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Research Article

On the Convergence of Solutions of Certain Third-Order Differential Equations

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We establish sufficient conditions for the convergence of solutions of a certain third-order nonlinear differential equations. By constructing a Lyapunov function as the basic tool, some results which exist in the relevant literature are generalized.

1. Introduction

As well known, in the investigation of qualitative behaviors of solutions, stability, convergence, boundedness, oscillation, and so forth of solutions are very important problems in theory and applications of differential equations. For example, in applied sciences, some practical problems concerning mechanics, the engineering technique fields, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information theory, and so forth are associated with certain higher-order linear or nonlinear differential equations. Ever since Lyapunov [1] proposed his famous theory on the stability of motion, For some papers published on the qualitative behaviors of solutions of nonlinear second-and third-order differential equations, the readers can referee to the papers of Afuwape and Omeike [2, 3], Ezeilo [4, 5], Meng [6], Tejumola [7, 8], Tunç [9–11], Omeike [12], and the references listed in these papers as well as one can refer to the books of Reissig et al. [13, 14]. The motivation for the present work has been inspired basically by the paper of Afuwape and Omeike [2] and the papers listed above. Our aim here is to extend the results established by Afuwape and Omeike [2] to nonlinear differential equation (1.4) for the convergence of all solutions of this equation. In 2008, Afuwape and Omeike [2] considered third-order nonlinear differential equations of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}), \tag{1.1}$$

and by introducing a Lyapunov function they discussed the convergence of solutions for this equation. During establishment of the results, Afuwape and Omeike [2] defined the following relations with respect to the functions g and h:

$$0 < b \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq b_0 < \infty, \tag{1.2}$$

for any pair of constants y_2, y_1 $(y_2 \neq y_1)$ and

$$0 < \delta \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq kab, \tag{1.3}$$

for any pair of constants x_2, x_1 $(x_2 \neq x_1)$, where $k < 1$ is a positive constant.

In this paper, we consider nonlinear differential equation of the form

$$\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}), \tag{1.4}$$

where the functions f, g, h, and p are continuous in their respective arguments, with the functions f, g, and h are not necessarily differentiable. In addition to (1.2) and (1.3) we assume that

$$0 < a \leq \frac{f(z_2) - f(z_1)}{z_2 - z_1} \leq a_0 < \infty, \tag{1.5}$$

for any pair of constants z_2, z_1 $(z_2 \neq z_1)$.

By convergence of solutions we mean, any two solutions $x_1(t), x_2(t)$ of (1.4) are said to converge to each other if

$$x_2(t) - x_1(t) \longrightarrow 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \longrightarrow 0, \qquad \ddot{x}_2(t) - \ddot{x}_1(t) \longrightarrow 0 \tag{1.6}$$

as $t \to \infty$.

2. Main Results

The following results are established.

Theorem 2.1. *Suppose that $f(0) = g(0) = h(0)$, and that*

(i) *there are constants $a > 0, a_0 > 0$ such that $f(z)$ satisfies inequalities* (1.5),

(ii) *there are constants $b > 0$, $b_0 > 0$ such that $g(y)$ satisfies inequalities* (1.2),

(iii) *there are constants $\delta > 0$, $k < 1$ such that for any ξ, η $(\eta \neq 0)$, the incrementary ratio for h satisfies*

$$\frac{(h(\xi + \eta) - h(\xi))}{\eta} \quad \textit{lies in } I_0 \tag{2.1}$$

with $I_0 = [\delta, kab]$,

(iv) *there is a continuous function* $\phi(t)$ *such that*

$$|p(t, x_2, y_2, z_2) - p(t, x_1, y_1, z_1)| \leq \phi(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|\} \tag{2.2}$$

holds for arbitrary $t, x_1, y_1, z_1, x_2, y_2, z_2$, *and satisfies*

$$\int_0^t \phi^{\nu}(\tau) d\tau \leq D_1 t \tag{2.3}$$

for some constant $D_1 > 0$, *where* ν *is a constant in the range* $1 \leq \nu \leq 2$.

Then all solutions of (1.4) *converge.*

A very important step in the proof of Theorem 2.1 will be to give estimate for any two solutions of (1.4). This in itself, being of independent interest, is giving as follows.

Theorem 2.2. *Let* $x_1(t), x_2(t)$ *be any two solutions of* (1.4). *Suppose that all the conditions of Theorem 2.1 are satisfied, then for each fixed* ν, *in the range* $1 \leq \nu \leq 2$, *there exist constants* D_2, D_3, *and* D_4 *such that for* $t_2 \geq t_1$,

$$S(t_2) \leq D_2 S(t_1) \exp\left\{-D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \phi^{\nu}(\tau) d\tau\right\}, \tag{2.4}$$

where

$$S(t) = \left\{[x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2\right\}. \tag{2.5}$$

We have the following corollaries when $x_1(t) = 0$ and $t_1 = 0$.

Corollary 2.3. *Suppose that* $p = 0$ *in* (1.4) *and suppose further that conditions (i), (ii), and (iii) of Theorem 2.1 hold, then the trivial solution of* (1.4) *is exponentially stable in the large.*

Also, if we put $\xi = 0$ in (2.1) with η $(\eta \neq 0)$ arbitrary, we get the following.

Corollary 2.4. *If* $p \neq 0$ *and hypotheses (i), (ii), and (iii) of Theorem 2.1 hold for arbitrary* $\eta(\eta \neq 0)$, *and* $\xi = 0$, *then there exists a constant* $D_5 > 0$ *such that every solution* $x(t)$ *of* (1.4) *satisfies*

$$|x(t)| \leq D_5, \qquad |\dot{x}(t)| \leq D_5, \qquad |\ddot{x}(t)| \leq D_5. \tag{2.6}$$

3. Preliminary Results

On setting $\dot{x} = y, \dot{y} = z$, (1.4) can be replaced by an equivalent system

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -f(z) - g(y) - h(x) + p(t, x, y, z). \tag{3.1}$$

Let $(x_i(t), y_i(t), z_i(t))$, $i = 1, 2$, be any two solutions of (3.1) such that

$$a \le \frac{f(z_2) - f(z_1)}{z_2 - z_1} \le a_0 \quad (z_2 \ne z_1),$$
$$b \le \frac{g(y_2) - g(y_1)}{y_2 - y_1} \le b_0 \quad (y_2 \ne y_1), \tag{3.2}$$
$$\delta \le \frac{h(x_2) - h(x_1)}{x_2 - x_1} \le kab \quad (x_2 \ne x_1),$$

where a_0, a, b_0, b, δ, and k are finite constants, and k will be determined later.

Our investigation rests mainly on the properties of the function, $W = W(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ defined by

$$\begin{aligned} 2W &= \beta(1-\beta)b^2(x_2 - x_1)^2 + \beta b(y_2 - y_1)^2 + \alpha b a^{-1}(y_2 - y_1)^2 \\ &\quad + \alpha a^{-1}(z_2 - z_1)^2 + \left\{(z_2 - z_1) + a(y_2 - y_1) + (1-\beta)b(x_2 - x_1)\right\}^2, \end{aligned} \tag{3.3}$$

where $0 < \beta < 1$ and $\alpha > 0$ are constants.

Following the argument used in [5], we can easily verify the following for W.

Lemma 3.1. *(i)* $W(0,0,0) = 0$.

(ii) There exist finite positive constants D_6, D_7 such that

$$D_6\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right\} \le W \le D_7\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right\}, \tag{3.4}$$

where

$$D_6 = \frac{1}{2}\min\left\{\beta(1-\beta)b^2, b\left(\beta + \alpha a^{-1}\right), \alpha a^{-1}\right\}, \tag{3.5}$$

and using the inequality $|x||y| \le (1/2)(x^2 + y^2)$,

$$D_7 = \frac{1}{2}\max\left\{b(1-\beta)(1+b+a), b(\beta + \alpha a^{-1}) + a(1 + a + b(1-\beta)), 1 + \alpha a^{-1} + a + b(1-\beta)\right\}. \tag{3.6}$$

If we define the function $W(t)$ by

$$W(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t)) \tag{3.7}$$

and using the fact that the solutions (x_i, y_i, z_i), $i = 1, 2$, satisfy (3.1), then $S(t)$ as defined in (2.5) becomes

$$S(t) = \left\{[x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2 + [z_2(t) - z_1(t)]^2\right\}. \tag{3.8}$$

Lemma 3.2. *Assume that the conditions (i), (ii), and (iii) of Theorem 2.1 are satisfied. Then, there exist positive finite constants D_8 and D_9 such that*

$$\frac{dW}{dt} \leq -2D_8 S + D_9 S^{1/2}|\theta|, \tag{3.9}$$

where $\theta = p(t, x_2, y_2, z_2) - p(t, x_1, y_1, z_1)$.

Proof of Lemma 3.2

Differentiating the function W in (3.3) along the system (3.1) we obtain

$$\dot{W} = \frac{dW}{dt} = -W_1 - W_2 - W_3 - W_4 - W_5 - W_6 - W_7 + W_8, \tag{3.10}$$

in which

$$\begin{aligned}
W_1 &= \Big\{\gamma_1 b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 + \eta_1 a\big[G(y_2,y_1) - b(1-\beta)\big](y_2-y_1)^2 \\
&\quad + \xi_1 \alpha a^{-1} F(z_2,z_1)(z_2-z_1)^2 + (F(z_2,z_1) - a)(z_2-z_1)^2\Big\}, \\
W_2 &= \Big\{\gamma_2 b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 + \xi_2 \alpha a^{-1} F(z_2,z_1)(z_2-z_1)^2 \\
&\quad + \left(1+\alpha a^{-1}\right)(x_2-x_1)(z_2-z_1)H(x_2,x_1)\Big\}, \\
W_3 &= \Big\{\gamma_3 b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 + \eta_2 a\big[G(y_2,y_1) - b(1-\beta)\big](y_2-y_1)^2 \\
&\quad + a(x_2-x_1)(y_2-y_1)H(x_2,x_1)\}, \\
W_4 &= \Big\{\gamma_4 b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 + \xi_3 \alpha a^{-1} F(z_2,z_1)(z_2-z_1)^2 \\
&\quad + b(1-\beta)(x_2-x_1)(z_2-z_1)[F(z_2,z_1) - a]\}, \\
W_5 &= \Big\{\gamma_5 b(1-\beta)H(x_2,x_1)(x_2-x_1)^2 + \eta_3 a\big[G(y_2,y_1) - b(1-\beta)\big](y_2-y_1)^2 \\
&\quad + b(1-\beta)(x_2-x_1)(y_2-y_1)\big[G(y_2,y_1) - b\big]\}, \\
W_6 &= \Big\{\xi_4 \alpha a^{-1} F(z_2,z_1)(z_2-z_1)^2 + \eta_4 a\big[G(y_2,y_1) - b(1-\beta)\big](y_2-y_1)^2 \\
&\quad + \left(1+\alpha a^{-1}\right)(y_2-y_1)(z_2-z_1)\big[G(y_2,y_1) - b\big]\Big\}, \\
W_7 &= \Big\{\xi_5 \alpha a^{-1} F(z_2,z_1)(z_2-z_1)^2 + \eta_5 a\big[G(y_2,y_1) - b(1-\beta)\big](y_2-y_1)^2 \\
&\quad + a(y_2-y_1)(z_2-z_1)[F(z_2,z_1) - a]\}, \\
W_8 &= \Big\{b(1-\beta)(x_2-x_1) + a(y_2-y_1) + \left(1+\alpha a^{-1}\right)(z_2-z_1)\Big\}\theta(t),
\end{aligned} \tag{3.11}$$

with

$$\begin{aligned} F(z_2, z_1) &= \frac{f(z_2) - f(z_1)}{z_2 - z_1} \quad (z_2 \neq z_1), \\ G(y_2, y_1) &= \frac{g(y_2) - g(y_1)}{y_2 - y_1} \quad (y_2 \neq y_1), \\ H(x_2, x_1) &= \frac{h(x_2) - h(x_1)}{x_2 - x_1} \quad (x_2 \neq x_1), \end{aligned} \tag{3.12}$$

and ξ_i, η_i, and γ_i, $(i = 1, 2, 3, 4, 5)$ are strictly positive constants such that

$$\sum_{i=1}^{5} \xi_i = 1, \qquad \sum_{i=1}^{5} \eta_i = 1, \qquad \sum_{i=1}^{5} \gamma_i = 1. \tag{3.13}$$

Also, let us denote $F(z_2, z_1), G(y_2, y_1)$, and $H(x_2, x_1)$ simply by F, G, and H, respectively. For strictly positive constants k_1, k_2, k_3, k_4, k_5, and k_6 conveniently chosen later, we get

$$\begin{aligned}
&\left(1 + \alpha a^{-1}\right)(x_2 - x_1)(z_2 - z_1)H \\
&\quad = \left\{ k_1 \left(1 + \alpha a^{-1}\right)^{1/2} H^{1/2}(x_2 - x_1) + \frac{1}{2} k_1^{-1} \left(1 + \alpha a^{-1}\right)^{1/2} H^{1/2}(z_2 - z_1) \right\}^2 \\
&\qquad - k_1^2 \left(1 + \alpha a^{-1}\right) H (x_2 - x_1)^2 - \frac{1}{4} k_1^{-2} \left(1 + \alpha a^{-1}\right) H (z_2 - z_1)^2, \\
&a(x_2 - x_1)(y_2 - y_1)H \\
&\quad = \left\{ k_2 a^{1/2} H^{1/2}(x_2 - x_1) + \frac{1}{2} k_2^{-1} a^{1/2} H^{1/2}(y_2 - y_1) \right\}^2 \\
&\qquad - k_2^2 a H (x_2 - x_1)^2 - \frac{1}{4} k_2^{-2} a H (y_2 - y_1)^2, \\
&b(1 - \beta)(x_2 - x_1)(z_2 - z_1)[F - a] \\
&\quad = \left\{ \frac{1}{2} k_3^{-1} b^{1/2} (1 - \beta)^{1/2} [F - a]^{1/2} (x_2 - x_1) + k_3 b^{1/2} (1 - \beta)^{1/2} [F - a]^{1/2} (z_2 - z_1) \right\}^2 \\
&\qquad - \frac{1}{4} k_3^{-2} b (1 - \beta) [F - a] (x_2 - x_1)^2 - k_3^2 b (1 - \beta) [F - a] (z_2 - z_1)^2, \\
&b(1 - \beta)(x_2 - x_1)(y_2 - y_1)[G - b] \\
&\quad = \left\{ k_4 b^{1/2} (1 - \beta)^{1/2} [G - b]^{1/2} (x_2 - x_1) + \frac{1}{2} k_4^{-1} b^{1/2} (1 - \beta)^{1/2} [G - b]^{1/2} (y_2 - y_1) \right\}^2 \\
&\qquad - k_4^2 b (1 - \beta) [G - b] (x_2 - x_1)^2 - \frac{1}{4} k_4^{-2} b (1 - \beta) [G - b] (y_2 - y_1)^2,
\end{aligned}$$

$$
\begin{aligned}
&\left(1+\alpha a^{-1}\right)(y_2-y_1)(z_2-z_1)[G-b]\\
&\quad=\left\{k_5\left(1+\alpha a^{-1}\right)^{1/2}[G-b]^{1/2}(y_2-y_1)+\frac{1}{2}k_5^{-1}\left(1+\alpha a^{-1}\right)^{1/2}[G-b]^{1/2}(z_2-z_1)\right\}^2\\
&\qquad-k_5^2\left(1+\alpha a^{-1}\right)[G-b](y_2-y_1)^2-\frac{1}{4}k_5^{-2}\left(1+\alpha a^{-1}\right)[G-b](z_2-z_1)^2,\\
&a(y_2-y_1)(z_2-z_1)[F-a]\\
&\quad=\left\{\frac{1}{2}k_6^{-1}a^{1/2}[F-a]^{1/2}(y_2-y_1)+k_6a^{1/2}[F-a]^{1/2}(z_2-z_1)\right\}^2\\
&\qquad-\frac{1}{4}k_6^{-2}a[F-a](y_2-y_1)^2-k_6^2a[F-a](z_2-z_1)^2
\end{aligned}
\tag{3.14}
$$

Thus,

$$
\begin{aligned}
W_2&=\left\{k_1\left(1+\alpha a^{-1}\right)^{1/2}H^{1/2}(x_2-x_1)+\frac{1}{2}k_1^{-1}\left(1+\alpha a^{-1}\right)^{1/2}H^{1/2}(z_2-z_1)\right\}^2\\
&\quad+\left\{\gamma_2b(1-\beta)H-k_1^2\left(1+\alpha a^{-1}\right)H\right\}(x_2-x_1)^2\\
&\quad+\left\{\xi_2\alpha a^{-1}F-\frac{1}{4}k_1^{-2}\left(1+\alpha a^{-1}\right)H\right\}(z_2-z_1)^2,\\
W_3&=\left\{k_2a^{1/2}H^{1/2}(x_2-x_1)+\frac{1}{2}k_2^{-1}a^{1/2}H^{1/2}(y_2-y_1)\right\}^2\\
&\quad+\left\{\gamma_3b(1-\beta)H-k_2^2aH\right\}(x_2-x_1)^2\\
&\quad+\left\{\eta_2a[G-b(1-\beta)]-\frac{1}{4}k_2^{-2}aH\right\}(y_2-y_1)^2,\\
W_4&=\left\{\frac{1}{2}k_3^{-1}b^{1/2}(1-\beta)^{1/2}[F-a]^{1/2}(x_2-x_1)+k_3b^{1/2}(1-\beta)^{1/2}[F-a]^{1/2}(z_2-z_1)\right\}^2\\
&\quad+\left\{\gamma_4b(1-\beta)H-\frac{1}{4}k_3^{-2}b(1-\beta)[F-a]\right\}(x_2-x_1)^2\\
&\quad+\left\{\xi_3\alpha a^{-1}F-k_3^2b(1-\beta)[F-a]\right\}(z_2-z_1)^2,\\
W_5&=\left\{k_4b^{1/2}(1-\beta)^{1/2}[G-b]^{1/2}(x_2-x_1)+\frac{1}{2}k_4^{-1}b^{1/2}(1-\beta)^{1/2}[G-b]^{1/2}(y_2-y_1)\right\}^2\\
&\quad+\left\{\gamma_5b(1-\beta)H-k_4^2b(1-\beta)[G-b]\right\}(x_2-x_1)^2\\
&\quad+\left\{\eta_3a[G-b(1-\beta)]-\frac{1}{4}k_4^{-2}b(1-\beta)[G-b]\right\}(y_2-y_1)^2,
\end{aligned}
$$

$$\begin{aligned}
W_6 = &\left\{k_5\left(1+\alpha a^{-1}\right)^{1/2}[G-b]^{1/2}(y_2-y_1)+\frac{1}{2}k_5^{-1}\left(1+\alpha a^{-1}\right)^{1/2}[G-b]^{1/2}(z_2-z_1)\right\}^2 \\
&+\left\{\eta_4 a[G-b(1-\beta)]-k_5^2\left(1+\alpha a^{-1}\right)[G-b]\right\}(y_2-y_1)^2 \\
&+\left\{\xi_4\alpha a^{-1}F-\frac{1}{4}k_5^{-2}\left(1+\alpha a^{-1}\right)[G-b]\right\}(z_2-z_1)^2, \\
W_7 = &\left\{\frac{1}{2}k_6^{-1}a^{1/2}[F-a]^{1/2}(y_2-y_1)+k_6 a^{1/2}[F-a]^{1/2}(z_2-z_1)\right\}^2 \\
&+\left\{\eta_5 a[G-b(1-\beta)]-\frac{1}{4}k_6^{-2}a[F-a]\right\}(y_2-y_1)^2 \\
&+\left\{\xi_5\alpha a^{-1}F-k_6^2 a[F-a]\right\}(z_2-z_1)^2.
\end{aligned} \tag{3.15}$$

Moreover, in view of (3.2), we obtain for all $x_i, z_i (i = 1, 2)$ in $\mathfrak{R}$,

$$W_2 \geq 0, \tag{3.16}$$

if

$$k_1^2 \leq \frac{\gamma_2(1-\beta)ab}{(\alpha+a)} \quad \text{with } H \leq \frac{4\xi_2\gamma_2\alpha(1-\beta)a^2 b}{(\alpha+a)^2}, \tag{3.17}$$

and for all x_i, y_i $(i = 1, 2)$ in $\mathfrak{R}$,

$$W_3 \geq 0, \tag{3.18}$$

if

$$k_2^2 \leq \frac{\gamma_3(1-\beta)b}{a} \quad \text{with } H \leq \frac{4\eta_2\gamma_3\beta(1-\beta)b^2}{a}. \tag{3.19}$$

Combining all the inequalities in (3.16) and (3.18), we have for all $x_i, y_i, z_i (i = 1, 2\ )$ in $\mathfrak{R}$,

$$W_2 \geq 0, \qquad W_3 \geq 0, \tag{3.20}$$

if

$$H \leq kab \quad \text{with } k = \min\left\{\frac{4\xi_2\gamma_2\alpha(1-\beta)a}{(\alpha+a)^2}, \frac{4\eta_2\gamma_3\beta(1-\beta)b}{a^2}\right\} < 1. \tag{3.21}$$

Also, for all $x_i, z_i (i = 1, 2)$in $\mathfrak{R}$,

$$W_4 \geq 0, \tag{3.22}$$

if

$$\frac{a_0 - a}{4\gamma_4\delta} \leq k_3^2 \leq \frac{\xi_3\alpha}{(1-\beta)b(a_0 - a)}, \tag{3.23}$$

for all $x_i, y_i (i = 1, 2)$ in $\mathfrak{R}$,

$$W_5 \geq 0, \tag{3.24}$$

if

$$\frac{(1-\beta)(b_0 - b)}{4\beta a\eta_3} \leq k_4^2 \leq \frac{\delta\gamma_5}{(b_0 - b)}, \tag{3.25}$$

for all $y_i, z_i (i = 1, 2)$ in $\mathfrak{R}$,

$$W_6 \geq 0, \tag{3.26}$$

if

$$\frac{(\alpha + a)(b_0 - b)}{4\xi_4\alpha a} \leq k_5^2 \leq \frac{\eta_4\beta b a^2}{(\alpha + a)(b_0 - b)}, \tag{3.27}$$

and for all $y_i, z_i (i = 1, 2)$ in $\mathfrak{R}$,

$$W_7 \geq 0, \tag{3.28}$$

if

$$\frac{a_0 - a}{4\eta_5\beta b} \leq k_6^2 \leq \frac{\xi_5\alpha}{a(a_0 - a)}. \tag{3.29}$$

Further

$$W_1 \geq 2D_{10}\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right\}, \tag{3.30}$$

where $2D_{10} = \min\{\gamma_1 b\delta(1-\beta), \eta_1 ab\beta, \xi_1\alpha\}$, on the other hand

$$W_8 \leq D_{11}\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right\}^{1/2}|\theta(t)|, \tag{3.31}$$

where $D_{11} = 2\max\{b(1-\beta), a, (1 + \alpha a^{-1})\}$.

Bringing together the estimates just obtained for $W_1, W_2, W_3, W_4, W_5, W_6, W_7$, and W_8 in (3.10) and using (3.8), we have

$$\frac{dW}{dt} \leq -2D_{10}S(t) + D_{11}S^{1/2}(t)|\theta(t)|. \tag{3.32}$$

This completes the proof of Lemma 3.2.

4. Proof of Theorem 2.2

This follows directly from [5], on using inequality (3.32). Let ν be any constant in the range $1 \leq \nu \leq 2$. Set $2\mu = 2 - \nu$, so that $0 \leq \mu \leq 1/2$. We rewrite (3.32) in the form

$$\frac{dW}{dt} + D_{10}S \leq -D_{10}S + D_{11}S^{1/2}|\theta| = D_{11}S^{\mu}W^{*}, \tag{4.1}$$

where

$$W^{*} = \left(|\theta| - D_{12}S^{1/2}\right)S^{1/2-\mu}, \tag{4.2}$$

with $D_{12} = D_{10}D_{11}^{-1}$, considering the two cases

(i) $|\theta| < D_{12}S^{1/2}$ and

(ii) $|\theta| \geq D_{12}S^{1/2}$

separately. If $|\theta| < D_{12}S^{1/2}$, then $W^{*} < 0$. On the other hand, if $|\theta| \geq D_{12}S^{1/2}$, then the definition of W^{*} in (4.2) gives at least

$$W^{*} \leq S^{(1/2-\mu)}|\theta| \tag{4.3}$$

and also $S^{1/2} \leq |\theta|/D_{12}$. This implies that

$$S^{1/2(1-2\mu)} \leq \left[\frac{|\theta|}{D_{12}}\right]^{(1-2\mu)}. \tag{4.4}$$

Therefore

$$S^{1/2(1-2\mu)}|\theta| \leq \left[\frac{|\theta|}{D_{12}}\right]^{(1-2\mu)} \times |\theta|, \tag{4.5}$$

from which together with W^{*}, we obtain

$$W^{*} \leq D_{13}|\theta|^{2(1-\mu)}, \tag{4.6}$$

where $D_{13} = D_{12}^{(2\mu-1)}$. Again due to (4.1) and using the estimate on W^* for W^*, we have

$$\begin{aligned} \frac{dW}{dt} + D_{10}S &\le D_{11}D_{13}S^{\mu}|\theta|^{2(1-\mu)} \\ &\le D_{14}S^{\mu}\phi^{2(1-\mu)}S^{(1-\mu)}, \end{aligned} \tag{4.7}$$

where $D_{14} = 3^{1-\mu}D_{11}D_{13}$, which follows from

$$\begin{aligned} |\theta| &= \left|p(t,x_2,y_2,z_2) - p(t,x_1,y_1,z_1)\right| \\ &\le \phi(t)\left\{|x_2 - x_1| + \left|y_2 - y_1\right| + |z_2 - z_1|\right\}. \end{aligned} \tag{4.8}$$

In view of the fact that $\nu = 2(1-\mu)$, we obtain

$$\frac{dW}{dt} \le -D_{10}S + D_{14}\phi^{\nu}S, \tag{4.9}$$

and on using inequality (3.4), we have

$$\frac{dW}{dt} + \left(D_{15} - D_{16}\phi^{\nu}(t)\right)W \le 0 \tag{4.10}$$

for some positive constants D_{15} and D_{16}. On integrating (4.10) from t_1 to t_2 $(t_2 \ge t_1)$, we have

$$W(t_2) \le W(t_1)\exp\left\{-D_{15}(t_2 - t_1) + D_{16}\int_{t_2}^{t_2}\phi^{\nu}(\tau)d\tau\right\}. \tag{4.11}$$

Again, using Lemma 3.1, we obtain (2.4), with $D_2 = D_7D_6^{-1}, D_3 = D_{15}$, and $D_4 = D_{16}$. This completes the proof of Theorem 2.2.

5. Proof of Theorem 2.1

This follows from the estimate (2.4) and the condition (2.3) on $\phi(t)$. Choose $D_1 = D_3D_4^{-1}$ in (2.3). From the estimate (2.4), if

$$\int_{t_1}^{t_2}\phi^{\nu}(\tau)d\tau \le D_3D_4^{-1}(t_2 - t_1), \tag{5.1}$$

then the exponential index remains negative for all $t_2 - t_1 \ge 0$. Then, as $t = (t_2 - t_1) \to \infty$, we have $S(t) \to 0$, and this gives

$$x_2 - x_1 \longrightarrow 0, \qquad y_2 - y_1 \longrightarrow 0, \qquad z_2 - z_1 \longrightarrow 0, \tag{5.2}$$

as $t \to \infty$. This completes the proof of Theorem 2.1.

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