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Research Article

Symmetric Positive Solutions for Nonlinear Singular Fourth-Order Eigenvalue Problems with Nonlocal Boundary Condition

Fuyi Xu[1] and Jian Liu[2]

[1] *School of Science, Shandong University of Technology, Zibo, Shandong 255049, China*
[2] *School of Statistics and Mathematics Science, Shandong Economics University, Jinan, 250014 Shandong, China*

Correspondence should be addressed to Fuyi Xu, xfy_02@163.com

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We investigate nonlinear singular fourth-order eigenvalue problems with nonlocal boundary condition $u^{(4)}(t) - \lambda h(t) f(t, u, u'') = 0$, $0 < t < 1$, $u(0) = u(1) = \int_0^1 a(s)u(s)ds$, $u''(0) = u''(1) = \int_0^1 b(s)u''(s)ds$, where $a, b \in L^1[0,1]$, $\lambda > 0$, h may be singular at $t = 0$ and/or 1. Moreover $f(t, x, y)$ may also have singularity at $x = 0$ and/or $y = 0$. By using fixed point theory in cones, an explicit interval for λ is derived such that for any λ in this interval, the existence of at least one symmetric positive solution to the boundary value problem is guaranteed. Our results extend and improve many known results including singular and nonsingular cases. The associated Green's function for the above problem is also given.

1. Introduction

Boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics and so on, and the existence of positive solutions for such problems has become an important area of investigation in recent years. To identify a few, we refer the reader to [1–7] and references therein.

At the same time, a class of boundary value problems with nonlocal boundary conditions appeared in heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. Such problems include two-point, three-point, multipoint boundary value problems as special cases and have attracted the attention of Gallardo [1], Karakostas and Tsamatos [2], and Lomtatidze and Malaguti [3] (and see the references therein). For more information about the general theory of integral equations and their relation to boundary value problems we refer the reader to the book of Corduneanu [8] and Agarwal and O'Regan [9].

Motivated by the works mentioned above, in this paper, we study the existence of symmetric positive solutions of the following fourth-order nonlocal boundary value problem (BVP):

$$
\begin{aligned}
&u^{(4)}(t) - \lambda h(t) f\left(t, u, u''\right) = 0, \quad 0 < t < 1, \\
&u(0) = u(1) = \int_0^1 a(s) u(s) ds, \\
&u''(0) = u''(1) = \int_0^1 b(s) u''(s) ds,
\end{aligned} \tag{1.1}
$$

where $a, b \in L^1[0,1]$, $\lambda > 0$, h may be singular at $t = 0$ and/or 1. Moreover $f(t, x, y)$ may also have singularity at $x = 0$ and/or $y = 0$.

The main features of this paper are as follows. Firstly, comparing with [4–7], we discuss the boundary value problem with nonlocal boundary conditions, that is, BVP (1.1) including fourth-order two-point, three-point, multipoint boundary value problems as special cases. Secondly, comparing with [4–7], we discuss the boundary value problem when nonlinearity f contains second-derivatives u''. Thirdly, here we not only allow h have singularity at $t = 0$ and/or 1 but also allow $f(t, x, y)$ have singularity at $x = 0$ and/or $y = 0$. Finally, in [4–7], authors only studied the existence of positive solutions. However, they did not further provide characters of positive solutions, such as symmetry. It is now natural to consider the existence of symmetric positive solutions. To our knowledge, no paper has considered the existence of symmetric positive solutions and nonlinearity f with singularity at $x = 0$ and (or) $y = 0$ for fourth-order equation with nonlocal boundary condition. Hence we improve and generalize the results of [4–7] to some degree, and so it is interesting and important to study the existence of symmetric positive solutions for problem (1.1). The arguments are based upon a specially constructed cone and the fixed point theory for cones.

Let K be a cone in a Banach space E and let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, and $\overline{K}_{r,R} = \{x \in K : r \leq \|x\| \leq R\}$, where $0 < r < R < +\infty$.

Our main tool of this paper is the following fixed point theorem.

Lemma 1.1 (see [10]). *Let K be a positive cone in real Banach space E, $0 < r < R < +\infty$, and let $T : \overline{K}_{r,R} \to K$ be a completely continuous operator and such that*

(i) $\|Tx\| \leq \|x\|$ *for* $x \in \partial K_R$;

(ii) *there exists* $e \in \partial K_1$ *such that* $x \neq Tx + me$ *for any* $x \in \partial K_r$ *and* $m > 0$.

Then T has a fixed point in $\overline{K}_{r,R}$.

Remark 1.2. If (i) and (ii) are satisfied for $x \in \partial K_r$ and $x \in \partial K_R$, respectively, then Lemma 1.1 is still true.

The following concept will also be utilized.

Definition 1.3. If $u : [0,1] \to R$ is a continuous function and $u(t) = u(1-t)$ for $t \in [0,1]$, then one says that u is symmetric on $[0,1]$.

2. Preliminaries and Lemmas

In this section, we present some lemmas that are important to prove our main results.

Lemma 2.1. *Suppose that* $d := \int_0^1 m(s)ds \neq 1$, $m \in L^1[0,1]$, $y \in C[0,1]$, *then BVP*

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = \int_0^1 m(s)u(s)ds \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 H(t,s)y(s)ds, \tag{2.3}$$

where

$$H(t,s) = G(t,s) + \frac{1}{1-d}\int_0^1 G(s,x)m(x)dx, \qquad G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.4}$$

Proof. Integrating both sides of (2.1) on $[0,t]$, we have

$$u'(t) = -\int_0^t y(s)ds + B. \tag{2.5}$$

Again integrating (2.5) from 0 to t, we get

$$u(t) = -\int_0^t (t-s)y(s)ds + Bt + A. \tag{2.6}$$

In particular,

$$u(1) = -\int_0^1 (1-s)y(s)ds + B + A, \qquad u(0) = A. \tag{2.7}$$

By (2.2) we get

$$B = \int_0^1 (1-s)y(s)ds. \tag{2.8}$$

By $G(s,x) = G(x,s)$ and (2.6), we can get

$$\begin{aligned}
A = u(0) &= \int_0^1 m(x)u(x)dx = \int_0^1 m(x)\left(-\int_0^x (x-s)y(s)ds + Bx + A\right)dx \\
&= \int_0^1 m(x)\left(-\int_0^x (x-s)y(s)ds + x\int_0^1 (1-s)y(s)ds\right)dx + A\int_0^1 m(x)dx \\
&= \int_0^1 m(x)\left(\int_0^x s(1-x)y(s)ds + \int_x^1 x(1-s)y(s)ds\right)dx + Ad \\
&= \int_0^1 m(x)\left(\int_0^1 G(s,x)y(s)ds\right)dx + Ad \\
&= \int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right)y(s)ds + Ad.
\end{aligned} \tag{2.9}$$

So,

$$A = \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right)y(s)ds. \tag{2.10}$$

By (2.6), (2.8), and (2.10), we have

$$\begin{aligned}
u(t) &= -\int_0^t (t-s)y(s)ds + Bt + A \\
&= -\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds + \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right)y(s)ds \\
&= \int_0^t s(1-t)y(s)ds + \int_t^1 t(1-s)y(s)ds + \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right)y(s)ds \\
&= \int_0^1 G(t,s)y(s)ds + \frac{1}{1-d}\int_0^1 \left(\int_0^1 G(s,x)m(x)dx\right)y(s)ds \\
&= \int_0^1 H(t,s)y(s)ds.
\end{aligned} \tag{2.11}$$

This completes the proof of Lemma 2.1. □

It is easy to verify the following properties of $H(t,s)$ and $G(t,s)$.

Lemma 2.2. *If $m(t) > 0$, and $d := \int_0^1 m(s)ds \in (0,1)$, then*

(1) $H(t,s) \geq 0$, $t,s \in [0,1]$, $H(t,s) > 0$, $t,s \in (0,1)$;

(2) $G(1-t,1-s) = G(t,s)$, $G(t,t) \leq G(t,s) \leq G(s,s)$, $t,s \in [0,1]$;

(3) $\gamma_1 H(s,s) \leq H(t,s) \leq H(s,s)$, *where* $\gamma_1 = \eta_1/(1-d+\eta_1) \in (0,1)$, $\eta_1 = \int_0^1 G(x,x)m(x)dx$.

So we may denote Green's function of boundary value problem

$$
\begin{aligned}
&-u''(t) = 0, \quad 0 < t < 1, \\
&u(0) = u(1) = \int_0^1 a(s)u(s)ds, \\
&-u''(t) = 0, \quad 0 < t < 1, \\
&u(0) = u(1) = \int_0^1 b(s)u(s)ds
\end{aligned}
\tag{2.12}
$$

by $H_1(t,s)$ and $H_2(t,s)$, respectively. By Lemma 2.1, we know that $H_1(t,s)$ and $H_2(t,s)$ can be written by

$$
\begin{aligned}
H_1(t,s) &= G(t,s) + \frac{1}{1-\int_0^1 a(s)ds}\int_0^1 G(s,x)a(x)d, \\
H_2(t,s) &= G(t,s) + \frac{1}{1-\int_0^1 b(s)ds}\int_0^1 G(s,x)b(x)dx.
\end{aligned}
\tag{2.13}
$$

Obviously, $H_1(t,s)$ and $H_2(t,s)$ have the same properties with $H(t,s)$ in Lemma 2.2.

Remark 2.3. For notational convenience, we introduce the following constants:

$$
\begin{gathered}
\alpha = \int_0^1 a(s)ds, \qquad \beta = \int_0^1 b(s)ds, \qquad \mu = \max_{0\le t\le 1}\int_0^1 H_1(t,s)ds, \\
\gamma = \frac{\eta}{1-\beta+\eta} \in (0,1), \qquad \eta = \int_0^1 G(x,x)b(x)dx, \\
L = (1+\mu)\int_0^1 H_2(s,s)h(s)ds, \qquad l = \min_{t\in[0,1]}\int_0^1 H_2(t,s)h(s)ds.
\end{gathered}
\tag{2.14}
$$

Obviously, $\mu > 0$, $0 < l < L < \infty$.

Now we define an integral operator $S : C[0,1] \to C[0,1]$ by

$$
(Sv)(t) = \int_0^1 H_1(t,\tau)v(\tau)d\tau.
\tag{2.15}
$$

Then, we have

$$
\begin{aligned}
&(Sv)''(t) = -v(t), \quad 0 < t < 1, \\
&Sv(0) = Sv(1) = \int_0^1 a(s)Sv(s)ds.
\end{aligned}
\tag{2.16}
$$

Lemma 2.4. *The fourth-order nonlocal boundary value problem* (1.1) *has a positive solution if and only if the following integral-differential boundary value problem*

$$\begin{aligned} &v''(t) + \lambda h(t) f(t, Sv(t), -v(t)) = 0, \quad 0 < t < 1, \\ &v(0) = v(1) = \int_0^1 b(s) v(s) ds \end{aligned} \tag{2.17}$$

has a positive solution.

Proof. In fact, if u is a positive solution of (BVP) (1.1), let $u(t) = Sv(t)$, then $v = -u''$. This implies that $u'' = -v$ is a solution of (2.17). Conversely, if v is a positive solution of (2.17). Let $u(t) = Sv(t)$, by (2.16), $u''(t) = (Sv)''(t) = -v$. Thus $u(t) = Sv(t)$ is a positive solution of (BVP) (1.1). This completes the proof of Lemma 2.4. □

So we will concentrate our study on (2.17). Let $C^+[0,1] = \{x \in C[0,1] : x \geq 0\}$ and

$$K = \left\{ x \in C^+[0,1] : x(t) \text{ is symmetric and concave function on } [0,1], \min_{t \in [0,1]} x(t) \geq \gamma \|x\| \right\}; \tag{2.18}$$

$\|\cdot\|$ is the supremum norm on $C^+[0,1]$. It is easy to see that K is a cone in $C[0,1]$ and $\overline{K}_{r,R} \subset K \subset C^+[0,1]$. Now we define an operator $T : K \setminus \{0\} \rightarrow C^+[0,1]$ by

$$(Tv)(t) = \lambda \int_0^1 H_2(t,s) h(s) f(s, Sv(s), -v(s)) ds, \quad t \in [0,1]. \tag{2.19}$$

Clearly v is a solution of the BVP (2.17) if and only if v is a fixed point of the operator T.

In the rest of the paper, we make the following assumptions:

(H_1) $a, b \in L^1[0,1]$, $a(t) \geq 0$, $b(t) \geq 0$, $a(1-t) = a(t)$, $b(1-t) = b(t)$, $\alpha, \beta \in (0,1)$;

(H_2) $h \in C((0,1), [0,+\infty))$, $h(1-t) = h(t)$, $0 < \int_0^1 H_2(s,s) h(s) ds < +\infty$;

(H_3) $f(t,u,v) \in C((0,1) \times (0,+\infty) \times (-\infty,0), [0,+\infty))$, $f((1-t),u,v) = f(t,u,v)$, and for any $0 < r_1 < R_1 < +\infty$, $0 < r_2 < R_2 < +\infty$

$$\lim_{n \to +\infty} \sup_{u \in \overline{K}_{r_1,R_1}, -v \in \overline{K}_{r_2,R_2}} \int_{D(n)} H_2(s,s) h(s) f(s, u(s), v(s)) ds = 0, \tag{2.20}$$

where $D(n) = [0, 1/n] \cup [(n-1)/n, 1]$.

Remark 2.5. If (H_1) holds, then for all $t, s \in [0,1]$, we have

$$H_1(1-t, 1-s) = H_1(t,s), \qquad H_2(1-t, 1-s) = H_2(t,s). \tag{2.21}$$

Lemma 2.6. *Assume that conditions* (H_1), (H_2), *and* (H_3) *hold. Then* $T : \overline{K}_{r,R} \to C^+[0,1]$ *is a completely continuous operator.*

Proof. Firstly, for any $r > 0$, we will show

$$\sup_{v\in\partial K_r} \lambda \int_0^1 H_2(s,s)h(s)f(s,Sv(s),-v(s))ds < +\infty. \tag{2.22}$$

At the same time, this implies that $T : K \setminus \{0\} \to C^+[0,1]$ is well defined.

In fact, by (H_3), for any $r > 0$, there exists a natural number m such that

$$\sup_{v\in\partial K_r} \lambda \int_{D(m)} H_2(s,s)h(s)f(s,Sv(s),-v(s))ds < 1. \tag{2.23}$$

For any $v \in \partial K_r$, let $v(t_0) = \max_{t\in[0,1]}|v(t)| = r$. It follows from the concavity of $v(t)$ on $[0,1]$ that

$$v(t) \geq \begin{cases} \dfrac{rt}{t_0}, & 0 \leq t \leq t_0, \\[2mm] \dfrac{r}{1-t_0}(1-t), & t_0 \leq t \leq 1. \end{cases} \tag{2.24}$$

So we obtain

$$v(t) \geq \begin{cases} rt, & 0 \leq t \leq t_0, \\ r(1-t), & t_0 \leq t \leq 1. \end{cases} \tag{2.25}$$

Consequently, from (2.25) for any $t \in [1/m,(m-1)/m]$, we have $r/m \leq v(t) \leq r$ and

$$\frac{l_m r}{m} = \frac{r}{m} \min_{t\in[1/m,(m-1)/m]} \int_0^1 H_1(t,s)ds \leq Sv(t) \leq r \max_{t\in[1/m,(m-1)/m]} \int_0^1 H_1(t,s)ds \leq \mu r, \tag{2.26}$$

where $l_m = \min_{t\in[1/m,(m-1)/m]} \int_0^1 H_1(t,s)ds$ and μ is defined in Remark 2.3. Let

$$M_1 = \max\left\{ f(t,x,y) : (t,x,y) \in \left[\frac{1}{m}, \frac{m-1}{\mathrm{m}}\right] \times \left[\frac{l_m r}{m}, \mu r\right] \times \left[-r, -\frac{r}{m}\right] \right\}. \tag{2.27}$$

By (H_1)–(H_3), we have

$$\begin{aligned}
&\sup_{v\in\partial K_r}\lambda\int_0^1 H_2(s,s)h(s)f(s,Sv(s),-v(s))ds \\
&\quad\le \sup_{v\in\partial K_r}\lambda\int_{D(m)} H_2(s,s)h(s)f(s,Sv(s),-v(s))ds \\
&\qquad+\sup_{v\in\partial K_r}\lambda\int_{1/m}^{(m-1)/m} H_2(s,s)h(s)f(s,Sv(s),-v(s))ds \\
&\quad\le 1+M_1\lambda\int_0^1 H_2(s,s)h(s)ds<+\infty,
\end{aligned} \tag{2.28}$$

that is, (2.22) holds. This also implies that $T(B)$ is uniformly bounded for any bounded set $B\subset \overline{K}_{r,R}$ from (2.28).

Next we prove that T is equicontinuous on $\overline{K}_{r,R}$. In fact, by (H_3) for any $\varepsilon>0$, there exists a natural number k such that

$$\sup_{v\in\overline{K}_{r,R}}\lambda\int_{D(k)} H_2(s,s)h(s)f(s,Sv(s),-v(s))ds<\frac{\varepsilon}{4}. \tag{2.29}$$

Let

$$M_2=\max\left\{f(t,x,y):(t,x,y)\in\left[\frac{1}{k},\frac{k-1}{k}\right]\times\left[\frac{l_k r}{k},\mu r\right]\times\left[-r,-\frac{r}{k}\right]\right\}, \tag{2.30}$$

where $l_k=\min_{t\in[1/k,(k-1)/k]}\int_0^1 H_1(t,s)ds$. Since $H_2(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for the above $\varepsilon>0$ and fixed $s\in[1/k,(k-1)/k]$, there exists $\delta>0$ such that

$$\left|H_2(t,s)-H_2(t',s)\right|\le(2\lambda LM_2)^{-1}(\mu+1)H_2(s,s)\varepsilon \tag{2.31}$$

for $|t-t'|<\delta$ and $t,t'\in[0,1]$. Consequently, when $|t-t'|<\delta$ and $t,t'\in[0,1]$, we have

$$\begin{aligned}
\left|Tv(t)-Tv(t')\right|&\le 2\sup_{v\in\overline{K}_{r,R}}\lambda\int_{D(k)} H_2(s,s)h(s)f(s,Sv(s),-v(s))ds \\
&\quad+\lambda\sup_{v\in\overline{K}_{r,R}}\int_{1/k}^{(k-1)/k}\left|H_2(t,s)-H_2(t',s)\right|h(s)f(s,Sv(s),-v(s))ds<\varepsilon.
\end{aligned} \tag{2.32}$$

This implies that $T(\overline{K}_{r,R})$ is equicontinuous. Then by the Arzela-Ascoli theorem $T:\overline{K}_{r,R}\to C^+[0,1]$ is compact.

Finally, we show that $T : \overline{K}_{r,R} \longrightarrow C^{+}[0,1]$ is continuous. Assume $v_n, v_0 \in \overline{K}_{r,R}$ and $\|v_n - v_0\| \longrightarrow 0 \ \ (n \longrightarrow \infty)$. Then $r \leq \|v_n\| \leq R$ and $r \leq \|v_0\| \leq R$. For any $\varepsilon > 0$, by (H_3), there exists a natural number $m > 0$ such that

$$\sup_{v\in\overline{K}_{r,R}} \lambda \int_{D(m)} H_2(s,s)h(s)f(s, Sv(s), -v(s))ds < \frac{\varepsilon}{4}. \tag{2.33}$$

On the other hand, by (2.25), for any $t \in [1/m, (m-1)/m]$, we have

$$\frac{r}{m} \leq v_n(t) \leq R, \quad \frac{l_m r}{m} \leq Sv_n(t) \leq \mu R, \quad n = 0, 1, 2, \ldots, \tag{2.34}$$

where $l_m = \min_{t\in[1/m,(m-1)/m]} \int_0^1 H_1(t,s)ds$ and μ is defined by Remark 2.3.

Since $f(t,x,y)$ is uniformly continuous in $[1/m, (m-1)/m] \times [l_m r/m, \mu R] \times [-R, -r/m]$, we have that

$$\lim_{n\to+\infty} \left| f(s, Sv_n(s), -v_n(s)) - f(s, Sv_0(s), -v_0(s)) \right| = 0 \tag{2.35}$$

holds uniformly on $s \in [1/m, (m-1)/m]$. Then the Lebesgue dominated convergence theorem yields that

$$\int_{1/m}^{(m-1)/m} \lambda H_2(s,s)h(s) \left| f(s, Sv_n(s), -v_n(s)) - f(s, Sv_0(s), -v_0(s)) \right| ds \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.36}$$

Thus for above $\varepsilon > 0$, there exists a natural number; N, for $n > N$, we have

$$\int_{1/m}^{(m-1)/m} \lambda H_2(s,s)h(s) \left| f(s, Sv_n(s), -v_n(s)) - f(s, Sv_0(s), -v_0(s)) \right| ds < \frac{\varepsilon}{2}. \tag{2.37}$$

It follows from (2.33) and (2.37) that when $n > N$,

$$\begin{aligned} \|Tv_n - Tv_0\| &\leq \int_{1/m}^{(m-1)/m} \lambda H_2(s,s)h(s) \left| f(s, Sv_n(s), -v_n(s)) - f(s, Sv_0(s), -v_0(s)) \right| ds \\ &\quad + 2 \sup_{v\in\overline{K}_{r,R}} \int_{D(m)} \lambda H_2(s,s)h(s) f(s, Sv(s), -v(s))ds \\ &< \frac{\varepsilon}{2} + 2 \times \frac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{2.38}$$

This implies that $T : \overline{K}_{r,R} \longrightarrow C^{+}[0,1]$ is continuous. Thus $T : \overline{K}_{r,R} \longrightarrow C^{+}[0,1]$ is completely continuous. This completes the proof of Lemma 2.6. □

Lemma 2.7. *One has* $T(\overline{K}_{r,R}) \subset K$.

Proof. For $v \in \overline{K}_{r,R}, t \in [0,1]$, we have

$$\begin{aligned}(Tv)(t) &= \lambda\int_0^1 H_2(t,s)h(s)f(s,Sv(s),-v(s))d\\ &\le \lambda\int_0^1 H_2(s,s)h(s)f(s,Sv(s),-v(s))ds.\end{aligned} \tag{2.39}$$

Thus

$$\|Tv\| \le \lambda\int_0^1 H_2(s,s)h(s)f(s,Sv(s),-v(s))ds. \tag{2.40}$$

On the other hand, by Lemma 2.2 we have

$$\begin{aligned}(Tv)(t) &= \lambda\int_0^1 H_2(t,s)h(s)f(s,Sv(s),-v(s))ds\\ &\ge \gamma\lambda\int_0^1 H_2(s,s)h(s)f(s,Sv(s),-v(s))ds\end{aligned} \tag{2.41}$$

which implies $\min_{t\in[0,1]}(Tv)(t) \ge \gamma\|Tv\|$.

In addition, for $v \in K,\ t \in [0,1]$, by Remark 2.5, (H_2), and (H_3) we have

$$\begin{aligned}(Sv)(1-t) &= \int_0^1 H_1(1-t,\tau)v(\tau)d\tau\\ &= \int_1^0 H_1(1-t,1-\tau)v(1-\tau)d(1-\tau)\\ &= \int_0^1 H_1(t,\tau)v(\tau)d\tau = (Sv)(t),\end{aligned} \tag{2.42}$$

$$\begin{aligned}(Tv)(1-t) &= \lambda\int_0^1 H_2(1-t,s)h(s)f(s,Sv(s),-v(s))ds,\\ &= \lambda\int_1^0 H_2(1-t,1-s)h(1-s)f(1-s,Sv(1-s),-v(1-s))d(1-s)\\ &= \lambda\int_0^1 H_2(t,s)h(s)f(s,Sv(s),-v(s))ds = (Tv)(t),\end{aligned} \tag{2.43}$$

that is, $(Tv)(1-t) = (Tv)(t)$, $t \in [0,1]$. Therefore, $(Tv)(t)$ is symmetric on $[0,1]$. Obviously, $(Tv)(t)$ is concave on $[0,1]$. Consequently, $T(\overline{K}_{r,R}) \subset K$. This completes the proof of Lemma 2.7. □

3. The Main Results

Our main results of this paper are as follows. For notational convenience, we let

$$f^0 = \limsup_{\substack{|x|+|y|\to 0\\ x>0,\, y<0}} \max_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|}, \qquad f_\infty = \liminf_{\substack{|x|+|y|\to +\infty\\ x>0,\, y<0}} \min_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|},$$
$$f^\infty = \limsup_{\substack{|x|+|y|\to +\infty\\ x>0,\, y<0}} \max_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|}, \qquad f_0 = \liminf_{\substack{|x|+|y|\to 0\\ x>0,\, y<0}} \min_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|}. \tag{3.1}$$

Theorem 3.1. *Suppose that conditions* (H_1), (H_2), *and* (H_3) *are satisfied. Further assume that the following condition* (H_4) *holds:*

(H_4) $Lf^0 < lf_\infty$.

Then the BVP (1.1) *has at least one symmetric positive solution for any*

$$\lambda \in \left(\frac{1}{lf_\infty}, \frac{1}{Lf^0}\right), \tag{3.2}$$

where L and l are defined by Remark 2.3.

Proof. Let λ satisfy (3.2) and at $\varepsilon > 0$ be chosen such that

$$f_\infty - \varepsilon > 0, \quad \frac{1}{(f_\infty - \varepsilon)l} \le \lambda \le \frac{1}{(f^0+\varepsilon)L}. \tag{3.3}$$

Next, by (H_4) there exists $r_0 > 0$ such that

$$f(t,x,y) \le \left(f^0+\varepsilon\right)(|x|+|y|), \quad \forall t \in [0,1],\ 0 < |x|+|y| < r_0,\ x>0,\ y<0. \tag{3.4}$$

Take $r = r_0/(\mu+1)$. Notice that

$$0 < |Sv| + |v| \le (\mu+1)\|v\| = (\mu+1)r = r_0, \quad 0 \le t \le 1. \tag{3.5}$$

It follows from (3.4) and (3.5) that, for any $v \in \partial K_r$,

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \lambda \int_0^1 H_2(t,s)h(s)f(s,Sv(s),-v(s))ds \\
&\le \lambda \int_0^1 H_2(s,s)h(s)f(s,Sv(s),-v(s))ds \\
&\le \lambda\left(f^0+\varepsilon\right)\int_0^1 H_2(s,s)h(s)(|Sv(s)|+|v(s)|)ds \\
&= \lambda\left(f^0+\varepsilon\right)(\mu+1)r\int_0^1 H_2(s,s)h(s)ds \\
&= \lambda L\left(f^0+\varepsilon\right)r \le r = \|v\|.
\end{aligned} \tag{3.6}$$

Thus, $\|Tv\| \le \|v\|$, for all $v \in \partial K_r$.

On the other hand, for the above ε, by (H_4), there exists $R_0 > 0$ such that

$$f(t,x,y) > (f_\infty - \varepsilon)(|x|+|y|), \quad t\in[0,1],\ |x|+|y| \ge R_0,\ x>0,\ y<0. \tag{3.7}$$

Let $R = \max\{2r, \gamma^{-1}R_0\}$ and $\varphi(t) \equiv 1$, $t\in[0,1]$. Then $R > r$ and $\varphi(t) \in \partial K_1$.

In the following we show $v \ne Tv + m\varphi$ $(m>0)$. Otherwise, there exists $v_0 \in \partial K_R$ and $m_0 > 0$ such that $v_0 = Tv_0 + m_0\varphi$. Let $\xi = \min\{v_0(t) : t\in[0,1]\}$ and notice that for any $s\in[0,1]$,

$$|Sv_0(s)|+|v_0(s)| \ge \min_{s\in[0,1]}[|Sv_0(s)|+|v_0(s)|] \ge \min_{s\in[0,1]}|v_0(s)| \ge \gamma\|v_0\| \ge R_0. \tag{3.8}$$

Consequently for any $t\in[0,1]$, we have

$$\begin{aligned}
v_0(t) &= \lambda\int_0^1 H_2(t,s)h(s)f(s,Sv_0(s),-v_0(s))ds + m_0\varphi(t) \\
&= \lambda\int_0^1 H_2(t,s)h(s)f(s,Sv_0(s),-v_0(s))ds + m_0 \\
&\ge \lambda(f_\infty-\varepsilon)\int_0^1 H_2(t,s)h(s)(|Sv_0(s)|+|v_0(s)|)ds + m_0 \\
&\ge \lambda(f_\infty-\varepsilon)\int_0^1 H_2(t,s)h(s)v_0(s)ds + m_0 \\
&\ge \lambda(f_\infty-\varepsilon)\xi\min_{t\in[0,1]}\int_0^1 H_2(t,s)h(s)ds + m_0 \ge \xi + m_0 > \xi.
\end{aligned} \tag{3.9}$$

This implies that $\xi > \xi$, which is a contradiction. It follows from Lemma 1.1 that T has a fixed point v^* with $r < |v^*| < R$. Thus v^* is a symmetric positive solution of the BVP (2.17). Consequently, by Lemma 2.4, one can obtain that BVP (1.1) has a symmetric positive solution. This completes the proof of Theorem 3.1. □

Remark 3.2. Since $0 < l < L < +\infty$, we easily obtain $0 < 1/lf_\infty < 1$, $1/Lf^0 > 1$. Thus $1 \in (1/lf_\infty, 1/Lf^0)$; so when $\lambda = 1$, Theorem 3.1 always holds.

Remark 3.3. From Theorem 3.1, we can see that $f(t,x,y)$ need not be superlinear or sublinear. In fact, Theorem 3.1 still holds, if one of the following conditions is satisfied.

(i) If $f_\infty = \infty$, $f^0 > 0$, then for each $\lambda \in (0, 1/Lf^0)$.

(ii) If $f_\infty = \infty$, $f^0 = 0$, then for each $\lambda \in (0, +\infty)$.

(iii) If $f_\infty > l^{-1} > 0$, $f^0 = 0$, then for each $\lambda \in (1/lf_\infty, +\infty)$.

Theorem 3.4. *Suppose that conditions (H_1), (H_2), and (H_3) are satisfied. Further assume that the following condition (H_5) holds.*

(H_5) $Lf^\infty < lf_0$.

Then the BVP (1.1) has at least one symmetric positive solution for any

$$\lambda \in \left(\frac{1}{lf_0}, \frac{1}{Lf^\infty}\right), \tag{3.10}$$

where L and l are defined by Remark 2.3.

Proof. Let λ satisfy (3.10) and let $\varepsilon_1 > 0$ be chosen such that $L^{-1} - \varepsilon_1 > 0$ and $\lambda f^\infty < L^{-1} - \varepsilon_1$. By (H_5), there exists $\mu R_0'$ such that

$$f(t,x,y) \le \frac{1}{\lambda}\left(L^{-1} - \varepsilon_1\right)\left(|x| + |y|\right), \quad |x| + |y| \ge \mu R_0',\ x > 0,\ y < 0,\ t \in [0,1]. \tag{3.11}$$

Let

$$M_0 = \sup_{v \in \partial K_{R_0'}} \lambda \int_0^1 H_2(s,s)h(s)f(s, Sv(s), -v(s))ds. \tag{3.12}$$

Then $M_0 < +\infty$ by (2.22). Take $R_1 > max\{R_0', M_0/L\varepsilon_1\}$, then $M_0 < LR_1\varepsilon_1$.

Notice that $v \in \partial K_{R_0'}$ implies that

$$v(t) \le \|v\| = R_0', \qquad Sv(t) \le \max_{t \in [0,1]} \int_0^1 H_1(t,s)ds\|v\| = \mu R_0'. \tag{3.13}$$

So for any $v \in \partial K_{R_1}$, let

$$D(Sv, -v) = \{t \in [0,1] : (Sv, -v) \in [\mu R_0', +\infty) \times (-\infty, -R_0']\}, \tag{3.14}$$

and then for any $t \in D(Sv, -v)$, clearly $\mu R_0' \leq |Sv| + |v| \leq (\mu + 1)\|v\| = (\mu + 1)R_1$.

In addition, for any $v \in \partial K_{R_1}$, let $v_1(t) = \min\{v(t), R_0'\}$, then $v_1 \in \partial K_{R_0'}$. Thus, for any $v \in \partial K_{R_1}$, we have

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \lambda \int_0^1 H_2(t,s)h(s)f(s, Sv(s), -v(s))ds \\
&\leq \max_{t\in[0,1]} \lambda \int_{D(Sv,-v)} H_2(t,s)h(s)f(s, Sv(s), -v(s))ds \\
&\quad + \lambda \int_{[0,1]/D(Sv,-v)} H_2(s,s)h(s)f(s, Sv(s), -v(s))ds \\
&\leq \frac{1}{\lambda}\left(L^{-1} - \varepsilon_1\right)\lambda \int_0^1 H_2(s,s)h(s)(|Sv(s)| + |v(s)|)ds \\
&\quad + \lambda \int_0^1 H_2(s,s)h(s)f(s, Sv_1(s), -v_1(s))ds \\
&\leq \left(L^{-1} - \varepsilon_1\right)(\mu+1)R_1 \int_0^1 H_2(s,s)h(s)ds + M_0 \\
&\leq \left(L^{-1} - \varepsilon_1\right)LR_1 + M_0 < R_1 = \|v\|.
\end{aligned} \tag{3.15}$$

Therefore $\|Tv\| \leq \|v\|$ for any $v \in \partial K_{R_1}$.

Next, let λ satisfy (3.10). Choose $\varepsilon_2 > 0$ such that $l^{-1} + \varepsilon_2 < \lambda f_0$. Then from (H_5), there exists $0 < \delta < (\mu + 1)R_1$ such that

$$f(t, x, y) \geq \frac{1}{\lambda}\left(l^{-1} + \varepsilon_2\right)(|x| + |y|), \quad 0 < |x| + |y| \leq \delta,\ x > 0,\ y < 0,\ t \in [0,1]. \tag{3.16}$$

Let $r_1 = \delta/(\mu + 1)$ and $\phi(t) \equiv 1$, $t \in [0,1]$, then $r_1 < R_1$ and $\phi \in \partial K_1$.

Now we prove $v \neq Tv + m\phi$ $(m > 0)$. Otherwise, there exists $v_0 \in \partial K_{r_1}$ and $m_0 > 0$ such that $v_0 = Tv_0 + m_0\phi$. Let $\zeta = \min\{v_0(t) : t \in [0,1]\}$ and apply that

$|v_0(s)| + |Sv_0(s)| < (\mu + 1)r_1 = \delta$; then for any $t \in [0,1]$, we have

$$\begin{aligned}
v_0(t) &= \lambda \int_0^1 H_2(t,s)h(s)f(s, Sv_0(s), -v_0(s))ds + m_0\phi(t) \\
&= \lambda \int_0^1 H_2(t,s)h(s)f(s, Sv_0(s), -v_0(s))ds + m_0 \\
&\geq \frac{1}{\lambda}\left(l^{-1} + \varepsilon_2\right)\lambda \int_0^1 H_2(t,s)h(s)(|Sv_0(s)| + |v_0(s)|)ds + m_0 \\
&\geq \left(l^{-1} + \varepsilon_2\right) \int_0^1 H_2(t,s)h(s)v_0(s)ds + m_0 \\
&\geq \left(l^{-1} + \varepsilon_2\right)\zeta \min_{t\in[0,1]} \int_0^1 H_2(t,s)h(s)ds + m_0 \\
&\geq (1 + l\varepsilon_2)\zeta + m_0 > \zeta.
\end{aligned} \tag{3.17}$$

This implies that $\zeta > \zeta$, which is a contradiction. It follows from Lemma 1.1 that T has a fixed point v^{**} with $r_1 < |v^{**}| < R_1$. Thus v^{**} is a symmetric positive solution of the BVP (2.17). By Lemma 2.4, BVP (1.1) has a symmetric positive solution. □

Remark 3.5. From Theorem 3.4 we can see that the conclusions still hold, if one of the following conditions is satisfied.

(i) If $f^\infty < L^{-1}$, $f_0 = \infty$, then for each $\lambda \in (0, 1/Lf^\infty)$.

(ii) If $f^\infty = 0$, $f_0 = +\infty$, then for each $\lambda \in (0, +\infty)$.

(iii) If $f^\infty = 0$, $f_0 > l^{-1} > 0$, then for each $\lambda \in (1/lf_0, +\infty)$.

Remark 3.6. Because of singularity of h, f, it seems to be difficult to prove our results by using the norm-type expansion and compression theorem. In addition, we need to point out that we not only obtain the existence of symmetric positive solutions of BVP (1.1) but also get the explicit interval about λ, which is different from the previous papers (see [4, 5]).

Remark 3.7. If conditions (H_1), (H_2) and the following condition (H_3') are satisfied,

(H_3') $f(t,u) \in C((0,1) \times (0,+\infty), [0,+\infty))$, $f((1-t),u) = f(t,u)$, and for any $0 < r < R < +\infty$,

$$\lim_{n\to+\infty} \sup_{u\in\overline{K}_{r,R}} \int_{D(n)} H(s,s)h(s)f(s,u(s))ds = 0, \tag{3.18}$$

where $D(n) = [0, 1/n] \cup [(n-1)/n, 1]$. Then our results can be applied to the second-order singular boundary value problem with nonlocal boundary condition:

$$\begin{gathered} u''(t) + \lambda h(t) f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 a(s) u(s) ds, \end{gathered} \tag{3.19}$$

under the corresponding to conditions of Theorems 3.1 and 3.4.

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