which a large body of existing predator-prey theory is built, assume that the per capita rate of predation depends on the prey numbers only. There is growing explicit biological and physiological evidence [3–8] that in many situations, especially when predators have to search and share or compete for food, a more suitable general predator-prey model should be based on the "ratio-dependent" theory.

Arditi and Ginzburg [9] proposed the following predator-prey model with ratio-dependent type functional response:

$$\begin{aligned} \frac{dx}{dt} &= ax\left(1-\frac{x}{k}\right)-\frac{cxy}{my+x}, \\ \frac{dy}{dt} &= y\left(-d+\frac{fx}{my+x}\right), \\ x(0) &> 0, \qquad y(0) > 0. \end{aligned} \tag{1.2}$$

It was known that the functional response can depend on predator density in other ways. One of the more widely known ones is due to Hassell and Varley [10]. A general predator-prey model with Hassell-Varley tape functional response may take the following form:

$$\begin{aligned} \frac{dx}{dt} &= x\left(a-\frac{x}{k}\right)-\frac{cxy}{my^{r}+x}, \\ \frac{dy}{dt} &= y\left(-d+\frac{fx}{my^{r}+x}\right), \quad r \in (0,1), \\ x(0) &> 0, \qquad y(0) > 0. \end{aligned} \tag{1.3}$$

This model is appropriate for interactions, where predators form groups and have applications in biological control. System (1.3) can display richer and more plausible dynamics. In a typical predator-prey interaction where predators do not form groups, one can assume that $\gamma = 1$, producing the so-called ratio-dependent predator-prey dynamics [11]. For terrestrial predators that form a fixed number of tight groups, it is often reasonable to assume that $\gamma = 1/2$. For aquatic predators that form a fixed number of tight groups, $\gamma = 1/3$ may be more appropriate. Recently, Hsu [11] presents a systematic analysis on the above system.

On the other hand, when the size of the population is rarely small or the population has nonoverlapping generation, the discrete time models are more appropriate than the continuous ones [12–24]. This motivated us to propose and study the discrete analogous of predator-prey system (1.3):

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{a(k)-b(k)x(k)-\frac{c(k)y(k)}{m(k)y^{r}(k)+x(k)}\right\}, \\ y(k+1) &= y(k)\exp\left\{-d(k)+\frac{f(k)x(k)}{m(k)y^{r}(k)+x(k)}\right\}, \end{aligned} \tag{1.4}$$

where $r \in (0,1)$; $\{a(k)\}$, $\{b(k)\}$, $\{c(k)\}$, $\{d(k)\}$, $\{m(k)\}$, $\{f(k)\}$ are all bounded nonnegative sequences. For the rest of the paper, we use the following notations: for any bounded sequence $\{g(k)\}$, set

$$g^u = \sup_{k \in N} g(k), \qquad g^l = \inf_{k \in N} g(k). \tag{1.5}$$

By the biological meaning, we will focus our discussion on the positive solution of system of (1.3). Thus, we require that

$$x(0) > 0, \qquad y(0) > 0. \tag{1.6}$$

2. Permanence

In order to establish the persistent result for system (1.4), we make some preparations.

Definition 2.1. System (1.4) said to be permanent if there exist positive constants m and M, which are independent of the solution of system (1.4), such that for any positive solution $\{x(k), y(k)\}$ of system (1.4) satisfies

$$m \leq \liminf_{k \to +\infty} \{x(k), y(k)\} \leq \limsup_{k \to +\infty} \{x(k), y(k)\} \leq M. \tag{2.1}$$

Lemma 2.2 (see [23]). *Assume that $\{x(k)\}$ satisfies $x(k) > 0$ and*

$$x(k+1) \leq x(k) \exp\{a(k) - b(k)x(k)\} \tag{2.2}$$

for $k \in N$, where $a(k)$ and $b(k)$ are all nonnegative sequences bounded above and below by positive constants. Then

$$\limsup_{k \to +\infty} x(k) \leq \frac{1}{b^l} \exp(a^u - 1). \tag{2.3}$$

Lemma 2.3 (see [23]). *Assume that $\{x(k)\}$ satisfies*

$$x(k+1) \geq x(k) \exp\{a(k) - b(k)x(k)\}, \quad k \geq N_0, \tag{2.4}$$

$\limsup_{k \to +\infty} x(k) \leq x^*$ *and $x(N_0) > 0$, where $a(k)$ and $b(k)$ are all nonnegative sequences bounded above and below by positive constants and $N_0 \in N$. Then*

$$\liminf_{k \to +\infty} x(k) \geq \frac{a^l \exp\{a^l - b^u x^*\}}{b^u}. \tag{2.5}$$

Theorem 2.4. *Assume that*

$$a^l - \frac{c^u M_2^{1-r}}{m^l} > 0, \tag{H_1}$$

$$f^l > d^u \tag{H_2}$$

hold, then system (1.4) *is permanent, that is, for any positive solution* $\{x(k),\ y(k)\}$ *of system* (1.4), *one has*

$$\begin{aligned} m_1 &\le \liminf_{k\to+\infty} x(k) \le \limsup_{k\to+\infty} x(k) \le M_1, \\ m_2 &\le \liminf_{k\to+\infty} x(k) \le \limsup_{k\to+\infty} y(k) \le M_2, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} m_1 &= \frac{a^l - (c^u M_2^{1-r}/m^l)}{b^u} \exp\left\{ a^l - \frac{c^u M_2^{1-r}}{m^l} - b^u M_1 \right\}, \\ m_2 &= \min\left\{ \left\{ \frac{(f^l - d^u)m_1}{m^u d^u} \right\}^{1/r}, \left\{ \frac{(f^l - d^u)m_1}{m^u d^u} \right\}^{1/r} \exp\left\{ -d^u + \frac{f^l m_1}{m^u M_2^r + m_1} \right\} \right\}, \\ M_1 &= \frac{1}{b^l} \exp(a^u - 1), \\ M_2 &= \left\{ \frac{f^u M_1}{m^l d^l} \right\}^{1/r} \exp\left\{ -d^l + f^u \right\}. \end{aligned} \tag{2.7}$$

Proof. We divided the proof into four claims.
Claim 1. From the first equation of (1.4), we have

$$x(k+1) \le x(k) \exp\{a(k) - b(k)x(k)\}. \tag{2.8}$$

By Lemma 2.2, we have

$$\limsup_{k\to+\infty} x(k) \le \frac{1}{b^l} \exp(a^u - 1) \overset{\text{def}}{=} M_1. \tag{2.9}$$

Above inequality shows that for any $\varepsilon > 0$, there exists a $k_1 > 0$, such that

$$x(k+1) \le M_1 + \varepsilon, \quad \forall k \ge k_1. \tag{2.10}$$

Claim 2. We divide it into two cases to prove that

$$\limsup_{k\to+\infty} y(k) \le M_2. \tag{2.11}$$

Case (i)

There exists an $l_0 \ge k_1$, such that $y(l_0+1) \ge y(l_0)$. Then by the second equation of system (1.4), we have

$$-d(l_0) + \frac{f(l_0)x(l_0)}{m(l_0)y^r(l_0) + x(l_0)} \ge 0. \tag{2.12}$$

Hence,

$$-d(l_0) + \frac{f(l_0)x(l_0)}{m(l_0)y^r(l_0)} \ge 0, \tag{2.13}$$

therefore,

$$y^r(l_0) \le \frac{(l_0)x(l_0)}{m(l_0)d(l_0)} \le \frac{f^u(M_1+\varepsilon)}{m^l d^l}, \tag{2.14}$$

and so,

$$y(l_0) \le \left\{\frac{f^u(M_1+\varepsilon)}{m^l d^l}\right\}^{1/r}. \tag{2.15}$$

It follows that

$$\begin{aligned} y(l_0+1) &= y(l_0)\exp\left\{-d(l_0) + \frac{f(l_0)x(l_0)}{m(l_0)y^r(l_0)+x(l_0)}\right\} \\ &\le \left\{\frac{f^u(M_1+\varepsilon)}{m^l d^l}\right\}^{1/r} \exp\left\{-d^l + f^u\right\} \overset{\text{def}}{=} M_{2\varepsilon}. \end{aligned} \tag{2.16}$$

We claim that

$$y(k) \le M_{2\varepsilon} \quad \forall k \ge l_0. \tag{2.17}$$

By a way of contradiction, assume that there exists a $p_0 > l_0$ such that $y(p_0) > M_{2\varepsilon}$. Then $p_0 \ge l_0 + 2$. Let $y(\widetilde{p}_0) \ge l_0 + 2$ be the smallest integer such that $y(\widetilde{p}_0) \ge M_{2\varepsilon}$. Then $y(\widetilde{p}_0) > y(\widetilde{p}_0 - 1)$. The above argument produces that $y(\widetilde{p}_0) \le M_{2\varepsilon}$, a contradiction. This prove the claim.

Case (ii)

We assume that $y(k+1) < y(k)$ for all $K \geq K_1$. Since $y(k)$ is nonincreasing and has a lower bound 0, we know that $\lim_{k\to+\infty} y(k)$ exists, denoted by $\overline{y}$, then

$$\lim_{k\to+\infty} y(k) = \overline{y}. \tag{2.18}$$

We claim that

$$\overline{y} \leq \left\{ \frac{f^u(M_1+\varepsilon)}{m^l d^l} \right\}^{1/r}. \tag{2.19}$$

By a way of contradiction, assume that $\overline{y} > \{f^u(M_1+\varepsilon)/m^l d^l\}^{1/r}$. Taking limit in the second equation in system (1.4) gives

$$\lim_{k\to+\infty} \left\{ -d(k) + \frac{f(k)x(k)}{m(k)y^r(k)+x(k)} \right\} = 0, \tag{2.20}$$

which is a contradiction since for $K > K_1$

$$-d(k) + \frac{f(k)x(k)}{m(k)y^r(k)+x(k)} \leq -d^l + \frac{f^u(M_1+\varepsilon)}{m^l \overline{y}^r} < 0. \tag{2.21}$$

This prove the claim, then we have

$$\limsup_{k\to+\infty} y(k) = \lim_{k\to+\infty} y(k) = \overline{y} \leq \left\{ \frac{f^u(M_1+\varepsilon)}{m^l d^l} \right\}^{1/r}. \tag{2.22}$$

Combining Cases (i) and (ii), we see that

$$\limsup_{k\to+\infty} y(k) \leq M_{2\varepsilon}. \tag{2.23}$$

Let $\varepsilon \to 0$, we have

$$\limsup_{k\to+\infty} y(k) \leq \left\{ \frac{f^u M_1}{m^l d^l} \right\}^{1/r} \exp\left\{ -d^l + f^u \right\} = M_2. \tag{2.24}$$

Claim 3 ($\liminf_{k\to\infty} x(k) \geq m_1$). Conditions (H_1) imply that for enough small positive constant ε, we have

$$a^l - \frac{c^u (M_2+\varepsilon)^{1-r}}{m^l} > 0. \tag{2.25}$$

For above ε, it follows form Claims 1 and 2 that there exists a k_2 such that for all $k > k_2$

$$x(k) \leq M_1 + \varepsilon, \qquad y(k) \leq M_2 + \varepsilon. \tag{2.26}$$

From the first equation of (1.4), we have

$$x(k+1) \geq x(k) \exp\left\{ a^l - \frac{c^u (M_2 + \varepsilon)^{1-r}}{m^l} - b^u x(k) \right\}. \tag{2.27}$$

By applying Lemma 2.3 to above inequality, we have

$$\liminf_{k \to +\infty} x(k) \geq \frac{a^l - \left(c^u (M_2 + \varepsilon)^{1-r} / m^l \right)}{b^u} \exp\left\{ a^l - \frac{c^u (M_2 + \varepsilon)^{1-r}}{m^l} - b^u (M_1 + \varepsilon) \right\}. \tag{2.28}$$

Setting $\varepsilon \to 0$ in (2.28) leads to

$$\liminf_{k \to +\infty} x(k) \geq \frac{a^l - \left(c^u M_2^{1-r} / m^l \right)}{b^u} \exp\left\{ a^l - \frac{c^u M_2^{1-r}}{m^l} - b^u M_1 \right\} \stackrel{\text{def}}{=} m_1. \tag{2.29}$$

This ends the proof of Claim 3.

Claim 4. For any small positive constant $\varepsilon < m_1/2$, from Claims 1–3, it follows that there exists a $k_3 > k_2$ such that for all $k > k_3$

$$x(k) \geq m_1 - \varepsilon, \qquad x(k) \leq M_1 + \varepsilon, \qquad y(k) \leq M_2 + \varepsilon. \tag{2.30}$$

We present two cases to prove that

$$\liminf_{k \to +\infty} y(k) \geq m_2 \tag{2.31}$$

Case (i)

There exists an $n_0 \geq k_3$ such that $y(n_0 + 1) \leq y(n_0)$, then

$$-d(n_0) + \frac{f(n_0) x(n_0)}{m(n_0) y^r(n_0) + x(n_0)} \leq 0. \tag{2.32}$$

Hence

$$y(n_0) \geq \left\{ \frac{(f^l - d^u)(m_1 - \varepsilon)}{m^u d^u} \right\}^{1/r} \stackrel{\text{def}}{=} c_{1\varepsilon}, \tag{2.33}$$

and so,

$$y(n_0+1) \geq \left\{\frac{(f^l-d^u)(m_1-\varepsilon)}{m^u d^u}\right\}^{1/r} \exp\left\{-d^u + \frac{f^l(m_1-\varepsilon)}{m^u(M_2+\varepsilon)^r + (m_1-\varepsilon)}\right\} \stackrel{\text{def}}{=} c_{2\varepsilon}. \tag{2.34}$$

Set

$$m_{2\varepsilon} = \min\{c_{1\varepsilon}, c_{2\varepsilon}\}. \tag{2.35}$$

We claim that $y(k) \geq m_{2\varepsilon}$ for $k \geq n_0$. By a way of contradiction, assume that there exists a $q_0 \geq n_0$, such that $y(q_0) < m_{2\varepsilon}$. Then $q_0 \geq n_0 + 2$. Let $\tilde{q}_0 \geq n_0 + 2$ be the smallest integer such that $y(\tilde{q}_0) < m_{2\varepsilon}$. Then $y(\tilde{q}_0) < y(\tilde{q}_0 - 1)$, which implies that $y(q_0) \leq m_{2\varepsilon}$, a contradiction, this proves the claim.

Case (ii)

We assume that $y(k+1) > y(k)$ for all $k > k_3$. According to (2.30), $\lim_{k\to+\infty} y(k)$ exists, denoted by $\underline{y}$, then

$$\lim_{k\to+\infty} y(k) = \underline{y}. \tag{2.36}$$

We claim that

$$\underline{y} \geq m_{2\varepsilon}. \tag{2.37}$$

By the way of contradiction, assume that $\underline{y} < m_{2\varepsilon}$. Taking limit in the second equation in system (1.4) gives

$$\lim_{k\to+\infty}\left\{-d(k) + \frac{f(k)x(k)}{m(k)y^r(k) + x(k)}\right\} = 0, \tag{2.38}$$

which is a contradiction since for $k > k_3$,

$$-d(k) + \frac{f(k)x(k)}{m(k)y^r(k) + x(k)} \geq -d^u + \frac{f^l(m_1-\varepsilon)}{m^u \underline{y}^r + (m_1-\varepsilon)} > 0. \tag{2.39}$$

The above analysis show that

$$\liminf_{k\to+\infty} y(k) \geq m_{2\varepsilon}. \tag{2.40}$$

Letting $\varepsilon \to 0$, we have

$$\liminf_{k\to+\infty} y(k) \geq m_2, \tag{2.41}$$

where

$$m_2 = \min\left\{\left\{\frac{(f^l - d^u)m_1}{m^u d^u}\right\}^{1/r}, \left\{\frac{(f^l - d^u)m_1}{m^u d^u}\right\}^{1/r} \exp\left\{-d^u + \frac{f^l m_1}{m^u M_2^r + m_1}\right\}\right\}. \quad (2.42)$$

According to Claims 1–4, we can easily find that the result of Theorem 2.4 holds. □

3. Global Attractivity

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold. Assume further that there exist positive constants* α, β, *and* δ *such that*

$$\alpha \min\left\{b^l, \frac{2}{M_1} - b^u\right\} - \alpha\frac{c^u M_2^{1-(r/2)}}{4m^l m_2} - \beta\frac{f^u M_1^{1/2}}{4m_1 m_2^{r/2}} > \delta, \quad (H_3)$$

$$\beta \min\left\{\frac{f^l m^l m_1 r}{(m^u M_2^r + M_1)^2 M_2^{1-r}}, \frac{2}{M_2} - \frac{f^u M_1^{1/2} r}{4m_2 m_1^{1/2}}\right\} - \alpha\frac{c^u M_1^{1/2}}{4m^l m_2^r m_1^{1/2}} - \alpha\frac{c^u M_2^r (1-r)}{4m_1 m_2^r} > \delta. \quad (H_4)$$

Then system (1.4) *with initial condition* (1.6) *is globally attractive, that is, for any two positive solutions* $(x_1(k), y_1(k))$ *and* $(x_2(k), y_2(k))$ *of system* (1.4), *one has*

$$\lim_{k\to+\infty} |x_1(k) - x_2(k)| = 0, \qquad \lim_{k\to+\infty} |y_1(k) - y_2(k)| = 0. \quad (3.1)$$

Proof. From conditions (H_3) and (H_4), there exists an enough small positive constant $\varepsilon < \min\{m_1/2, m_2/2\}$ such that

$$\begin{aligned}
&\alpha \min\left\{b^l, \frac{2}{M_1+\varepsilon} - b^u\right\} - \alpha\frac{c^u (M_2+\varepsilon)^{1-(r/2)}}{4m^l (m_2-\varepsilon)} - \beta\frac{f^u (M_1+\varepsilon)^{1/2}}{4(m_1-\varepsilon)(m_2-\varepsilon)^{r/2}} > \delta, \\
&\beta \min\left\{\frac{f^l m^l (m_1-\varepsilon) r}{[m^u (M_2+\varepsilon)^r + (M_1+\varepsilon)]^2 (M_2+\varepsilon)^{1-r}}, \frac{2}{M_2+\varepsilon} - \frac{f^u (M_1+\varepsilon)^{1/2} r}{4(m_2-\varepsilon)(m_1-\varepsilon)^{1/2}}\right\} \\
&\quad - \alpha\frac{c^u (M_1+\varepsilon)^{1/2}}{4m^l (m_2-\varepsilon)^r (m_1-\varepsilon)^{1/2}} - \alpha\frac{c^u (M_2+\varepsilon)^r (1-r)}{4(m_1-\varepsilon)(m_2-\varepsilon)^r} > \delta.
\end{aligned} \quad (3.2)$$

Since (H_1) and (H_2) hold, for any positive solutions $(x_1(k), y_1(k))$ and $(x_2(k), y_2(k))$ of system (1.4), it follows from Theorem 2.4 that

$$\begin{aligned}
&m_1 \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_1, \\
&m_2 \le \liminf_{k\to+\infty} y_i(k) \le \limsup_{k\to+\infty} y_i(k) \le M_2, \quad i = 1, 2.
\end{aligned} \quad (3.3)$$

For above ε and (3.3), there exists a $k_4 > 0$ such that for all $k > k_4$,

$$m_1 - \varepsilon \le x_i(k) \le M_1 + \varepsilon, \qquad m_2 - \varepsilon \le x_i(k) \le M_2 + \varepsilon, \quad i = 1, 2. \tag{3.4}$$

Let

$$V_1(k) = |\ln x_1(k) - \ln x_2(k)|. \tag{3.5}$$

Then from the first equation of system (1.3), we have

$$\begin{aligned} V_1(k+1) &= |\ln x_1(k+1) - \ln x_2(k+1)| \\ &\le |\ln x_1(k) - \ln x_2(k) - b(k)(x_1(k) - x_2(k))| \\ &\quad + c(k)\left|\frac{y_1(k)}{m(k)y_1^r(k) + x_1(k)} - \frac{y_2(k)}{m(k)y_2^r(k) + x_2(k)}\right|. \end{aligned} \tag{3.6}$$

Using the Mean Value Theorem, we get

$$\begin{gathered} x_1(k) - x_2(k) = \exp(\ln x_1(k)) - \exp(\ln x_2(k)) = \xi_1(k)(\ln x_1(k) - \ln x_2(k)), \\ y_1^{1-r}(k) - y_2^{1-r}(k) = (1-r)\xi_2^{-r}(k)\big(y_1(k) - y_2(k)\big), \end{gathered} \tag{3.7}$$

where $\xi_1(k)$ lies between $x_1(k)$ and $x_2(k)$, $\xi_2(k)$ lies between $y_1(k)$ and $y_2(k)$.

It follows from (3.6), (3.7) that

$$\begin{aligned} V_1(k+1) \le{} & |\ln x_1(k) - \ln x_2(k)| - \left(\frac{1}{\xi_1(k)} - \left|\frac{1}{\xi_1(k)} - b(k)\right|\right)|x_1(k) - x_2(k)| \\ & + \left|\frac{c(k)y_1(k)}{\big(m(k)y_1^r(k) + x_1(k)\big)\big(m(k)y_2^r(k) + x_2(k)\big)}\right| |x_1(k) - x_2(k)| \\ & + \left|\frac{c(k)x_1(k)}{\big(m(k)y_1^r(k) + x_1(k)\big)\big(m(k)y_2^r(k) + x_2(k)\big)}\right| \big|y_1(k) - y_2(k)\big| \\ & + \left|\frac{c(k)m(k)y_1^r(k)y_2^r(k)}{\big(m(k)y_1^r(k) + x_1(k)\big)\big(m(k)y_2^r(k) + x_2(k)\big)} \frac{1-r}{\xi_2^r(k)}\right| \big|y_1(k) - y_2(k)\big|. \end{aligned} \tag{3.8}$$

And so, for $k > k_4$

$$\begin{aligned}
\Delta V_1 \le & -\min\left\{b^l, \frac{2}{M_1+\varepsilon} - b^u\right\}|x_1(k) - x_2(k)| \\
& + \frac{c^u (M_2+\varepsilon)^{1-(r/2)}}{4m^l (m_2-\varepsilon)^{r/2}(m_1-\varepsilon)}|x_1(k) - x_2(k)| \\
& + \frac{c^u (M_1+\varepsilon)^{1/2}}{4m^l (m_2-\varepsilon)^{r}(m_1-\varepsilon)^{1/2}}\left|y_1(k) - y_2(k)\right| \\
& + \frac{c^u (M_2+\varepsilon)^{r}(1-r)}{4(m_1-\varepsilon)(m_2-\varepsilon)^{r}}\left|y_1(k) - y_2(k)\right|.
\end{aligned} \tag{3.9}$$

Let

$$V_2(k) = \left|\ln y_1(k) - \ln y_2(k)\right|. \tag{3.10}$$

Then from the second equation of system (1.4), we have

$$\begin{aligned}
V_2(k+1) &= \left|\ln y_1(k+1) - \ln y_2(k+1)\right| \\
&= \left|\ln y_1(k) - \ln y_2(k) + f(k)\left(\frac{x_1(k)}{m(k)y_1^r(k)+x_1(k)} - \frac{x_2(k)}{m(k)y_2^r(k)+x_2(k)}\right)\right| \\
&\le \left|\ln y_1(k) - \ln y_2(k) - \frac{f(k)m(k)x_1(k)\left(y_1^r(k) - y_2^r(k)\right)}{\left(m(k)y_1^r(k)+x_1(k)\right)\left(m(k)y_2^r(k)+x_2(k)\right)}\right| \\
&\quad + \left|\frac{f(k)m(k)y_1^r(k)(x_1(k)-x_2(k))}{\left(m(k)y_1^r(k)+x_1(k)\right)\left(m(k)y_2^r(k)+x_2(k)\right)}\right|.
\end{aligned} \tag{3.11}$$

Using the Mean Value Theorem, we get

$$\begin{gathered}
y_1(k) - y_2(k) = \exp\left(\ln y_1(k)\right) - \exp\left(\ln y_2(k)\right) = \xi_3(k)\left(\ln y_1(k) - \ln y_2(n)\right), \\
y_1^r(k) - y_2^r(k) = r\xi_4^{r-1}(k)\left(y_1(k) - y_2(k)\right),
\end{gathered} \tag{3.12}$$

where $\xi_3(k)$, $\xi_4(k)$ lie between $y_1(k)$ and $y_2(k)$, respectively. Then, it follows from (3.11), (3.12) that for $k > k_4$,

$$\begin{aligned}
\Delta V_2 \le& -\left(\frac{1}{\xi_3(k)} - \left|\frac{1}{\xi_3(k)} - \frac{f(k)m(k)x_1(k)r}{\left(m(k)y_1^r(k)+x_1(k)\right)\left(m(k)y_2^r(k)+x_2(k)\right)\xi_4^{1-r}(k)}\right|\right)\\
&\times\left|y_1(k)-y_2(k)\right|\\
&+\frac{f(k)m(k)y_1^r(k)}{\left(m(k)y_1^r(k)+x_1(k)\right)\left(m(k)y_2^r(k)+x_2(k)\right)}\left|x_1(k)-x_2(k)\right|\\
\le& -\min\left\{\frac{f^l m^l(m_1-\varepsilon)r}{\left[m^u(M_2+\varepsilon)^r+(M_1+\varepsilon)\right]^2(M_2+\varepsilon)^{1-r}}, \frac{2}{M_2+\varepsilon} - \frac{f^u(M_1+\varepsilon)^{1/2}r}{4(m_2-\varepsilon)(m_1-\varepsilon)^{1/2}}\right\}\\
&\times\left|y_1(k)-y_2(k)\right|\\
&+\frac{f^u(M_1+\varepsilon)^{r/2}}{4(m_1-\varepsilon)(m_2-\varepsilon)^{r/2}}\left|x_1(k)-x_2(k)\right|.
\end{aligned} \tag{3.13}$$

Now we define a Lyapunov function as follows:

$$V(k) = \alpha V_1(k) + \beta V_2(k). \tag{3.14}$$

Calculating the difference of V along the solution of system (1.4), for $k > k_4$, it follows from (3.9) and (3.13) that

$$\begin{aligned}
\Delta V \le& \left[-\alpha\min\left\{b^l, \frac{2}{M_1+\varepsilon} - b^u\right\} - \alpha\frac{c^u(M_2+\varepsilon)^{1-(r/2)}}{4m^l(m_2-\varepsilon)^{r/2}(m_1-\varepsilon)} - \beta\frac{f^u(M_1+\varepsilon)^{r/2}}{4(m_1-\varepsilon)(m_2-\varepsilon)^{r/2}}\right]\\
&\times\left|x_1(k)-x_2(k)\right|\\
&-\left[\beta\min\left\{\frac{f^l m^l(m_1-\varepsilon)r}{\left[m^u(M_2+\varepsilon)^r+(M_1+\varepsilon)\right]^2(M_2+\varepsilon)^{1-r}}, \frac{2}{M_2+\varepsilon} - \frac{f^u(M_1+\varepsilon)^{1/2}r}{4(m_2-\varepsilon)(m_1-\varepsilon)^{1/2}}\right\}\right.\\
&\left.-\alpha\frac{c^u(M_1+\varepsilon)^{1/2}}{4m^l(m_2-\varepsilon)^r(m_1-\varepsilon)^{1/2}} - \alpha\frac{c^u(M_2+\varepsilon)^r(1-r)}{4(m_1-\varepsilon)(m_2-\varepsilon)^r}\right] \times \left|y_1(k)-y_2(k)\right|\\
\le& -\delta\left(\left|x_1(k)-x_2(k)\right| + \left|y_1(k)-y_2(k)\right|\right).
\end{aligned} \tag{3.15}$$

Summating both sides of the above inequalities from k_4 to k, we have

$$\sum_{p=k_4}^{k}\left(V(p+1)-v(p)\right) \le -\delta\sum_{p=k_4}^{k}\left(\left|x_1(p)-x_2(p)\right| + \left|y_1(p)-y_2(p)\right|\right), \tag{3.16}$$

which implies

$$V(k+1)+\delta\sum_{p=k_4}^{k}\left(\left|x_1(p)-x_2(p)\right|+\left|y_1(p)-y_2(p)\right|\right)\leq V(k_4). \tag{3.17}$$

It follows that

$$\sum_{p=k_4}^{k}\left(\left|x_1(p)-x_2(p)\right|+\left|y_1(p)-y_2(p)\right|\right)\leq \frac{V(k_4)}{\delta}. \tag{3.18}$$

Using the fundamental theorem of positive series, there exists small enough positive constant $\varepsilon>0$ such that

$$\sum_{p=k_4}^{+\infty}\left(\left|x_1(p)-x_2(p)\right|+\left|y_1(p)-y_2(p)\right|\right)\leq \frac{V(k_4)}{\delta}, \tag{3.19}$$

which implies that

$$\lim_{k\to+\infty}\left(\left|x_1(k)-x_2(k)\right|+\left|y_1(k)-y_2(k)\right|\right)=0, \tag{3.20}$$

that is

$$\lim_{k\to+\infty}\left|x_1(k)-x_2(k)\right|=0,\qquad \lim_{k\to+\infty}\left|y_1(k)-y_2(k)\right|=0. \tag{3.21}$$

This completes the proof of Theorem 3.1. □

4. Extinction of the Predator Species

This section is devoted to study the extinction of the predator species y.

Theorem 4.1. *Assume that*

$$-d^l+f^u<0. \tag{H_5}$$

Then, the species y will be driven to extinction, and the species x is permanent, that is, for any positive solution $(x(k),y(k))$ of system (1.4),

$$\begin{gathered}\lim_{k\to+\infty}y(k)=0,\\ m_*\leq\liminf_{k\to+\infty}x(k)\leq\limsup_{k\to+\infty}x(k)\leq M_1,\end{gathered} \tag{4.1}$$

where

$$m_* = \frac{a^l}{b^u} \exp\left\{a^l - b^u M_1\right\},$$
$$M_1 = \frac{1}{b^l} \exp(a^u - 1). \tag{4.2}$$

Proof. For condition (H_5), there exists small enough positive $\gamma > 0$, such that

$$-d^l + f^u < -\gamma < 0 \tag{4.3}$$

for all $k \in N$, from (4.3) and the second equation of the system (1.4), one can easily obtain that

$$\begin{aligned} y(k+1) &= y(k) \exp\left\{-d(k) + \frac{f(k)x(k)}{m(k)y(k) + x(k)}\right\} \\ &< y(k) \exp\left\{-d^l + f^u\right\} \\ &< y(k) \exp\{-\gamma\}. \end{aligned} \tag{4.4}$$

Therefore,

$$y(k+1) < y(0) \exp\{-k\gamma\}, \tag{4.5}$$

which yields

$$\lim_{k \to +\infty} y(k) = 0. \tag{4.6}$$

From the proof of Theorem 3.1, we have

$$\limsup_{k \to +\infty} x(k) \le M_1. \tag{4.7}$$

For enough small positive constant $\varepsilon > 0$,

$$a^l - \frac{c^u \varepsilon^{1-r}}{m^l} > 0. \tag{4.8}$$

For above ε, from (2.9) and (4.6), there exists a $k_5 > 0$ such that for all $k > k_5$,

$$x(k) < M_1 + \varepsilon, \quad y(k) < \varepsilon. \tag{4.9}$$

From the first equation of (1.4), we have

$$x(k+1) \geq x(k) \exp\left\{a^l - \frac{c^u \varepsilon^{1-r}}{m^l} - b^u x(k)\right\}. \tag{4.10}$$

By Lemma 2.3, we have

$$\liminf_{k\to+\infty} x(k) \geq \frac{a^l - (c^u \varepsilon^{1-r}/m^l)}{b^u} \exp\left\{a^l - \frac{c^u \varepsilon^{1-r}}{m^l} - b^u (M_1 + \varepsilon)\right\}. \tag{4.11}$$

Setting $\varepsilon \to 0$ in (4.11) leads to

$$\liminf_{k\to+\infty} x(k) \geq \frac{a^l}{b^u} \exp\left\{a^l - b^u M_1\right\} \stackrel{\text{def}}{=} m_*. \tag{4.12}$$

The proof of Theorem 4.1 is completed. □

5. Example

The following example shows the feasibility of the main results.

Example 5.1. Consider the following system:

$$\begin{aligned} x(k+1) &= x(k) \exp\left\{1.41 + 0.12\cos(k) - 1.78x(k) - \frac{0.33y(k)}{2.16y^{1/2}(k) + x(k)}\right\}, \\ y(k+1) &= y(k) \exp\left\{-0.62 + \frac{1.79x(k)}{2.16y^{1/2}(k) + x(k)}\right\}. \end{aligned} \tag{5.1}$$

One could easily see that there exist positive constants $\alpha = 0.01,\ \beta = 0.05,\ \delta = 0.001$ such that

$$\begin{gathered} a^l - \frac{c^u M_2^{1-r}}{m^l} \approx 2.3281 > 0, \\ f^l > d^u \approx 1.1700 > 0, \\ \alpha \min\left\{b^l, \frac{2}{M_1} - b^u\right\} - \alpha \frac{c^u M_2^{1-(r/2)}}{4m^l m_2} - \beta \frac{f^u M_1^{1/2}}{4m_1 m_2^{r/2}} \approx 0.0011 > \delta, \\ \beta \min\left\{\frac{f^l m^l m_1 r}{(m^u M_2^r + M_1)^2 M_2^{1-r}}, \frac{2}{M_2} - \frac{f^u M_1^{1/2} r}{4m_2 m_1^{1/2}}\right\} - \alpha \frac{c^u M_1^{1/2}}{4m^l m_2^r m_1^{1/2}} - \alpha \frac{c^u M_2^r (1-r)}{4m_1 m_2^r} \approx 0.0107 > \delta. \end{gathered} \tag{5.2}$$

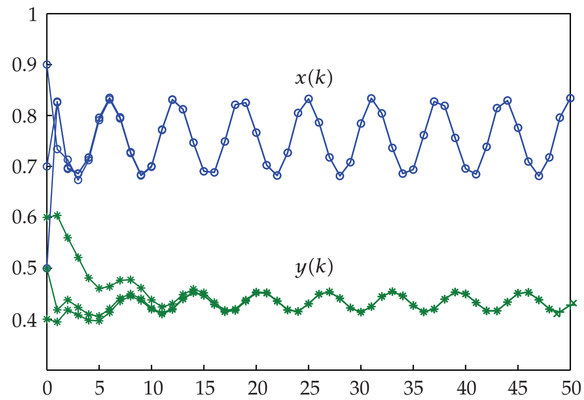

Figure 1: Dynamics behavior of system (1.4) with initial conditions $(x(0), y(0)) = (0.9, 0.6)$, $(0.7, 0.4)$, $(0.5, 0.5)$, respectively.

Clearly, conditions (H_1)–(H_4) are satisfied. It follows from Theorems 2.4 and 3.1, that the system is permanent and globally attractive. Numerical simulation from Figure 1 shows that solutions do converge and system is permanent.

6. Conclusion

In this paper, we have obtained sufficient conditions for the permanence and global attractivity of the system (1.4), where $r \in (0,1)$. If $r = 1$ in the system (1.4), the system (1.4) is a discrete ratio-dependent predator-prey model with Holling-II functional response, in this case, HUO and LI gave sufficient conditions for the permanence of the system in [24], however, they did not provide the condition for the extinction of the predator species y. In this paper, Theorem 2.4 gives the same conditions as that of Huo and Li's condition for the permanence of the system. Furthermore, Theorem 4.1 gives sufficient conditions which ensure the extinction the predator of the system (1.4) when $r = 1$. If $a^l - c^u/m^l > 0$ holds, then the prey species x is permanence. If $r = 0$ in the system of (1.4), the system is a discrete predator-prey model with Holling-II function response, Theorem 4.1 also holds for the case $r = 0$.

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