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Research Article

On Some Properties of Focal Points

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We consider some dynamical properties of two-dimensional maps having an inverse with vanishing denominator. We put in evidence a link between a fixed point of a map with fractional inverse and a focal point of this inverse.

1. Introduction

A specific class of maps, defined by $x' = F(x, y)$, $y' = G(x, y)$, with at least one of the components F or G defined by a fractional rational function, has been studied in the work in [1–10], and has very interesting properties. Some peculiar dynamical properties have been evidenced and observed in iterated maps (or in one of the inverses) having a vanishing denominator or having the form $0/0$ in a point of $\mathbb{R}^2$. This characteristic has revealed new types of singularities in the phase plane, such as focal points and prefocal curves. The presence of these sets may cause new kinds of bifurcations generated by contacts between them and other singularities, which give rise to new dynamic phenomena and new structures of basins and invariant sets.

In this paper, which is a continuation of a former study [10], we present results on such singularities. We can distinguish typical scenarios between the map and its inverse. We recall definitions and main properties as given in some papers by Bischi et al. (see, e.g., [2–8]). A *prefocal curve* of a map is a set of points, whose image by one inverse is reduced into a single point called *focal point*.

We put in evidence necessary conditions and/or sufficient for a focal point to be a fixed point of the inverse map.

This paper is organized as follows. In Section 2 we present some basic definitions and facts about the key role of focal points and prefocal curves. In Section 3, we give our main results about the link between the property of the basin of attraction of a map T being connected (resp., nonconnected) and focal points of T^{-1} being inside (resp., outside) of this attraction basin, and the link existing between fixed point of this map and the focal point of its inverse. Section 4 is devoted to applications to illustrate our results, two examples are considered with one focal point and two focal points.

2. Definitions and Geometric Localization of Focal Point, Prefocal Curve

In this paper, we use the notation introduced in [6–8]. Definitions and generic properties given here concern fractional maps $(x, y) \to (x', y') = T(x, y)$ of the form:

$$T : \begin{cases} x' = F(x, y), \\ y' = \dfrac{N(x, y)}{D(x, y)}, \end{cases} \tag{2.1}$$

where it is assumed that the functions F, N, and D are polynomial functions defined from $\mathbb{R}^2$ to $\mathbb{R}^2$. The *set of nondefinition* of T is given by

$$\delta_s = \left\{(x, y) \in \mathbb{R}^2 \mid D(x, y) = 0\right\}. \tag{2.2}$$

In the following, we will suppose that δ_s is a smooth curve in the plane. The initial conditions belong to E, given by

$$E = \mathbb{R}^2 - \bigcup_{k=0}^{\infty} T^{-k}(\delta_s). \tag{2.3}$$

In order to define the concepts of *focal point* and *prefocal curve*, we consider a smooth arc γ transverse to δ_s and we study the shape of its image under T, that is, $T(\gamma)$.

Definition 2.1 (see [6]). Consider the map (2.1). A point $Q = (x_0, y_0)$ is a focal point if at least one component of T takes the form $0/0$ in Q and there exist smooth simple arcs $\gamma(\tau)$, with $\gamma(0) = Q$, such that $\lim_{\tau \to 0} T(\gamma(\tau))$ is finite. The set of all these finite values, obtained in different arcs $\gamma(\tau)$, is called a prefocal curve δ_Q.

We can compute prefocal curve and the focal point analytically by the following method (see [6]), provided that the inverse map is known explicitly. We search the set J_0' for which $\det(DT^{-1})$ vanishes, and we calculate the images, under T^{-1}, points of J_0'. If J_0' contains a curve δ such that $T^{-1}(\delta)$ is reduced to a point Q, thus δ is a prefocal curve for the map T and Q is then the associated focal point.

The presence of a vanishing denominator induces important effects on the geometrical and dynamical properties of the map. In this case, we need to locate geometrically the focal point in the phase plane. This concept is stated and proved in the following proposition [10].

Proposition 2.2. *Let $T(x, y) = (F(x, y), N(x, y)/D(x, y))$ be a fractional rational map with a unique polynomial inverse T^{-1}. If $T^{-1}(\delta_s)$ crosses transversely δ_s in a point $Q = (x_0, y_0)$, where δ_s is the set of nondefinition of T, then Q is a focal point of T.*

Corollary 2.3. *Let Q be a focal point of T, if $\gamma(\tau) = T^{-1}(\delta_s)$ and $\gamma(0) = Q$, then* $\lim_{\tau\to 0} T(\gamma(\tau))$ *belongs to $\delta_s \cap \delta_Q$.*

Some of such behaviors can also be observed in maps without a vanishing denominator, but their inverses have a vanishing denominator.

Corollary 2.4. *Let T be a polynomial map and T^{-1} its unique fractional inverse. If $T(\delta_s)$ crosses transversely δ_s in a point $Q = (x_0, y_0)$, where δ_s is the set of nondefinition of T^{-1}, then Q is a focal point of T^{-1}.*

3. Properties of Focal Points and Prefocal Curves

We now describe our main results concerning several focal points in more detail, we confirm and extend some others for the simplest case with one focal point as stated in [10], and similar properties continue to hold.

The following proposition shows the link between the structure of the basin of attraction of an attractor of T and the fact that the focal points of T^{-1} are inside or outside the basin.

Proposition 3.1. *Let T be a smooth two-dimensional map, with a unique fractional and rational inverse T^{-1}, δ_s being its set of nondefinition, and Q_i, $i = 1, 2, \ldots, k$, its focal points. Let D be the basin of attraction of an attracting set A of T of period-1. Suppose that for any $i = 1, 2, \ldots, k$, $\delta_{s,i} \cap D \neq \phi$, such that $\delta_{s,i}$ is the branch of $\delta_s = \cup_{i=1}^{k}(\delta_{s,i})$ containing the focal point Q_i. Therefore, the basin D is connected, if and only if all focal points Q_i belong to D.*

Proof. Let D_0 be the immediate basin of A, F the boundary of the basin $D = \cup_{n\geq 0} T^{-n}(D_0)$, and δ_{Q_i} the prefocal curves of T^{-1} associated, respectively, with focal points Q_i, $i = 1, 2, \ldots, k$.

First, we recall the following implications:

(1) $\delta_s \cap D_0 \neq \phi \Rightarrow T(\delta_s) \cap D_0 \neq \phi$,

(2) $Q_i \in D_0 \Leftrightarrow T^{-1}\{Q_i\} = \delta_{Q_i} \subset T^{-1}(D_0)$,

(3) $Q_i \notin D_0 \Leftrightarrow \delta_{Q_i} \cap T^{-1}(D_0) = \phi$,

(4) $Q_i \in F \Rightarrow T^{-1}\{Q_i\} = \delta_{Q_i} \subset T^{-1}(F) \supseteq F$, and in this case $T^{-1}F = F$ (as T restricted to $\mathbb{R}^2 - \{\cup_i \delta_{Q_i}\}$ is invertible).

Let us assume that all $\delta_{s,i}$ cross the immediate basin D_0 of A. Otherwise, if a branch $\delta_{s,i}$ crosses an island D_{n_0} without crossing D_0, we reason on this island as follows.

We denote by $D_{0,i}^1$ and $D_{0,i}^2$ the parts of D_0 located on either side of $\delta_{s,i}$ with $D_{0,i}^1 \cap D_{0,i}^2 = D_0 \cap \delta_{s,i}$, $D_0 = D_{0,i}^1 \cup D_{0,i}^2 = \cup_{i=1}^{k}(D_{0,i}^1 \cup D_{0,i}^2)$. Since T has a unique inverse and that $T^{-1}(\delta_{s,i})$ is the set of points at infinity of T^{-1} plus δ_{Q_i}, then $T^{-1}(D_{0,i}^1)$ and $T^{-1}(D_{0,i}^2)$ are on either side of δ_{Q_i}.

(1) We show that if D is connected, then all Q_i belong to D.

Suppose the contrary, in other words there exists $Q_{i_0} \notin D$, this implies that $Q_{i_0} \notin D_0$ and therefore, according to the equivalence (3) mentioned above, $T^{-1}\{Q_{i_0}\} = \delta_{Q_{i_0}}$ has no intersection with $T^{-1}(D_0) = T^{-1}(D_{0,i_0}^1 \cup D_{0,i_0}^2) = T^{-1}(D_{0,i_0}^1) \cup T^{-1}(D_{0,i_0}^2)$. Besides, as $T^{-1}(D_{0,i_0}^1)$ and $T^{-1}(D_{0,i_0}^2)$ are on either side of $\delta_{Q_{i_0}}$, this implies that $T^{-1}(D_{0,i_0}^1) \cap T^{-1}(D_{0,i_0}^2) = \phi$. Therefore, $T^{-1}(D_0)$ is not connected and then $D = \cup_{n\geq 0} T^{-n}(D_0)$ is not connected. This contradicts the hypothesis, D is connected, then $Q_{i_0} \in D$.

(2) Conversely, let us prove now that if all Q_i belong to D, then D is connected.

Suppose that all Q_i belong to D_0. If a focal point Q_{i_0} belongs to an island D_{n_0}, we reason on this island as follows.

Let $Q_i \in D_0$, from the equivalence (2), we have $T^{-1}\{Q_i\} = \delta_{Q_i} \subset T^{-1}(D_0) = T^{-1}(D^1_{0,i}) \cup T^{-1}(D^2_{0,i})$. Since $T^{-1}(D^1_{0,i})$ and $T^{-1}(D^2_{0,i})$ are on either side of δ_{Q_i}, therefore, in order that $T^{-1}(D_0)$ contains δ_{Q_i}, it is necessary that $T^{-1}(D^1_{0,i})$ and $T^{-1}(D^2_{0,i})$ merge, this means that $T^{-1}(D^1_{0,i}) \cap T^{-1}(D^2_{0,i}) = \delta_{Q_i}$. This is true for all prefocal curves δ_{Q_i}, then $T^{-1}(D_0)$ is connected. Since, by the definition of the attraction basin, $T^{-1}(D_0)$ contains D_0, and taking into account that D_0 is the biggest connected component containing the attractor, then D_0 contains $T^{-1}(D_0)$. This means that $T^{-1}(D_0) = D_0$. Therefore, $T^{-n}(D_0) = D_0$ and then $D = \cup_{n\geq 0} T^{-n}(D_0) = D_0$ is connected. □

Remark 3.2. The proposition does not apply for the case of the periodic attractors of period $k > 1$. Intuitively, some analogies with Proposition 3.1, as considering T^k, would be possible, but more work is needed to precise this fact.

The proposition which follows, localizes a focal point of T^{-1}, when it is a fixed point of T.

Proposition 3.3. *Let T be a smooth planar map, let T^{-1} be, its unique and fractional inverse and let Q be a focal point of T^{-1}. Then, Q is a fixed point of T if and only if $Q \in \delta_Q$.*

Proof. Q fixed point of $T \Leftrightarrow T(Q) = Q \Leftrightarrow T(Q) \in \{Q\} \Leftrightarrow Q \in T^{-1}\{Q\} = \delta_Q$. □

The following proposition made the link between prefocal curves of T^{-1} and of T^{-2} associated with the point focal Q and the fact that this last is a fixed point of T.

Proposition 3.4. *Let T be a smooth planar map and T^{-1} its unique and fractional inverse. Let δ_Q and Δ_Q be the prefoal curves of T^{-1} and of T^{-2} associated with the focal point Q. Then we can claim that:*

$$Q \text{ fixed point of } T \Leftrightarrow \delta_Q \cap \Delta_Q \neq \phi. \tag{3.1}$$

Proof. (1) Let us show that $\delta_Q \cap \Delta_Q \neq \phi$ is a necessary condition for Q to be a fixed point of T. Let $(x, y) \in \delta_Q$, then the Jacobian $JT_{(x,y)} = 0$, thus $JT^2_{(x,y)} = JT_{T(x,y)} \cdot JT_{(x,y)} = JT_Q \cdot JT_{(x,y)} = 0$. Besides, as Q is a fixed point of T, for all $(x, y) \in \delta_Q$ we have

$$T^2(x, y) = T(T(x, y)) = T(Q) = Q. \tag{3.2}$$

Thus, one showed that $\delta_Q \subset \Delta_Q$.

(2) Now let us show that $\delta_Q \cap \Delta_Q \neq \phi$ is a sufficient condition for Q to be a fixed point of T. Let $(x, y) \in \delta_Q \cap \Delta_Q$ then we obtain

$$T(x, y) = Q = T^2(x, y) = T(T(x, y)) = T(Q). \tag{3.3}$$

Consequently, Q is a fixed point of T. □

We can give another necessary condition so that a focal point of T^{-1} is a fixed point of T, given by this proposition.

Proposition 3.5. *Let T be a smooth planar map and T^{-1} its unique and fractional inverse, and let Q be a focal point of T^{-1}. If Q is a fixed point of T, then an eigenvalue of the Jacobian matrix of T in Q is null.*

Proof. Let Q be a focal point of T^{-1}, this implies that $JT = 0$ on δ_Q. As Q is a fixed point of T, consequently, $Q \in \delta_Q$ and then $JT_Q = 0$. This last ($JT_Q = 0$) is necessary and sufficient to have a null eigenvalue, being the determinant of a matrix equal to the product of their eigenvalues. □

4. Applications

Example 4.1. Case of a polynomial mapping, having a unique fractional inverse with a unique focal point.

We consider this map

$$T(x,y) = \begin{cases} x' = \dfrac{(1-a)\left(b\cdot x - x^2\right) + x\cdot y - b\cdot y}{a} + c, \\ y' = x, \end{cases} \tag{4.1}$$

where a, b, and c are real parameters. Its unique inverse is given by

$$T^{-1}(x,y) = \begin{cases} x' = y, \\ y' = (1-a)y + a\dfrac{x-c}{y-b}, \end{cases} \tag{4.2}$$

The nondefinition set δ_s of T^{-1} is given by

$$y = b, \tag{4.3}$$

However, T^{-1} has a prefocal curve δ_Q of equation

$$x = b \tag{4.4}$$

associated with the focal point $Q = (c, b)$.

First let us fix $a = 0.5$, $c = 1.5$ and we vary the parameter b. We have the following situations.

(1) For $b = 1.490$, the map T has a saddle fixed point C of coordinates (1.469,1.469), whose invariant stable manifold delimits the basin of attraction of a stable focus F of coordinates (1.021,1.021). We can see in Figure 1 that the prefocal curve δ_Q and the focal point $Q = (1.500, 1.490)$ are outside the basin that this last is not connected and is constituted of an immediate basin containing F and of islands with a shape of crescents.

(2) For the value $b = 1.500$, the focal point $Q = (1.500, 1.500)$ is on the prefocal curve δ_Q. From Proposition 3.3, Q is then a fixed point of T. It is merged with the saddle fixed point $C = (1.500, 1.500)$, whose invariant stable manifold delimits the basin of attraction of the stable focus F. in Figure 2, we can see that there exists a contact between the boundaries

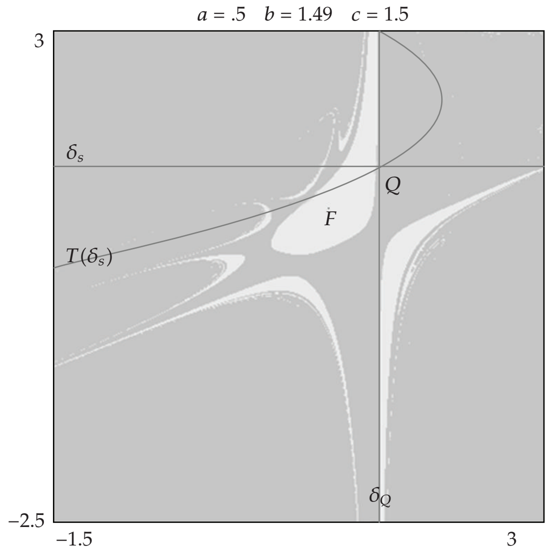

Figure 1: $b = 1.490$ basin of attraction non connected.

of the immediate basin and islands, so the closure of the total basin is connected. We can see also that the prefocal curve delimits a part of the basin of attraction, which implies that Q is on the boundary of the basin.

(3) For the value $b = 1.501$, the iterates of focal point Q converge toward the attractor, thus the focal point Q and prefocal curve δ_Q are inside the basin. We can see, in Figure 3, that the attractor is now an invariant closed curve, resulting from a Neimark-Hopf bifurcation for the value $b = 1.500$. We see that islands aggregate to the immediate basin. Therefore, the basin is connected. This value of the parameter b is at the same time, a basin-bifurcation value "connected ⟨--⟩ nonconnected" (as established in Proposition 3.1) and Neimark-Hopf bifurcation value.

Example 4.2. Case of polynomial mapping, having a unique fractional inverse with two focal points is considered.

We consider the following example:

$$T(x,y) = \begin{cases} x' = y, \\ y' = x \cdot y^2 - b \cdot y^2 - a \cdot x + a \cdot b \cdot y, \end{cases} \tag{4.5}$$

where a and b are real parameters. Its unique inverse is given by

$$T^{-1}(x,y) = \begin{cases} x' = \dfrac{y + b \cdot x^2 - a \cdot b \cdot x}{-a + x^2}, \\ y' = x. \end{cases} \tag{4.6}$$

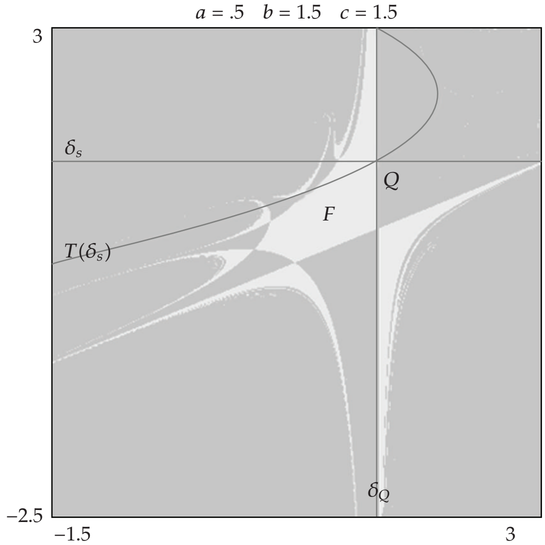

Figure 2: $b = 1.500$ basin-bifurcation value.

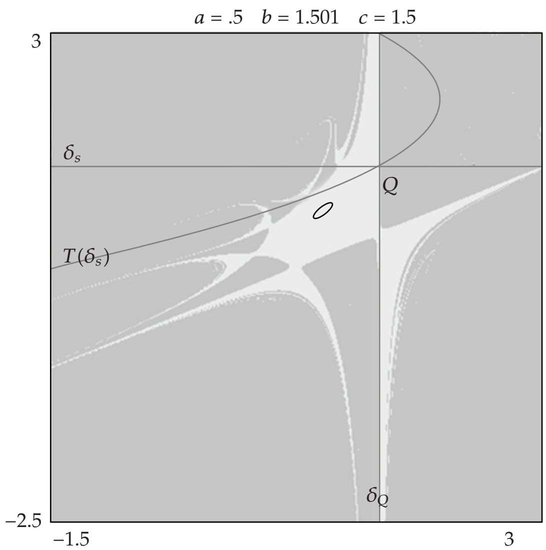

Figure 3: $b = 1.501$ basin of attraction connected.

The set of nondefinition δ_s of T^{-1} is constituted of two branches $\delta_{s,1} = \{x = \sqrt{a}\}$ and $\delta_{s,2} = \{x = -\sqrt{a}\}$:

$$\delta_s = \delta_{s,1} \cup \delta_{s,2}, \tag{4.7}$$

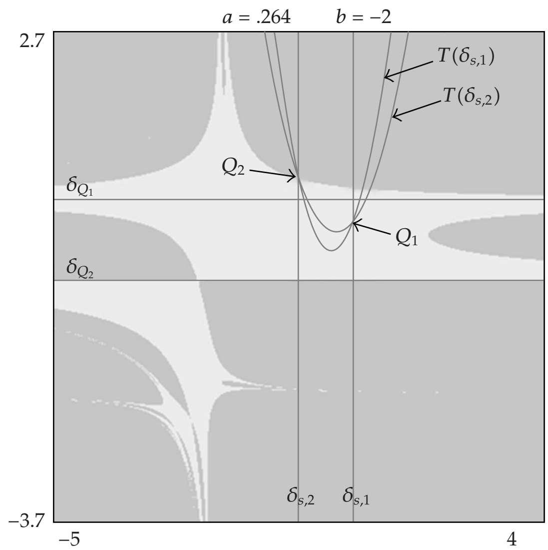

Figure 4: $a = 0.264$ basin of attraction connected.

T^{-1} has two prefocal curves:

$$\delta_{Q_1} : y = \sqrt{a}, \qquad \delta_{Q_2} : y = -\sqrt{a}, \tag{4.8}$$

associated, respectively, with the two focal points:

$$Q_1 = \left(\sqrt{a}, a \cdot b \left(\sqrt{a} - 1\right)\right), \qquad Q_2 = \left(-\sqrt{a}, -a \cdot b \left(\sqrt{a} + 1\right)\right), \tag{4.9}$$

(I) Let us check that $\delta_{Q_i} \cap \Delta_{Q_i} = \phi$, $i = 1, 2$, if and only if Q_i is not a fixed point of T (Proposition 3.4). We have $JT^2_{(x,y)} = JT_{T(x,y)} \cdot JT_{(x,y)} = ((xy^2 - by^2 - ax + aby)^2 - a) \cdot (y^2 - a) = 0$ this implies that

$$y = \pm\sqrt{a}, \qquad x = \frac{by^2 - aby \pm \sqrt{a}}{y^2 - a}. \tag{4.10}$$

The images of these curves, by T^2, give us

$$T^2\left(x = \frac{by^2 - aby + \sqrt{a}}{y^2 - a}, y\right) = T\left(y, \sqrt{a}\right) = Q_1, \qquad T^2\left(x, y = \sqrt{a}\right) = T(Q_1). \tag{4.11}$$

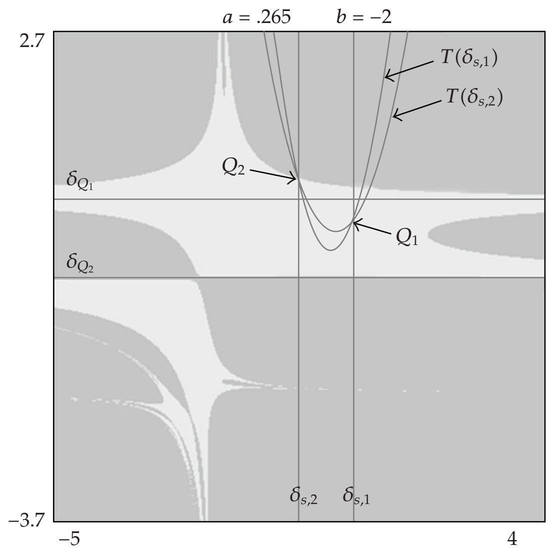

Figure 5: $a = 0.265$ basin of attraction non connected.

We deduce that $y = \sqrt{a}$ is a prefocal curve of T^{-2} associated with the focal point $T(Q_1)$, and hence if Q_1 is not fixed point of T, Δ_{Q_1} is equal to

$$\Delta_{Q_1} = \left\{ \left(x = \frac{by^2 - aby + \sqrt{a}}{y^2 - a}, y \right) \right\} \tag{4.12}$$

with $\delta_{Q_1} \cap \Delta_{Q_1} = \phi$. When Q_1 is a fixed point of T, Δ_{Q_1} is equal to

$$\Delta_{Q_1} = \left\{ \left(x = \frac{by^2 - aby + \sqrt{a}}{y^2 - a}, y \right) \right\} \cup \left\{ \left(x, y = \sqrt{a} \right) \right\}. \tag{4.13}$$

Thus $\delta_{Q_1} \subset \Delta_{Q_1}$.

The result is quite similar for the focal point Q_2.

(II) Study of the basin bifurcation "connected ⟨--⟩ nonconnected"

Fix $b = -2$ and we vary the parameter a.

(1) For the value $a = 0.264$, we see in Figure 4 that the basin of attraction is connected, and the two prefocal curves and the two focal points are inside the basin.

(2) For the value $a = 0.265$, we see in Figure 5 that the basin of attraction is not more connected. δ_{Q_2} and Q_2 are outside of the basin (Proposition 3.1).

(3) For the value $a = 0.583$, we can see in Figure 6 that the basin is not connected and the two prefocal curves do not belong to the basin.

(III) We can obtain basin bifurcations curves, when the focal points Q_1 and Q_2 are fixed points of T on the boundary of the basin.

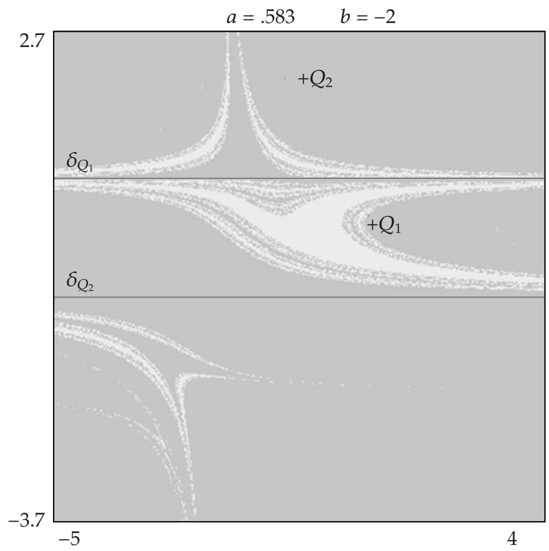

Figure 6: $a = 0.583$ basin of attraction non connected.

T has three fixed points A, B, C of coordinates

$$A = (0,0),$$
$$B = \left(\frac{b+\sqrt{b^2-4ab+4a+4}}{2}, \frac{b+\sqrt{b^2-4ab+4a+4}}{2}\right),$$
$$C = \left(\frac{b-\sqrt{b^2-4ab+4a+4}}{2}, \frac{b-\sqrt{b^2-4ab+4a+4}}{2}\right). \quad (4.14)$$

According to Proposition 3.3, $Q_1 = (\sqrt{a}, a \cdot b\ (\sqrt{a}-1))$ is a fixed point of T, if and only if $Q_1 \in \delta_{Q_1}$, then $a \cdot b\ (\sqrt{a}-1) = \sqrt{a}$, which gives the curve $b_1(a) = 1/(a-\sqrt{a})$ for $a \neq 0$ and $a \neq 1$.

For $a = 0$, $Q_1 = (0,0)$ is an attractive fixed point (with eigenvalues $\lambda_1 = \lambda_2 = 0$), hence Q_1 is not on the boundary of a basin. For $a = 1$, $Q_1 = (1,0)$ is not a fixed point of T.

By identification of Q_1 with the fixed points B, C for $b \in b_1(a)$, we obtain

$$Q_1 = (\sqrt{a}, \sqrt{a}) = B \quad \text{for } a \in]0,1[\cup]1.6826, +\infty[,$$
$$Q_1 = (\sqrt{a}, \sqrt{a}) = C \quad \text{for } a \in]1, 1.6826[. \quad (4.15)$$

Similarly, $Q_2 = (-\sqrt{a}, -a \cdot b\ (\sqrt{a}+1))$ is a fixed point of T if and only if $Q_2 \in \delta_{Q_2}$, then $-a \cdot b\ (\sqrt{a}+1) = -\sqrt{a}$, which gives $b_2(a) = 1/(a+\sqrt{a})$ for $a \neq 0$. For $a = 0$, $Q_2 = (0,0)$ is an attractive fixed point.

By identification of Q_2 with fixed points B, C for $b \in b_2(a)$, we obtain

$$Q_2 = (\sqrt{a}, \sqrt{a}) = C \quad \text{for } a \in]0, +\infty[. \quad (4.16)$$

In the parameter plane (a, b), the curves $b_1(a)$ and $b_2(a)$ can be bifurcation curves "connected $\langle$--$\rangle$ nonconnected" (when an attractor exists and that the fixed points B, C are on the boundary of the basin of this attractor), if between these two curves, a focal point cannot reach the boundary of the attraction basin without being a fixed point. Otherwise, it will be basin bifurcations curves, in the sense that the number of connected components of the basin changes while crossing them.

5. Conclusion

Fractional maps have suggested interesting analysis of the dynamic behavior and showed that they have their own characteristics, and have revealed new types of singularities considered as theoretical tools for analyzing the bifurcation phenomena. In this paper, we have presented complementary results that appear not to have been given in the works cited therein. We have proved some results, the first permits to localize geometrically the focal point in the phase plane. The second result makes the link between the property of the basin of attraction of an attractor of a map T with a unique inverse being connected (resp., nonconnected) and focal points of T^{-1} being inside (resp., outside) of this attraction basin. The third and last result gives the link between fixed point and focal point of T and T^{-1}. It would be interesting to study these same properties for maps with several inverses.

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