

Research Article

Existence and Exponential Stability of Periodic Solution of High-Order Hopfield Neural Network with Delays on Time Scales

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On time scales, by using the continuation theorem of coincidence degree theory and constructing some suitable Lyapunov functions, the periodicity and the exponential stability are investigated for a class of delayed high-order Hopfield neural networks (HHNNs), which are new and complement of previously known results. Finally, an example is given to show the effectiveness of the proposed method and results.

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1. Introduction

Consider the following HHNNs with time-varying delays:

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t - \gamma_{ij}(t))) \\ & + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - \nu_{ijl}(t))) \\ & + I_i(t), \quad i = 1, 2, \dots, n, \end{aligned} \quad (1.1)$$

where n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the i th unit at the time t , $c_i(t)$ represents the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the

neural network, $\gamma_{ij}(t) \geq 0$, $\sigma_{ijl}(t) \geq 0$, and $v_{ijl}(t) \geq 0$ correspond to the transmission delays, $I_i(t)$ denote the external inputs at time t , and f_j and g_j are the activation functions of signal transmission.

Due to the fact that high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks, high-order neural networks have been the object of intensive analysis by numerous authors in the recent years. In particular, there have been extensive results on the problem of the existence and stability of equilibrium points and periodic solutions of HHNNs (1.1) in the literature. We refer the reader to [1–9] and the references cited therein. In fact, both continuous and discrete systems are very important in implementing and applications. The theory of calculus on time scales (see [10, 11] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [12] in order to unify continuous and discrete analysis, and it has a tremendous potential for application and has recently received much attention since his foundational work. Therefore, it is meaningful to study that on time scales which can unify the continuous and discrete situations.

Our purpose of this paper is to consider the model

$$\begin{aligned} x_i^\Delta(t) = & -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t - \gamma_{ij}(t))) \\ & + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - v_{ijl}(t))) \\ & + I_i(t), \quad i = 1, 2, \dots, n, t \in \mathbb{T}, \end{aligned} \quad (1.2)$$

where \mathbb{T} is an ω -periodic time scale which has the subspace topology inherited from the standard topology on \mathbb{R} , $I(t) = (I_1(t), I_2(t), \dots, I_n(t))^T$ is an input periodic vector function with period ω ; that is, there exists $\omega > 0$ such that $I_i(t + \omega) = I_i(t)$ ($i = 1, 2, \dots, n$) for all $t \in (0, +\infty) \cap \mathbb{T}$, and $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))^T$, and $g(x) = (g_1(x_1), g_2(x_2), \dots, g_n(x_n))^T$ is the activation function of the neurons.

System (1.2) is supplemented with initial values given by

$$x_i(s) = \varphi_i(s), \quad s \in [-\theta, 0] \cap \mathbb{T}, \quad i = 1, 2, \dots, n, \quad (1.3)$$

where $\varphi_i(\cdot)$ denotes continuous ω -periodic function defined on $[-\theta, 0] \cap \mathbb{T}$, $\theta = \max\{\gamma, \sigma, v\}$, $\gamma = \max_{1 \leq i, j \leq n} \{\gamma_{ij}^+\}$, $\sigma = \max_{1 \leq i, j, l \leq n} \{\sigma_{ijl}^+\}$, $v = \max_{1 \leq i, j, l \leq n} \{v_{ijl}^+\}$, $\gamma_{ij}^+ = \max_{t \in [0, \omega] \cap \mathbb{T}} \gamma_{ij}(t)$, $\sigma_{ijl}^+ = \max_{t \in [0, \omega] \cap \mathbb{T}} \sigma_{ijl}(t)$, $v_{ijl}^+ = \max_{t \in [0, \omega] \cap \mathbb{T}} v_{ijl}(t)$, $i, j, l = 1, 2, \dots, n$. To the best of our knowledge, this is the first paper to study the stability and existence of periodic solutions of (1.2).

Throughout this paper, we assume the following.

- (H₁) For $i, j, l = 1, 2, \dots, n$, $c_i(t)$, $a_{ij}(t)$, $b_{ijl}(t)$, $I_i(t)$, $\gamma_{ij}(t)$, $\sigma_{ijl}(t)$, $v_{ijl}(t)$ are positive continuous periodic functions with period $\omega > 0$, and $c_i(t)$ is regressive.
- (H₂) There exist positive constants $M_j, N_j, j = 1, 2, \dots, n$ such that $|f_j(x)| \leq M_j, |g_j(x)| \leq N_j$ for $j = 1, 2, \dots, n, x \in \mathbb{R}$.

(H₃) Functions $f_j(u), g_j(u)$ ($j = 1, 2, \dots, n$) satisfy the Lipschitz condition; that is, there exist constants $L_j, H_j > 0$ such that $|f_j(u_1) - f_j(u_2)| \leq L_j|u_1 - u_2|, |g_j(u_1) - g_j(u_2)| \leq H_j|u_1 - u_2|, j = 1, 2, \dots, n.$

2. Preliminaries

In this section, we will first recall some basic definitions and lemmas which are used in what follows.

Let \mathbb{T} be a nonempty closed subset (time scale) of \mathbb{R} . The forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t. \quad (2.1)$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum m , then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is right-dense continuous provided that it is continuous at right-dense point in \mathbb{T} and its left-side limits exist at left-dense points in \mathbb{T} . If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on \mathbb{T} .

For $y : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$ to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| [y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s] \right| < \varepsilon|\sigma(t) - s| \quad (2.2)$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t , then y is continuous at t .

A function $r : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive if

$$1 + \mu(t)r(t) \neq 0 \quad (2.3)$$

for all $t \in \mathbb{T}^k$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau)) \Delta \tau \right\} \quad \text{for } s, t \in \mathbb{T}, \quad (2.4)$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1 + hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases} \quad (2.5)$$

Let $p, q : \mathbb{T} \rightarrow \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu p q, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q). \quad (2.6)$$

Then the generalized exponential function has the following properties.

Lemma 2.1 (see [10]). *Assume that $p, q : \mathbb{T} \rightarrow \mathbb{R}$ are two regressive functions, then*

- (i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (iii) $e_p(t, \sigma(s)) = e_p(t, s)/(1 + \mu(s)p(s))$;
- (iv) $1/e_p(t, s) = e_{\ominus p}(t, s)$;
- (v) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;
- (vi) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (vii) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;
- (viii) $e_p(t, s)/e_q(t, s) = e_{p \ominus q}(t, s)$.

Lemma 2.2 (see [10]). *Assume that $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$, then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)). \quad (2.7)$$

Let y be right-dense continuous. If $Y^\Delta(t) = y(t)$, then one defines the delta integral by

$$\int_a^t y(s) \Delta s = Y(t) - Y(a). \quad (2.8)$$

Lemma 2.3. *If $a, b \in \mathbb{T}, \alpha, \beta \in \mathbb{R}$ and $f, g \in C(\mathbb{T}, \mathbb{R})$, then*

- (1) $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t$;
- (2) *If $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t) \Delta t \geq 0$;*
- (3) *If $|f(t)| \leq g(t)$ on $[a, b] := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t) \Delta t| \leq \int_a^b g(t) \Delta t$.*

In this paper, one assumes that $\bar{k} = \min\{[0, +\infty) \cap \mathbb{T}\}$. Clearly, from Lemma 2.3, one can obtain Lemma 2.4.

Lemma 2.4. *If $f, g \in C(\mathbb{T}, \mathbb{R})$, and $f(t) \leq g(t)$ on $t \in [\bar{k}, \bar{k} + \omega)$, then $\int_{\bar{k}}^{\bar{k} + \omega} f(t) \Delta t \leq \int_{\bar{k}}^{\bar{k} + \omega} g(t) \Delta t$.*

In the proof of our main result, one will use the following three lemmas which can be found in [13, 14].

Lemma 2.5 (see [13]). *Let $t_1, t_2 \in [\bar{k}, \bar{k} + \omega) \cap \mathbb{T}$, and $t \in \mathbb{T}$. If $f : \mathbb{T} \rightarrow \mathbb{R}$ is ω -periodic, then $f(t) \leq f(t_1) + \int_{\bar{k}}^{\bar{k} + \omega} |f^\Delta(s)| \Delta s$, and $f(t) \geq f(t_2) - \int_{\bar{k}}^{\bar{k} + \omega} |f^\Delta(s)| \Delta s$.*

Lemma 2.6 ([14], Cauchy-Schwarz inequality on time scales). *Let $a, b \in \mathbb{T}$. For rd -continuous functions $f, g : [a, b] \rightarrow \mathbb{R}$, one has $(\int_a^b |f(t)g(t)|\Delta t)^2 \leq \int_a^b |f(t)|^2 \Delta t \int_a^b |g(t)|^2 \Delta t$.*

Lemma 2.7. *Let $r : \mathbb{T} \rightarrow \mathbb{R}$ be right-dense continuous and regressive. $a \in \mathbb{T}$, and $y_a \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^\Delta(t) = r(t)y(t) + h(t), \quad y(a) = y_a \quad (2.9)$$

is given by

$$y(t) = e_r(t, a)y_a + \int_a^t e_r(t, \sigma(s))h(s)\Delta s. \quad (2.10)$$

For the sake of convenience, one introduces the following notations:

$$\begin{aligned} \bar{c}_i &= \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} c_i(t)\Delta t, & \bar{a}_{ij} &= \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} a_{ij}(t)\Delta t, & \bar{I}_i &= \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} I_i(t)\Delta t, \\ \overline{b}_{ijl} &= \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} b_{ijl}(t)\Delta t, & c_i^+ &= \max_{t \in [\bar{k}, \bar{k}+\omega] \cap \mathbb{T}} |c_i(t)|, & c_i^- &= \min_{t \in [\bar{k}, \bar{k}+\omega] \cap \mathbb{T}} |c_i(t)|, \\ a_{ij}^+ &= \max_{t \in [\bar{k}, \bar{k}+\omega] \cap \mathbb{T}} |a_{ij}(t)|, & b_{ijl}^+ &= \max_{t \in [\bar{k}, \bar{k}+\omega] \cap \mathbb{T}} |b_{ijl}(t)|, & I_i^+ &= \max_{t \in [\bar{k}, \bar{k}+\omega] \cap \mathbb{T}} |I_i(t)|, \end{aligned} \quad (2.11)$$

$$\begin{aligned} A_i(t) &= -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t - \gamma_{ij}(t))) \\ &+ \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - \nu_{ijl}(t))) + I_i(t), \quad i, j, l = 1, 2, \dots, n. \end{aligned}$$

Obliviously, for system (1.2), finding the periodic solutions is equivalent to finding those of the following boundary-value problem:

$$\begin{aligned} x_i^\Delta(t) &= -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t - \gamma_{ij}(t))) \\ &+ \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - \nu_{ijl}(t))) + I_i(t), \\ &t \in [\bar{k}, \bar{k} + \omega] \cap \mathbb{T}, \quad i = 1, 2, \dots, n, \\ x_i(\bar{k}) &= x_i(\bar{k} + \omega), \quad i = 1, 2, \dots, n. \end{aligned} \quad (2.12)$$

Now, one states Mawhin's continuous theorem [15].

Theorem 2.8. Let X and Z be two Banach spaces and let L be a Fredholm mapping of index zero. let $\bar{\Omega} \subset \mathbb{X}$ be an open bounded set and let $N : \bar{\Omega} \rightarrow \mathbb{Z}$ be a continuous operator which is L -compact on $\bar{\Omega}$. Assume that

- (1) for each $\lambda \in (0, 1)$, $x \in \partial\bar{\Omega} \cap \text{Dom } L$, $Lx \neq \lambda Nx$;
- (2) for each $x \in \partial\bar{\Omega} \cap \text{Ker } L$, $QNx \neq 0$;
- (3) $\deg(JNQx, \bar{\Omega} \cap \text{Ker } L, 0) \neq 0$, where $JQN : \text{Ker } L \rightarrow \text{Ker } L$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom } L$.

In order to apply Theorem 2.8 to system (2.12), we first define

$$\begin{aligned} X = Z &= \left\{ x = (x_1, x_2, \dots, x_n)^T \in C(\mathbb{T}, \mathbb{R}^n) : x(t + \omega) = x(t), t \in \mathbb{T} \right\}, \\ \|x\| &= \left\| (x_1, x_2, \dots, x_n)^T \right\| = \sum_{i=1}^n \max_{t \in [\bar{k}, \bar{k} + \omega] \cap \mathbb{T}} |x_i(t)| \end{aligned} \quad (2.13)$$

for any $x \in X$ (or Z). Then X and Z are Banach spaces with the norm $\|\cdot\|$. Let

$$Nx = (A_1(t), A_2(t), \dots, A_n(t))^T, \quad x \in X \quad (2.14)$$

$$Lx = (x_1^\Delta, x_2^\Delta, \dots, x_n^\Delta)^T \quad (2.15)$$

$$\begin{aligned} Px &= \left(\frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} x_1(t) \Delta t, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} x_2(t) \Delta t, \dots, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} x_n(t) \Delta t \right)^T, \\ Qz &= \left(\frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} z_1(t) \Delta t, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} z_2(t) \Delta t, \dots, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} z_n(t) \Delta t \right)^T, \quad z \in Z. \end{aligned} \quad (2.16)$$

Then it follows that

$$\begin{aligned} \text{Ker } L &= \left\{ (x_1, x_2, \dots, x_n)^T \in \mathbb{X} : (x_1, x_2, \dots, x_n)^T = (w_1(t), w_2(t), \dots, w_n(t))^T \in \mathbb{R}^n, t \in \mathbb{T} \right\}, \\ \text{Im } L &= \left\{ z \in \mathbb{Z} : \left(\frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} z_1(t) \Delta t, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} z_2(t) \Delta t, \dots, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k} + \omega} z_n(t) \Delta t \right)^T = 0 \right\} \end{aligned} \quad (2.17)$$

is closed in Z , $\dim \text{Ker } L = n = \text{codim Im } L$. It is not difficult to show that P and Q are continuous and satisfy $\text{Im } P = \text{Ker } L$, $\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)$. It is easy to see that $\text{Im } L$ is closed in Z , which leads to the following lemma.

Lemma 2.9. Let L and N be defined by (2.14) and (2.15), respectively, then L is a Fredholm operator of index zero.

Lemma 2.10. *Let L and N be defined by (2.14) and (2.15), respectively, suppose that Ω is an open bounded subset of $\text{Dom } L$, then N is L -compact on $\overline{\Omega}$.*

Proof. Through an easy computation, we find that the inverse $K_P : \text{Im } L \rightarrow \text{Ker } P \cap \text{Dom } L$ of L_P has the form

$$(K_P z)(t) = \int_{\bar{k}}^t z(s) \Delta s - \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} \int_{\bar{k}}^t z(s) \Delta s \Delta t. \tag{2.18}$$

Thus, the expression of QNx is

$$QNx = \left(\frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} A_1(t) \Delta t, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} A_2(t) \Delta t, \dots, \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} A_n(t) \Delta t \right)^T, \tag{2.19}$$

and then

$$\begin{aligned} K_P(I - Q)Nx = & \begin{pmatrix} \int_{\bar{k}}^t A_1(s) \Delta s \\ \vdots \\ \int_{\bar{k}}^t A_n(s) \Delta s \end{pmatrix} \\ & - \begin{pmatrix} \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} \int_{\bar{k}}^t A_1(s) \Delta s \Delta t \\ \vdots \\ \frac{1}{\omega} \int_{\bar{k}}^{\bar{k}+\omega} \int_{\bar{k}}^t A_n(s) \Delta s \Delta t \end{pmatrix} \\ & - \begin{pmatrix} \left(\frac{t - \bar{k}}{\omega} - \frac{1}{(\omega)^2} \int_{\bar{k}}^{\bar{k}+\omega} (t - \bar{k}) \Delta t \right) \int_{\bar{k}}^{\bar{k}+\omega} A_1(s) \Delta s \\ \vdots \\ \left(\frac{t - \bar{k}}{\omega} - \frac{1}{(\omega)^2} \int_{\bar{k}}^{\bar{k}+\omega} (t - \bar{k}) \Delta t \right) \int_{\bar{k}}^{\bar{k}+\omega} A_n(s) \Delta s \end{pmatrix}. \end{aligned} \tag{2.20}$$

Thus, QN and $K_P(I - Q)N$ are continuous. Since X is a Banach space, it is not difficult to show that $K_P(I - Q)N(\overline{\Omega})$ is compact. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L -compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. The proof of Lemma 2.10 is completed. \square

3. Existence of Periodic Solution

In this section, we study the existence of periodic solution of (1.2) based on Mawhin's continuation theorem.

Theorem 3.1. *Assume that (H_1) - (H_2) hold, then system (1.2) has at least one ω -periodic solution.*

Proof. Based on the Lemma 2.9 and Lemma 2.10, now, what we need to do is just to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0, 1)$, we have

$$\begin{aligned}
x_i^\Delta(t) = \lambda & \left[-c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t - \gamma_{ij}(t))) \right. \\
& + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - \nu_{ijl}(t))) \\
& \left. + I_i(t), t \in \mathbb{T} \cap [\bar{k}, \bar{k} + \omega] \right], \quad i = 1, 2, \dots, n, \\
x_i(\bar{k}) & = x_i(\bar{k} + \omega), \quad i = 1, 2, \dots, n.
\end{aligned} \tag{3.1}$$

For the sake of convenience, defined $\|x\|_2$ by

$$\|x\|_2 = \left(\int_{\bar{k}}^{\bar{k}+\omega} |x(t)|^2 \Delta t \right)^{1/2} \tag{3.2}$$

for $x \in C(\mathbb{T}, \mathbb{R})$. Suppose that $(x_1(t), x_2(t), \dots, x_n(t))^T \in X$ is a solution of system (3.1) for a certain $\lambda \in (0, 1)$. Integrating (3.1) over $[\bar{k}, \bar{k} + \omega]$, we obtain

$$\int_{\bar{k}}^{\bar{k}+\omega} A_i(t) \Delta t = 0. \tag{3.3}$$

Hence

$$\begin{aligned}
& \int_{\bar{k}}^{\bar{k}+\omega} c_i(s)x_i(s) \Delta s \\
& = \left[\int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^n a_{ij}(t)f_j(x_j(t - \gamma_{ij}(t))) + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - \nu_{ijl}(t))) + I_i(t) \right] \Delta t, \\
& \qquad \qquad \qquad i = 1, 2, \dots, n.
\end{aligned} \tag{3.4}$$

Let $\zeta_i, \eta_i \in \mathbb{T} \cap [\bar{k}, \bar{k} + \omega]$, such that $x_i(\zeta_i) = \inf_{t \in \mathbb{T} \cap [\bar{k}, \bar{k} + \omega]} x_i(t)$, $x_i(\eta_i) = \sup_{t \in \mathbb{T} \cap [\bar{k}, \bar{k} + \omega]} x_i(t)$. Then by (3.4) and Lemma 2.4, we have

$$\begin{aligned}
\omega \bar{c}_i x_i(\zeta_i) &\leq \int_{\bar{k}}^{\bar{k} + \omega} \left[\sum_{j=1}^n a_{ij}(s) f_j(x_j(s - \gamma_{ij}(s))) \right. \\
&\quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) g_j(x_j(s - \sigma_{ijl}(s))) g_l(x_l(s - \nu_{ijl}(s))) + I_i(s) \right] \Delta s \\
&\leq \int_{\bar{k}}^{\bar{k} + \omega} \sum_{j=1}^n |a_{ij}(s)| |f_j(x_j(s - \gamma_{ij}(s)))| \Delta s \\
&\quad + \int_{\bar{k}}^{\bar{k} + \omega} \sum_{j=1}^n \sum_{l=1}^n |b_{ijl}(s)| |g_j(x_j(s - \sigma_{ijl}(s)))| |g_l(x_l(s - \nu_{ijl}(s)))| \Delta s + \int_{\bar{k}}^{\bar{k} + \omega} I_i(s) \Delta s \\
&\leq \omega \left[\sum_{j=1}^n a_{ij}^+ M_j + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ N_j N_l + I_i^+ \right], \quad i = 1, 2, \dots, n.
\end{aligned} \tag{3.5}$$

Hence $x_i(\zeta_i) \leq (1/\bar{c}_i) \{ [\sum_{j=1}^n a_{ij}^+ M_j + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ N_j N_l + I_i^+] \} := B_i$, $i = 1, 2, \dots, n$. By (3.4) and Lemma 2.4, we can also have

$$\begin{aligned}
\omega \bar{c}_i x_i(\eta_i) &\geq - \int_{\bar{k}}^{\bar{k} + \omega} \left[\sum_{j=1}^n a_{ij}(s) f_j(x_j(s - \gamma_{ij}(s))) \right. \\
&\quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) g_j(x_j(s - \sigma_{ijl}(s))) g_l(x_l(s - \nu_{ijl}(s))) + I_i(s) \right] \Delta s \\
&\geq - \int_{\bar{k}}^{\bar{k} + \omega} \sum_{j=1}^n |a_{ij}(s)| |f_j(x_j(s - \gamma_{ij}(s)))| \Delta s \\
&\quad - \int_{\bar{k}}^{\bar{k} + \omega} \sum_{j=1}^n \sum_{l=1}^n |b_{ijl}(s)| |g_j(x_j(s - \sigma_{ijl}(s)))| |g_l(x_l(s - \nu_{ijl}(s)))| \Delta s - \int_{\bar{k}}^{\bar{k} + \omega} |I_i(s)| \Delta s \\
&\geq -\omega \left[\sum_{j=1}^n a_{ij}^+ M_j + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ N_j N_l + I_i^+ \right], \quad i = 1, 2, \dots, n.
\end{aligned} \tag{3.6}$$

Hence $x_i(\eta_i) \geq (-1/\bar{c}_i) \{ [\sum_{j=1}^n a_{ij}^+ M_j + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ N_j N_l + I_i^+] \} = -B_i$, $i = 1, 2, \dots, n$. From (3.1), (3.4), and Lemma 2.6, we have

$$\begin{aligned}
& \int_{\bar{k}}^{\bar{k}+\omega} |x_i^\Delta(t)| \Delta t \\
& \leq \int_{\bar{k}}^{\bar{k}+\omega} |c_i(t)| |x_i(t)| \Delta t + \int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^n |a_{ij}(t)| |f_j(x_j(t - \gamma_{ij}(t)))| \Delta t \\
& \quad + \int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^n \sum_{l=1}^n |b_{ijl}(t)| |g_j(x_j(t - \sigma_{ijl}(t)))| |g_l(x_l(t - \nu_{ijl}(t)))| \Delta t + \int_{\bar{k}}^{\bar{k}+\omega} |I_i(t)| \Delta t \\
& \leq \left(\int_{\bar{k}}^{\bar{k}+\omega} |c_i(t)|^2 \Delta t \right)^{1/2} \left(\int_{\bar{k}}^{\bar{k}+\omega} |x_i(t)|^2 \Delta t \right)^{1/2} \\
& \quad + \sum_{j=1}^n \left(\int_{\bar{k}}^{\bar{k}+\omega} |a_{ij}(t)|^2 \Delta t \right)^{1/2} \left(\int_{\bar{k}}^{\bar{k}+\omega} |f_j(x_j(t - \gamma_{ij}(t)))|^2 \Delta t \right)^{1/2} \\
& \quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ \left(\int_{\bar{k}}^{\bar{k}+\omega} |g_j(x_j(t - \sigma_{ijl}(t)))|^2 \Delta t \right)^{1/2} \left(\int_{\bar{k}}^{\bar{k}+\omega} |g_l(x_l(t - \nu_{ijl}(t)))|^2 \Delta t \right)^{1/2} + I_i^+ \omega \\
& \leq (\omega)^{1/2} c_i^+ \|x_i\|_2 + \sum_{j=1}^n \omega a_{ij}^+ M_j + \sum_{j=1}^n \sum_{l=1}^n (\omega) b_{ijl}^+ N_j N_l + I_i^+ \omega.
\end{aligned} \tag{3.7}$$

From Lemma 2.7 and (3.1), for $i = 1, 2, \dots, n$, we can obtain

$$\begin{aligned}
x_i(t) &= e_{-\lambda c_i(t)}(t, \bar{k}) x_i(\bar{k}) \\
& \quad + \int_{\bar{k}}^t \lambda e_{-\lambda c_i(t)}(t, \sigma(s)) \left[\sum_{j=1}^n a_{ij}(s) f_j(x_j(s - \gamma_{ij}(s))) \right. \\
& \quad \quad \quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) g_j(x_j(s - \sigma_{ijl}(s))) g_l(x_l(s - \nu_{ijl}(s))) + I_i(s) \right] \Delta s.
\end{aligned} \tag{3.8}$$

Hence

$$\begin{aligned}
 |x_i(t)| &\leq \left| x_i(\bar{k}) \right| + \sum_{j=1}^n \int_{\bar{k}}^{\bar{k}+\omega} |a_{ij}(s)| |f_j(x_j(s - \gamma_{ij}(s)))| \Delta s \\
 &\quad + \sum_{j=1}^n \sum_{l=1}^n \int_{\bar{k}}^{\bar{k}+\omega} |b_{ijl}(s)| |g_j(x_j(s - \sigma_{ijl}(s)))| |g_l(x_l(s - \nu_{ijl}(s)))| \Delta s \\
 &\quad + \int_{\bar{k}}^{\bar{k}+\omega} |I_i(s)| \Delta s \tag{3.9} \\
 &\leq \left| x_i(\bar{k}) \right| + \sum_{j=1}^n \bar{a}_{ij} N_j \omega + \sum_{j=1}^n \sum_{l=1}^n \bar{b}_{ijl} N_j N_l \omega + \bar{I}_i \omega \\
 &=: u_i, \quad i = 1, 2, \dots, n,
 \end{aligned}$$

that is,

$$\|x_i\|_2 = \left(\int_{\bar{k}}^{\bar{k}+\omega} |x_i(t)|^2 \Delta t \right)^{1/2} \leq u_i(\omega)^{1/2}, \quad i = 1, 2, \dots, n. \tag{3.10}$$

Substituting (3.10) into (3.7), we have

$$\int_{\bar{k}}^{\bar{k}+\omega} |x_i^\Delta(t)| \Delta t \leq \omega c_i^+ u_i + \sum_{j=1}^n \omega a_{ij}^+ M_j + \sum_{j=1}^n \sum_{l=1}^n (\omega) b_{ijl}^+ N_j N_l + I_i^+ \omega, \quad i = 1, 2, \dots, n. \tag{3.11}$$

From Lemma 2.5, we have

$$\begin{aligned}
 x_i(t) &\leq x_i(\zeta_i) + \int_{\bar{k}}^{\bar{k}+\omega} |x_i^\Delta(t)| \Delta t, \quad i = 1, 2, \dots, n, \\
 x_i(t) &\geq x_i(\eta_i) - \int_{\bar{k}}^{\bar{k}+\omega} |x_i^\Delta(t)| \Delta t, \quad i = 1, 2, \dots, n.
 \end{aligned} \tag{3.12}$$

From (3.5), (3.6) and (3.11), there exist positive constants ξ_i ($i = 1, 2, \dots, n$) such that for $t \in [\bar{k}, \bar{k} + \omega] \cap \mathbb{T}$, $|x_i(t)| \leq \xi_i$, $i = 1, 2, \dots, n$. Clearly, ξ_i ($i = 1, 2, \dots, n$) is independent of λ . Denote $H^* = \sum_{i=1}^n \xi_i + C$, where $C > 0$ is taken sufficiently large so that

$$\min_{1 \leq i \leq n} \bar{c}_i H^* > n \max_{1 \leq i \leq n} \left(\left| \bar{I}_i \right| + \sum_{j=1}^n \left| \bar{a}_{ij} \right| M_j + \sum_{j=1}^n \sum_{l=1}^n \left| \bar{b}_{ijl} \right| N_j N_l \right). \tag{3.13}$$

Now we take $\Omega = \{(x_1(t), x_2(t), \dots, x_n(t))^T : \|(x_1(t), x_2(t), \dots, x_n(t))^T\| < H^*\}$. Thus (1) of Theorem 2.8 is satisfied. When $(x_1(t), x_2(t), \dots, x_n(t))^T \in \partial\Omega \cap \mathbb{R}^n$, $(x_1(t), x_2(t), \dots, x_n(t))^T$ is a constant vector in \mathbb{R}^n with $|x_1| + |x_2| + \dots + |x_n| = H^*$, then

$$QN(x_1(t), x_2(t), \dots, x_n(t))^T = \left(\begin{aligned} &-\bar{c}_i x_i(t) + \sum_{j=1}^n \bar{a}_{ij} f_j(x_j(t - \gamma_{ij}(t))) \\ &+ \sum_{j=1}^n \sum_{l=1}^n \bar{b}_{ijl} g_j(x_j(t - \sigma_{ijl}(t))) g_l(x_l(t - \nu_{ijl}(t))) + \bar{I}_i \end{aligned} \right)^{n \times 1}. \quad (3.14)$$

Therefore

$$\begin{aligned} \left\| QN(x_1(t), x_2(t), \dots, x_n(t))^T \right\| &= \sum_{i=1}^n \left| \bar{c}_i x_i(t) - \sum_{j=1}^n \bar{a}_{ij} f_j(x_j(t - \gamma_{ij}(t))) \right. \\ &\quad \left. - \sum_{j=1}^n \sum_{l=1}^n \bar{b}_{ijl} g_j(x_j(t - \sigma_{ijl}(t))) g_l(x_l(t - \nu_{ijl}(t))) - \bar{I}_i \right| \\ &\geq \sum_{i=1}^n \bar{c}_i |x_i(t)| - \sum_{i=1}^n \sum_{j=1}^n |\bar{a}_{ij}| M_j \\ &\quad - \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n \bar{b}_{ijl} N_j N_l - \sum_{i=1}^n |\bar{I}_i| \\ &\geq \sum_{i=1}^n (\bar{c}_i |x_i(t)|) \\ &\quad - \sum_{i=1}^n \left(|\bar{I}_i| + \sum_{j=1}^n |\bar{a}_{ij}| M_j + \sum_{j=1}^n \sum_{l=1}^n \bar{b}_{ijl} N_j N_l \right) \\ &\geq \min_{1 \leq i \leq n} (\bar{c}_i) \sum_{i=1}^n |x_i(t)| \\ &\quad - n \max_{1 \leq i \leq n} \left(|\bar{I}_i| + \sum_{j=1}^n |\bar{a}_{ij}| M_j + \sum_{j=1}^n \sum_{l=1}^n \bar{b}_{ijl} N_j N_l \right) \\ &> 0. \end{aligned} \quad (3.15)$$

Consequently,

$$QN(x_1(t), x_2(t), \dots, x_n(t))^T \neq (0, 0, \dots, 0)^T, \quad (3.16)$$

for $(x_1(t), x_2(t), \dots, x_n(t))^T \in \partial\Omega \cap \text{Ker } L$. This satisfies condition (2) of Theorem 2.8. Define $\Psi : \text{Ker } L \times [0, 1] \rightarrow X$ by

$$\Psi(x_1, x_2, \dots, x_n, \mu) = -\mu(x_1, x_2, \dots, x_n)^T + (1 - \mu)QN(x_1(t), x_2(t), \dots, x_n(t))^T \quad (3.17)$$

When $(x_1(t), x_2(t), \dots, x_n(t))^T \in \partial\Omega \cap \text{Ker } L$, $(x_1, x_2, \dots, x_n)^T$ is a constant vector in \mathbb{R}^n with $\sum_{i=1}^n |x_i| = H^*$, we easily have $\Psi(x_1, x_2, \dots, x_n, \mu) \neq (0, 0, \dots, 0)^T$. Therefore

$$\begin{aligned} \deg \left(QN(x_1(t), x_2(t), \dots, x_n(t))^T, \Omega \cap \text{Ker } L, (0, 0, \dots, 0)^T \right) \\ = \deg \left((-x_1(t), -x_2(t), \dots, -x_n(t))^T, \Omega \cap \text{Ker } L, (0, 0, \dots, 0)^T \right) \neq 0. \end{aligned} \quad (3.18)$$

Condition (3) of Theorem 2.8 is also satisfied. Thus, by Theorem 2.8 we can obtain that $Lx = Nx$ has at least one solution in X . That is, system (1.2) has at least one ω -periodic solution. The proof is complete. \square

4. Global Exponential Stability of Periodic Solution

In this section, we will construct suitable Lyapunov functions to study the global exponential stability of the periodic solution of (1.2) on time scales. So first we will introduce some definitions.

Definition 4.1 (see [12]). A function f from \mathbb{T} to \mathbb{R} is positively regressive if $1 + \mu(t)f(t) > 0$ for every $t \in \mathbb{T}$.

Denote \mathfrak{R}^+ is the set of positively regressive functions from \mathbb{T} to \mathbb{R} , and denote $\mathbb{T}^+ = [0, +\infty) \cap \mathbb{T}$.

Definition 4.2. The periodic solution $x^*(t)$ of system (1.2) is said to be exponentially stable if there exists a positive constant α with $-\alpha \in \mathfrak{R}^+$ such that for every $\delta \in \mathbb{T}$, there exists $N = N(\delta) \geq 1$ such that the solution $x(t)$ of (1.2) through $(\delta, \varphi(\delta))$ satisfies $\|x(t) - x^*(t)\| \leq N\|\varphi(\delta) - x^*(\delta)\|e_{-\alpha}(t, \delta)$, $t \in \mathbb{T}^+$ where $\|\varphi(\delta) - x^*(\delta)\| = \sum_{i=1}^n \max_{\delta \in [-\theta, 0] \cap \mathbb{T}} |\varphi_i(\delta) - x_i^*(\delta)|$.

Theorem 4.3. Assume that (H_1) – (H_3) hold. Suppose further that there exists n positive constants $\epsilon_i > 0, i = 1, 2, \dots, n$ such that

$$(H_4)$$

$$-\epsilon_i c_i^- + \sum_{j=1}^n a_{ij}^+ M_j e_j + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ (H_j N_l e_j + H_l N_j e_l) < 0, \quad i = 1, 2, \dots, n, \quad (4.1)$$

then the ω -periodic solution of system (1.2) is globally exponentially stable.

Proof. According to Theorem 3.1, we know that (1.2) has an ω -periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$. Suppose that $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is an arbitrary solution of (1.2). Then it follows from system (1.2) that

$$\begin{aligned} (x_i(t) - x_i^*(t))^\Delta &= -c_i(t)(x_i(t) - x_i^*(t)) + \sum_{j=1}^n a_{ij}(t) \left(f_j(x_j(t - \gamma_{ij}(t))) - f_j^*(x_j(t - \gamma_{ij}(t))) \right) \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) (g_j(x_j(t - \sigma_{ijl}(t))) g_l(x_l(t - \nu_{ijl}(t)))) \\ &\quad - g_j(x_j^*(t - \sigma_{ijl}(t))) g_l(x_l^*(t - \nu_{ijl}(t))), \quad i = 1, 2, \dots, n, \end{aligned} \quad (4.2)$$

with initial values given by

$$x_i(s) = \varphi_i(s), \quad s \in [-\theta, 0] \cap \mathbb{T}, \quad i = 1, 2, \dots, n, \quad (4.3)$$

where θ is defined as in(1.3) . If (H_4) holds, it can always find a small enough constant $\alpha > 0$ satisfying $\forall t \in \mathbb{T}, 1 - \mu(t)\alpha > 0$, namely, $-\alpha \in \mathfrak{R}^+$ such that

$$\begin{aligned} &(-c_i^- + \alpha)e_i + \sum_{j=1}^n a_{ij}^+ M_j e_\alpha(t, t - \gamma_{ij}(t)) e_j \\ &+ \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ [H_j N_l e_j e_\alpha(t, t - \sigma_{ijl}(t)) + H_l N_j e_l e_\alpha(t, t - \nu_{ijl}(t))] < 0, \quad i = 1, 2, \dots, n. \end{aligned} \quad (4.4)$$

We define a Lyapunov function $H = (h_1, h_2, \dots, h_n)^T$ by $h_i = e_\alpha(t, \delta) |x_i(t) - x_i^*(t)|, \delta \in [-\theta, 0] \cap \mathbb{T}, i = 1, 2, \dots, n$. In view of (4.2), we obtain

$$\begin{aligned} [h_i^\Delta(t)]^+ &= \alpha e_\alpha(t, \delta) |x_i(t) - x_i^*(t)| + e_\alpha(\sigma(t), \delta) \text{sign}(x_i(t) - x_i^*(t)) \\ &\quad \times \left\{ -c_i(t)(x_i(t) - x_i^*(t)) \right. \\ &\quad + \sum_{j=1}^n a_{ij}(t) \left(f_j(x_j(t - \gamma_{ij}(t))) - f_j(x_j^*(t - \gamma_{ij}(t))) \right) \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) (g_j(x_j(t - \sigma_{ijl}(t))) g_l(x_l(t - \nu_{ijl}(t)))) \\ &\quad \left. - g_j(x_j^*(t - \sigma_{ijl}(t))) g_l(x_l^*(t - \nu_{ijl}(t))) \right\} \end{aligned}$$

$$\begin{aligned}
& \leq \alpha e_\alpha(t, \delta) |x_i(t) - x_i^*(t)| + e_\alpha(\sigma(t), \delta) \\
& \quad \times \left\{ -c_i(t) |x_i(t) - x_i^*(t)| + \sum_{j=1}^n |a_{ij}(t)| |x_j(t - \gamma_{ij}(t)) - x_j^*(t - \gamma_{ij}(t))| M_j \right. \\
& \quad \left. + \sum_{j=1}^n \sum_{l=1}^n |b_{ijl}(t)| \left[|H_j N_l| |x_j(t - \sigma_{ijl}(t)) - x_j^*(t - \sigma_{ijl}(t))| \right. \right. \\
& \quad \left. \left. + H_l N_j |x_l(t - v_{ijl}(t)) - x_l^*(t - v_{ijl}(t))| \right] \right\} \\
& \leq e_\alpha(\sigma(t), \delta) \\
& \quad \times \left\{ (-c_i(t) + \alpha) |x_i(t) - x_i^*(t)| + \sum_{j=1}^n |a_{ij}(t)| |x_j(t - \gamma_{ij}(t)) - x_j^*(t - \gamma_{ij}(t))| M_j \right. \\
& \quad \left. + \sum_{j=1}^n \sum_{l=1}^n |b_{ijl}(t)| \left[|H_j N_l| |x_j(t - \sigma_{ijl}(t)) - x_j^*(t - \sigma_{ijl}(t))| \right. \right. \\
& \quad \left. \left. + H_l N_j |x_l(t - v_{ijl}(t)) - x_l^*(t - v_{ijl}(t))| \right] \right\} \\
& \leq [1 + \mu(t)\alpha] \left\{ (-c_i^- + \alpha) h_i(t) + \sum_{j=1}^n a_{ij}^+ M_j e_\alpha(t, t - \gamma_{ij}(t)) h_j(t - \gamma_{ij}(t)) \right. \\
& \quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ \times [H_j N_l e_\alpha(t, t - \sigma_{ijl}(t)) h_j(t - \sigma_{ijl}(t)) \right. \\
& \quad \left. + H_l N_j e_\alpha(t, t - v_{ijl}(t)) h_l(t - v_{ijl}(t))] \right\}, \\
& \qquad \qquad \qquad i = 1, 2, \dots, n.
\end{aligned} \tag{4.5}$$

Defining the curve $\rho = \{\underline{\omega}(l) : \underline{\omega}_i = \epsilon_i l, l > 0, i = 1, 2, \dots, n\}$ and the set $\Omega(\underline{\omega}) = \{\underline{u} : 0 \leq \underline{u} \leq \underline{\omega}\}$, $S_i(\underline{\omega}) = \{\underline{u} \in \Omega(\underline{\omega}) : \underline{u}_i = \underline{\omega}_i\}$, $i = 1, 2, \dots, n$. It is obvious that if $l < l'$, then $\Omega(\underline{\omega}(l)) \subset \Omega(\underline{\omega}(l'))$. We will prove that the zero solution of (4.2) is exponential stable, namely, there exists a constant $\beta > 0$ such that

$$\|x(t) - x^*(t)\| \leq N(\delta) e_{-\beta}(t, \delta) \|\varphi(t) - x^*(t)\|, \quad t \geq 0. \tag{4.6}$$

Let $e^M = \max_{1 \leq i \leq n} \{\epsilon_i\}$, $e^m = \min_{1 \leq i \leq n} \{\epsilon_i\}$, $l_0 = (1 - \delta) |\varphi_i - x_i^*|_0 / e^m$, where $-\delta \geq 0$ is a constant, $|\varphi_i - x_i^*|_0 = \max_{1 \leq i \leq n} \{\max_{\delta \in [-\theta, 0]} |\varphi_i(\delta) - x_i^*(\delta)|\}$. Then, $\{ |H| : |H| = e_\alpha(t, \delta) |\varphi(\delta) - x^*(\delta)|, -\theta \leq t \leq \delta \leq 0 \} \subset \Omega(\underline{\omega}(l_0))$, namely,

$$|h_i(\delta)| = e_\alpha(t, \delta) |\varphi_i(\delta) - x_i^*(\delta)| < \epsilon_i l_0, \quad -\theta \leq \delta \leq 0, \quad i = 1, 2, \dots, n. \tag{4.7}$$

We can claim that $|h_i(t)| < \epsilon_i l_0$ for $t > 0, i = 1, 2, \dots, n$. If it is not true, then there exist some i and $t_1 (t_1 > 0)$ such that $|h_i(t_1)| = \epsilon_i l_0, [h_i^\Delta(t_1)]^+ \geq 0$ and $|h_i(t)| \leq \epsilon_i l_0$ for $-\theta \leq t \leq t_1, i = 1, 2, \dots, n$. However, from (4.4) and (4.5), we get

$$\begin{aligned} [h_i^\Delta(t_1)]^+ &\leq [1 + \mu(t)\alpha] \left[(-c_i^- + \alpha)\epsilon_i + \sum_{j=1}^n a_{ij}^+ M_j e_\alpha(t, t - \gamma_{ij}(t)) \epsilon_j \right. \\ &\quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}^+ [H_j N_l e_\alpha(t, t - \sigma_{ijl}(t)) \epsilon_j + H_l N_j e_\alpha(t, t - \nu_{ijl}(t)) \epsilon_l] \right] l_0 < 0 \end{aligned} \quad (4.8)$$

for $t > 0, i = 1, 2, \dots, n$, this is a contradiction. So $|h_i(t)| < \epsilon_i l_0$, for $t \geq 0$. Also

$$|x_i(t) - x_i^*(t)| < e_{\Theta\alpha}(t, \delta) \epsilon_i l_0 \leq \frac{1}{e^m} e_{\Theta\alpha}(t, \delta) \epsilon_i (1 - \delta) |\varphi_i - x_i^*|_0, \quad i = 1, 2, \dots, n, t \geq 0, \quad (4.9)$$

which means that

$$\|x(t) - x^*(t)\| \leq \frac{1}{e^m} e_{\Theta\alpha}(t, \delta) e^M (1 - \delta) \|\varphi(\delta) - x^*(\delta)\|. \quad (4.10)$$

Denote $-\beta = \Theta\alpha = -\alpha/(1 + \mu\alpha) \in \mathfrak{R}^+, N = N(\delta) = (e^M/e^m)(1 - \delta) > 1$, in view of (4.10), we have

$$\|x(t) - x^*(t)\| \leq N e_{-\beta}(t, \delta) \|\varphi(s) - x^*(s)\|. \quad (4.11)$$

From Definition 4.2, the periodic solution of system (1.2) is globally exponentially stable. The proof is complete. \square

5. An Example

Let $\mathbb{T} = \mathbb{R}$, in this case, $x^\Delta(t) = dx(t)/dt$. Consider the following equation

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -c_i(t)x_i(t) + \sum_{j=1}^2 a_{ij}(t)f_j(x_j(t - \gamma_{ij}(t))) \\ &\quad + \sum_{j=1}^2 \sum_{l=1}^2 b_{ijl}(t)g_j(x_j(t - \sigma_{ijl}(t)))g_l(x_l(t - \nu_{ijl}(t))) + I_i(t), \quad i = 1, 2, \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} f_1(x_1) &= \sin\left(\frac{1}{2^{1/2}}x_1\right), & f_2(x_2) &= \sin\left(\frac{1}{2^{3/2}}x_2\right), \\ g_1(x_1) &= \arctan\left(\frac{1}{2^{1/2}}x_1\right), & g_2(x_2) &= \arctan\left(\frac{1}{2^{3/2}}x_2\right). \end{aligned} \quad (5.2)$$

Obviously, $f_i(x_i), g_i(x_i) (i = 1, 2)$ satisfy (H_1) and (H_2) , and

$$H_1 = H_2 = M_1 = M_2 = 1, \quad N_1 = N_2 = \frac{\pi}{2}. \quad (5.3)$$

Take

$$\begin{aligned} a_{11}(t) &= 1 + \cos(2\pi t), & a_{12}(t) &= 2 + \cos(2\pi t), & a_{21}(t) &= 2 + \cos(2\pi t), & a_{22}(t) &= 3 + \cos(2\pi t), \\ c_1(t) &= 20 + 5 \sin(2\pi t), & c_2(t) &= 33 + 16 \sin(2\pi t), & I_1(t) &= 1 + \sin(2\pi t), & I_2(t) &= 1 + \cos(2\pi t), \\ b_{111}(t) &= b_{222}(t) = \frac{1}{4} + \frac{1}{4} \sin(2\pi t), & b_{112}(t) &= b_{212}(t) = \frac{1}{3} + \frac{1}{3} \cos(2\pi t), \\ b_{121}(t) &= b_{221}(t) = \frac{1}{5} + \frac{1}{5} \cos(2\pi t), & b_{122}(t) &= b_{211}(t) = \frac{1}{6} + \frac{1}{6} \sin(2\pi t). \end{aligned} \quad (5.4)$$

One can verify that (H_1) is satisfied, and $\omega = 1$, $c_1^- = 15$, $c_2^- = 17$, $a_{11}^+ = 2$, $a_{12}^+ = 3$, $a_{21}^+ = 3$, $a_{22}^+ = 4$, $b_{111}^+ = b_{222}^+ = 1/2$, $b_{112}^+ = b_{212}^+ = 2/3$, $b_{121}^+ = b_{221}^+ = 2/5$, $b_{122}^+ = b_{211}^+ = 1/3$, so, if we take $\epsilon_1 = \epsilon_2 = 1$, we can obtain

$$\begin{aligned} -\epsilon_1 c_1^- + \sum_{j=1}^2 a_{1j}^+ M_j \epsilon_j + \sum_{j=1}^2 \sum_{l=1}^2 b_{1jl}^+ (H_j N_l \epsilon_j + H_l N_j \epsilon_l) &= -15 + 5 + \frac{19}{10} \pi < 0, \\ -\epsilon_2 c_2^- + \sum_{j=1}^2 a_{2j}^+ M_j \epsilon_j + \sum_{j=1}^2 \sum_{l=1}^2 b_{2jl}^+ (H_j N_l \epsilon_j + H_l N_j \epsilon_l) &= -17 + 7 + \frac{19}{10} \pi < 0. \end{aligned} \quad (5.5)$$

Condition (4.2) is satisfied. From Theorem 3.1 and 4.3, we know that (5.1) has at least one 1-periodic solution and this solution is exponential stability.

6. Conclusion

Sufficient conditions are derived to guarantee the stability and existence of periodic solutions for a class of delayed high-order Hopfield neural networks on time scales. To the best of our knowledge, the results presented here have been not appeared in the related literature. In fact, both continuous and discrete systems, are very important in implementing and applications. But it is troublesome to study the existence and stability of periodic solutions for continuous and discrete systems respectively. Therefore, it is meaningful to study that on time scales which can unify the continuous and discrete situations. Also, our methods used in this paper may be applied to some other systems.

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References

- [1] Z. Wang, J. Fang, and X. Liu, "Global stability of stochastic high-order neural networks with discrete and distributed delays," *Chaos, Solitons & Fractals*, vol. 36, no. 2, pp. 388–396, 2008.
- [2] B. Xiao and H. Meng, "Existence and exponential stability of positive almost periodic solutions for high-order Hopfield neural networks," *Applied Mathematical Modelling*, vol. 33, no. 1, pp. 532–542, 2009.
- [3] X. Yi, J. Shao, Y. Yu, and B. Xiao, "New convergence behavior of high-order Hopfield neural networks with time-varying coefficients," *Journal of Computational and Applied Mathematics*, vol. 219, no. 1, pp. 216–222, 2008.
- [4] B. Xu, Q. Wang, and X. Liao, "Stability analysis of high-order Hopfield type neural networks with uncertainty," *Neurocomputing*, vol. 71, no. 4–6, pp. 508–512, 2008.
- [5] X.-Y. Lou and B.-T. Cui, "Novel global stability criteria for high-order Hopfield-type neural networks with time-varying delays," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 144–158, 2007.
- [6] B. Xu, X. Liu, and X. Liao, "Global exponential stability of high order Hopfield type neural networks," *Applied Mathematics and Computation*, vol. 174, no. 1, pp. 98–116, 2006.
- [7] B. Xiao and H. Meng, "Existence and exponential stability of positive almost periodic solutions for high-order Hopfield neural networks," *Applied Mathematical Modelling*, vol. 33, no. 1, pp. 532–542, 2009.
- [8] B. Liu and L. Huang, "Existence and exponential stability of periodic solutions for a class of Cohen-Grossberg neural networks with time-varying delays," *Chaos, Solitons & Fractals*, vol. 32, no. 2, pp. 617–627, 2007.
- [9] F. Zhang and Y. Li, "Almost periodic solutions for higher-order Hopfield neural networks without bounded activation functions," *Electronic Journal of Differential Equations*, vol. 2007, no. 99, pp. 1–10, 2007.
- [10] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
- [11] M. Bohner and A. Peterson, Eds., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.
- [12] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.
- [13] M. Bohner, M. Fan, and J. Zhang, "Existence of periodic solutions in predator-prey and competition dynamic systems," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1193–1204, 2006.
- [14] F.-H. Wong, C.-C. Yeh, S.-L. Yu, and C.-H. Hong, "Young's inequality and related results on time scales," *Applied Mathematics Letters*, vol. 18, no. 9, pp. 983–988, 2005.
- [15] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, vol. 40 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1979.