Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 286031, 18 pages
doi:10.1155/2010/286031

*Research Article*

# Almost Periodic Solutions of a Discrete Mutualism Model with Feedback Controls

**Zheng Wang and Yongkun Li**

*Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China*

Correspondence should be addressed to Yongkun Li, yklie@ynu.edu.cn

Received 31 October 2009; Accepted 16 March 2010

Academic Editor: Leonid Berezansky



We consider a discrete mutualism model with feedback controls. Assuming that the coefficients in the system are almost periodic sequences, we obtain the existence and uniqueness of the almost periodic solution which is uniformly asymptotically stable.

## 1. Introduction

Two species cohabit a common habitat and each species enhances the average growth rate of the other; this type of ecological interaction is known as facultative mutualism [1]. A two species mutualism model can be described in the following form:

$$\begin{aligned} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} &= N_1(t) f_1(N_1(t), N_2(t)), \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} &= N_1(t) f_2(N_1(t), N_2(t)), \end{aligned} \tag{1.1}$$

where $f_1, f_2$ are continuously differentiable such that

$$\frac{\partial f_1}{\partial N_2} \geq 0, \qquad \frac{\partial f_2}{\partial N_1} \geq 0. \tag{1.2}$$

One of the simplest models which satisfies the above assumption is the traditional Lotka-Volterra two species mutualism model, which takes the form

$$\begin{aligned} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} &= N_1(t)(a_1 - b_1 N_1(t) + c_1 N_2(t)), \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} &= N_2(t)(a_2 - b_2 N_2(t) + c_2 N_1(t)). \end{aligned} \tag{1.3}$$

Since the above system could exhibit unbounded solutions [2, 3] and it is well known that in nature, with the restriction of resources, it is impossible for one species to survive if its density is too high. Thus, the above model is not so good in describing the mutualism of two species. Gopalsamy [4] has proposed the following model to describe the mutualism mechanism:

$$\begin{aligned} \frac{dN_1(t)}{dt} &= r_1 N_1(t)\left[\frac{K_1 + \alpha_1 N_2(t)}{1 + N_2(t)} - N_1(t)\right], \\ \frac{dN_2(t)}{dt} &= r_2 N_2(t)\left[\frac{K_2 + \alpha_2 N_1(t)}{1 + N_1(t)} - N_2(t)\right], \end{aligned} \tag{1.4}$$

where $r_i$ denotes the intrinsic growth rate of species $N_i$ and $\alpha_i > K_i, i = 1, 2$. The carrying capacity of species $N_i$ is $K_i$ in the absence of other species, while with the help of the other species, the carrying capacity becomes $(K_i + \alpha_i N_{3-i})/(1 + N_{3-i}), i = 1, 2$. The above mutualism can be classified as facultative, obligate, or a combination of both. For more details of mutualistic interactions, we refer to [5–9]. Realistic models require the inclusion of the effect of changing environment. This motivate us to consider the following nonautonomous model:

$$\begin{aligned} \frac{dN_1(t)}{dt} &= r_1(t) N_1(t)\left[\frac{K_1(t) + \alpha_1(t) N_2(t)}{1 + N_2(t)} - N_1(t)\right], \\ \frac{dN_2(t)}{dt} &= r_2(t) N_2(t)\left[\frac{K_2(t) + \alpha_2(t) N_1(t)}{1 + N_1(t)} - N_2(t)\right]. \end{aligned} \tag{1.5}$$

Since many authors [10, 11] have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations, then, discrete-timemodels can provide efficient computational types of continuous models for numerical simulations. It is reasonable to study the discrete-time mutualism model governed by difference equations. One of the ways of deriving difference equations modeling the dynamics of populations with nonoverlapping generations is based on appropriate modifications of the corresponding models with overlapping generations [4, 12]. In this approach, differential equations with piecewise constant arguments have been proved to be useful. Following the same idea and the same method in [4, 12], one can easily derive the following discrete analog of (1.5), which takes the form of

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n) + \alpha_1(n) x_2(n)}{1 + x_2(n)} - x_1(n) - b_1(n) u_1(n)\right]\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{r_2(n)\left[\frac{K_2(n) + \alpha_2(n) x_1(n)}{1 + x_1(n)} - x_2(n) - b_2(n) u_2(n)\right]\right\}. \end{aligned} \tag{1.6}$$

The exponential form of (1.6) is more biologically reasonable than that directly derived by replacing the differential by difference in (1.5). Feedback control is the basic mechanism by which systems, whether mechanical, electrical, or biological, maintain their equilibrium or homeostasis. During the last decade, a series of mathematical models have been established to describe the dynamics of feedback control systems [13–17].

In this paper, we are concerned with the following discrete mutualism model with feedback controls:

$$x_1(n+1) = x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)x_2(n)}{1+x_2(n)} - x_1(n) - b_1(n)u_1(n)\right]\right\},$$
$$x_2(n+1) = x_2(n)\exp\left\{r_2(n)\left[\frac{K_2(n)+\alpha_2(n)x_1(n)}{1+x_1(n)} - x_2(n) - b_2(n)u_2(n)\right]\right\},$$
$$\Delta u_1(n) = -a_1(n)u_1(n) + c_1(n)x_1(n),$$
$$\Delta u_2(n) = -a_2(n)u_2(n) + c_2(n)x_2(n). \tag{1.7}$$

To the best of our knowledge, though many works have been done for the population dynamic system with feedback controls, most of the works dealt with the continuous time model. For more results about the existence of almost periodic solutions of a continuous time system, we can refer to [18–22] and the references cited therein. There are few works that consider the existence of almost periodic solutions for discrete time population dynamic model with feedback controls. So, our main purpose of this paper is to study the existence and uniqueness of almost periodic solutions for the model (1.7).

Throughout this paper, we assume that

$(H)$ $\{r_i(n)\}, \{K_i(n)\}, \{\alpha_i(n)\}, \{a_i(n)\}, \{b_i(n)\}$, and $\{c_i(n)\}$ for $i = 1,2$ are bounded non-negative almost periodic sequences such that

$$0 < r_i^l \le r_i(n) \le r_i^u, \qquad 0 < K_i^l \le K_i(n) \le K_i^u, \qquad 0 < \alpha_i^l \le \alpha_i(n) \le \alpha_i^u,$$
$$0 < a_i^l \le a_i(n) \le a_i^u < 1, \qquad 0 \le b_i^l \le b_i(n) \le b_i^u, \qquad 0 \le c_i^l \le c_i(n) \le c_i^u, \tag{1.8}$$

and $\alpha_i > k_i$ for $i = 1,2$.

Here, for any bounded sequence $\{a(n)\}$, $a^u = \sup_{n\in N}\{a(n)\}$ and $a^l = \inf_{n\in N}\{a(n)\}$. By the biological meaning, we focus our discussion on the positive solution of the system (1.7). So it is assumed that the initial conditions of (1.7) are of the form

$$x_i(0) > 0, \quad u_i(0) > 0, \quad i = 1,2. \tag{1.9}$$

One can easily show that the solutions of (1.5) with the initial condition (1.9) are defined and remain positive for all $n \in Z^+ = \{0,1,2,\ldots\}$.

## 2. Preliminaries

In this section, we will introduce two definitions and a useful lemma.

*Definition 2.1* (see [23]). A sequence $x : Z \to R^k$ is called an almost periodic sequence if the $\varepsilon$- translation set of $x$:

$$E\{\varepsilon, x\} := \{\tau \in Z : |x(n+\tau) - x(n)| < \varepsilon\}, \tag{2.1}$$

for all $n \in Z$ is a relatively dense set in $Z$ for all $\varepsilon > 0$, that is, for any given $\varepsilon > 0$, there exists an integer $l > 0$ such that each discrete interval of length $l$ contains an integer $\tau = \tau(\varepsilon) \in E\{\varepsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \varepsilon, \tag{2.2}$$

for all $n \in Z$, $\tau$ is called the $\varepsilon$-translation number of $x(n)$.

*Definition 2.2* (see [23]). Let $f : Z \times D \to R^k$, where $D$ is an open set in $R^k$, $f(n,x)$ is said to be almost periodic in $n$ uniformly for $x \in D$, or uniformly almost periodic for short, if for any $\varepsilon > 0$ and any compact set $S$ in $D$, there exists a positive integer $l(\varepsilon, S)$ such that any interval of length $l(\varepsilon, S)$ contains an integer $\tau$ for which

$$\left| f(n+\tau, x) - f(n,x) \right| < \varepsilon \tag{2.3}$$

for all $n \in Z$ and $x \in S$. $\tau$ is called the $\varepsilon$-translation number of $f(n,x)$.

**Lemma 2.3** (see [23]). *$\{x(n)\}$ is an almost periodic sequence if and only if for any sequence $\{h'_k\} \subset Z$ there exists a subsequence $\{h_k\} \subset \{h'_k\}$ such that $x(n+h_k)$ converges uniformly on $n \in Z$ as $k \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

## 3. Persistence

In this section, we establish a persistence result for model (1.7).

**Proposition 3.1.** *Assume that $(H)$ holds. For every solution $(x_1(n), x_2(n), u_1(n), u_2(n))$ of* (1.7)

$$\limsup_{n\to\infty} x_1(n) < x_1^*, \quad \limsup_{n\to\infty} x_2(n) < x_2^*, \quad \limsup_{n\to\infty} u_1(n) < u_1^*, \quad \limsup_{n\to\infty} u_2(n) < u_2^*, \tag{3.1}$$

*where*

$$x_1^* = \frac{\alpha_1^u}{r_1^u} \exp\left[\alpha_1^u(r_1^u - 1)\right], \quad x_2^* = \frac{\alpha_2^u}{r_2^u} \exp\left[\alpha_2^u(r_2^u - 1)\right], \quad u_1^* = \frac{x_1^* c_1^u}{a_1^l}, \quad u_2^* = \frac{x_2^* c_2^u}{a_2^l}. \tag{3.2}$$

*Proof.* We first present two cases to prove that

$$\limsup_{n\to\infty} x_1(n) < x_1^*. \tag{3.3}$$

*Case 1.* By the first equation of model (1.7), $(H)$ and (1.9), we have

$$
\begin{aligned}
x_1(n+1) &= x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)x_2(n)}{1+x_2(n)} - x_1(n) - b_1(n)u_1(n)\right]\right\} \\
&< x_1(n)\exp\{r_1(n)[\alpha_1(n) - x_1(n) - b_1(n)u_1(n)]\} \\
&= x_1(n)\exp\left\{r_1(n)\alpha_1(n)\left[1-\frac{x_1(n)}{\alpha_1(n)} - \frac{b_1(n)u_1(n)}{\alpha_1(n)}\right]\right\}.
\end{aligned}
\tag{3.4}
$$

Then there exists an $l_0 \in N$ such that $x_1(l_0+1) \geq x_1(l_0)$. So, $1 - x_1(l_0)/\alpha_1(l_0) - (b_1(l_0)u_1(l_0)/\alpha_1(l_0)) \geq 0$. Hence, $x_1(l_0) \leq \alpha_1(l_0) \leq \alpha_1^u \leq x_1^*$, and

$$
\begin{aligned}
x_1(l_0+1) &< x_1(l_0)\exp\left\{r_1(l_0)\alpha_1(l_0)\left[1-\frac{x_1(l_0)}{\alpha_1(l_0)} - \frac{b_1(l_0)u_1(l_0)}{\alpha_1(l_0)}\right]\right\} \\
&\leq x_1(l_0)\exp\left[r_1^u\alpha_1^u\left(1-\frac{x_1(l_0)}{\alpha_1(l_0)}\right)\right] \\
&= \alpha_1(l_0)\frac{x_1(l_0)}{\alpha_1(l_0)}\exp\left[r_1^u\alpha_1^u\left(1-\frac{x_1(l_0)}{\alpha_1(l_0)}\right)\right] \\
&\leq \frac{\alpha_1^u}{r_1^u}\exp\left[\alpha_1^u(r_1^u-1)\right] \\
&= x_1^*.
\end{aligned}
\tag{3.5}
$$

Here we used $\max_{x\in R} x\exp(r(1-x)) = \exp(r-1)/r$ for $r > 0$. We claim that $x_1(n) \leq x_1^*$ for $n \geq l_0$.

In fact, if there exists an integer $m \geq n_0+2$ such that $x_1(m) > x_1^*$, and letting $m_1$ be the least integer between $n_0$ and $m$ such that $x_1(m_1) = \max_{n_0\leq n\leq m_1}\{x_1(n)\}$, then $m_1 \geq n_0+2$ and $x_1(m_1) > x_1(m_1-1)$ which implies $x_1(m_1) \leq x_1^* < x_1(m)$. This is impossible. The claim is proved.

*Case 2* ($x_1(n) \geq x_1(n+1)$ for $n \in N$). In particular, $\lim_{n\to\infty} x_1(n)$ exists, denoted by $\overline{x}_1$. We claim that $\overline{x}_1 \leq x_1^*$. By way of contradiction, assume that $\overline{x}_1 > x_1^*$. Taking $\lim_{n\to\infty}(1 - x_1(n)/\alpha_1(n) - b_1(n)u_1(n)/\alpha_1(n)) = 0$. Noting that $\alpha_1^u \leq x^*$, hence

$$
1-\frac{x_1(n)}{\alpha_1(n)} - \frac{b_1(n)u_1(n)}{\alpha_1(n)} \leq 1-\frac{x_1(n)}{\alpha_1(n)} \leq 1-\frac{\overline{x_1}}{\alpha_1^u} < 0,
\tag{3.6}
$$

for $n \in N$, which is a contradiction. This proves the claim.

We can prove that $\limsup_{n\to\infty} x_2(n) \leq x_2^*$ in the similar way. Therefore, for each $\varepsilon > 0$, there exists a large enough integer $n_0$ such that $x_i(n) \leq x_i^* + \varepsilon$, $i = 1,2$ whenever $n \geq n_0$. The proof of $\limsup_{n\to\infty} u_i(n) < u_i^* (i = 1,2)$ is very similar to that of Proposition 1 in [11]. Here we omit the details here. □

**Proposition 3.2.** *Assume that* $(H)$ *and* (1.6) *hold; furthermore,* $K_1^l - b_1^u u_1^* > 0$ *and* $K_2^l - b_2^u u_2^* > 0$, *where* $u_1^*$ *and* $u_2^*$ *are the same as those in Proposition 3.1. Then*

$$\liminf_{n\to\infty} x_1(n) > x_{1*}, \quad \liminf_{n\to\infty} x_2(n) > x_{2*}, \quad \liminf_{n\to\infty} u_1(n) > u_{1*}, \quad \liminf_{n\to\infty} u_2(n) > u_{2*}, \tag{3.7}$$

*where* $x_{1_*} = (K_1^l - b_1^u u_1^*)\exp[r_1^l(K_1^l - x_1^* - b_1^u u_1^*)]$, $x_{2_*} = (K_2^l - b_2^u u_2^*)\exp[r_2^l(K_2^l - x_2^* - b_2^u u_2^*)]$, $u_{1*} = c_1^l x_{1*}/a_1^u, u_{2*} = c_2^l x_{2*}/a_2^u$.

*Proof.* We also present two cases to prove that $\liminf_{n\to\infty} x_1(n) > x_{1*}$.

For any $\varepsilon > 0$ which satisfies $K_1^l - b_1^u u_1^* > 0$, according to Proposition 3.1, there exists $n_0 \in N$ such that

$$x_1(n) \le x_1^* + \varepsilon, \quad x_2(n) \le x_2^* + \varepsilon, \quad u_1(n) \le u_1^* + \varepsilon, \quad u_2(n) \le u_2^* + \varepsilon, \tag{3.8}$$

for $n \ge n_0$.

*Case 1.* There exists a positive integer $l_0 \ge n_0$ such that $x_1(l_0+1) \le x_1(l_0)$. Note that for $n \ge l_0$,

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n) + \alpha_1(n)x_2(n)}{1 + x_2(n)} - x_1(n) - b_1(n)u_1(n)\right]\right\} \\ &> x_1(n)\exp\{r_1(n)[K_1(n) - x_1(n) - b_1(n)u_1(n)]\} \\ &= x_1(n)\exp\left\{r_1(n)K_1(n)\left[1 - \frac{x_1(n)}{K_1(n)} - \frac{b_1(n)u_1(n)}{K_1(n)}\right]\right\} \\ &\ge x_1(n)\exp\left\{r_1(n)K_1(n)\left[1 - \frac{x_1(n)}{K_1(n)} - \frac{b_1^u(u_1^* + \varepsilon)}{K_1(n)}\right]\right\}. \end{aligned} \tag{3.9}$$

In particular, with $n = l_0$, we get

$$1 - \frac{x_1(l_0)}{K_1(l_0)} - \frac{b_1^u(u_1^* + \varepsilon)}{K_1(l_0)} \le 0, \tag{3.10}$$

which implies that $x_1(l_0) \ge K_1^l - b_1^u(u_1^* + \varepsilon)$. Then

$$x_1(l_0+1) > \left[K_1^l - b_1^u(u_1^* + \varepsilon)\right]\exp\left[r_1^l K_1^l\left(1 - \frac{x_1^* + \varepsilon}{K_1^l} - \frac{b_1^u(u_1^* + \varepsilon)}{K_1^l}\right)\right]. \tag{3.11}$$

Let $x_{1\varepsilon} = [K_1^l - b_1^u(u_1^* + \varepsilon)]\exp[r_1^l K_1^l(1 - (x_1^* + \varepsilon)/K_1^l - b_1^u(u_1^* + \varepsilon)/K_1^l)]$. We claim that $x_1(n) \ge x_{1\varepsilon}$ for $n \ge l_0$.

By a way of contradiction, assume that there exists a $p_0 \ge l_0$ such that $x_1(p_0) < x_{1\varepsilon}$. Then $p_0 \ge l_0 + 2$, let $p_1 \ge l_0 + 2$ be the smallest integer such that $x_1(p_1) < x_{1\varepsilon}$. Then $x(p_1 - 1) < x(p_1)$. The above argument produces that $x_1(p_1) \ge x_{1\varepsilon}$, a contradiction. This proves the claim.

*Case 2.* We assume that $x_1(n+1) > x_1(n)$ for all large $n$. Then $\lim_{n\to\infty} x_1(n)$ exists, denoted by $\underline{x}_1$. We claim that $\underline{x}_1 \geq K_1^l - b_1^u(u_1^* + \varepsilon)$. By way of contradiction, assume that $\underline{x}_1 < K_1^l - b_1^u(u_1^* + \varepsilon)$. Taking $\lim_{n\to\infty}(1 - x_1(n)/K_1(n) - b_1(n)u_1(n)/K_1(n)) = 0$, which is a contradiction, since

$$\liminf_{n\to\infty}\left(1 - \frac{x_1(n)}{K_1(n)} - \frac{b_1(n)u_1(n)}{K_1(n)}\right) \geq 1 - \frac{\underline{x}_1}{K_1^l} - \frac{b_1^u(u_1^* + \varepsilon)}{K_1^l} > 0. \tag{3.12}$$

Noting that $x_1^* \geq K_1^u \geq K_1^l$, we see that $K_1^l - b_1^u(u_1^* + \varepsilon) \geq x_{1\varepsilon}$, and $\lim_{\varepsilon\to 0} x_{1\varepsilon} = x_{1*}$. We can easily see that $\liminf_{n\to\infty} x_1(n) \geq x_{1*}$ holds. Similarly, we can prove that $\liminf_{n\to\infty} x_2(n) \geq x_{2*}$. Thus for any $\varepsilon > 0$ small enough, there exists a positive integer $n_0$, such that $x_i(n) \geq x_{i*} - \varepsilon > 0$ for $n \geq n_0$.

The proof of $\liminf_{n\to\infty} u_i(n) > u_{i*}, i = 1, 2$ is very similar to that of Proposition 2 in [11]. Here we omit the details. □

## 4. Main Results

For our purpose, we first introduce the following results which are given in Persistence.

**Lemma 4.1.** *Assume that* (1.9), $(H)$, $K_1^l - b_1^u u_1^* > 0$, *and* $K_2^l - b_2^u u_2^* > 0$ *hold, then*

$$x_{i_*} < \liminf_{n\to\infty} x_i(n) \leq \limsup_{n\to\infty} x_i(n) < x_i^*, \qquad u_{i_*} < \liminf_{n\to\infty} u_i(n) \leq \limsup_{n\to\infty} u_i(n) < u_i^*, \tag{4.1}$$

*where* $x_i^* = (\alpha_i^u/r_i^u)\exp[\alpha_i^u(r_i^u - 1)]$, $u_i^* = x_i^* c_i^u/a_i^l$, $x_{i*} = (K_i^l - b_i^u u_i^*)\exp[r_i^l(K_i^l - x_i^* - b_i^u u_i^*)]$, $u_{i*} = c_i^l x_{i*}/a_i^u$, $i = 1, 2$.

In [24], Zhang considered the following almost periodic difference system

$$x(n+1) = f(n, x(n)), \quad n \in Z^+, \tag{4.2}$$

where $f : Z \times S_B \to R^k$, $S_B = \{x \in R^k : \|x\| < B\}$, and $f(n, x)$ is almost periodic in $n$ uniformly for $x \in S_B$ and is continuous in $x$. Related to system (4.2), he also considered the following product system:

$$x(n+1) = f(n, x(n)), \qquad y(n+1) = f(n, y(n)) \tag{4.3}$$

and obtained the following theorem.

**Theorem 4.2** (see [24]). *Suppose that there exists a Lyapunov functional $V(n, x, y)$ defined for $n \in Z^+$, $\|x\| < B$, $\|y\| < B$ satisfying the following conditions:*

(i) $a(\|x - y\|) \leq V(n, x, y) \leq b(\|x - y\|)$, *where* $a, b \in \kappa$ *with* $\kappa = \{a \in C(R^+, R^+) : a(0) = 0$ *and* $a$ *is increasing*$\}$,

(ii) $|V(n, x_1, y_1) - V(n, x_2, y_2)| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\|)$, *where* $L > 0$ *is a constant,*

(iii) $\Delta V_{(4.3)}(n, x, y) \leq -aV(n, x, y)$, *where* $0 < a < 1$ *is a constant and* $\Delta V_{(4.3)}(n, x, y) = V(n+1, f(n, x), f(n, y)) - V(n, x, y)$.

*Moreover, if there exists a solution* $\varphi(n)$ *of* (4.2) *such that* $\|\varphi(n)\| \leq B^* < B$ *for* $n \in Z^+$*, then there exists a unique uniformly asymptotically stable almost periodic solution* $p(n)$ *of system* (4.2) *which is bounded by* $B^*$*. In particular, if* $f(n,x)$ *is periodic of period* $\omega$*, then there exists a unique uniformly asymptotically stable periodic solution of* (4.2) *of period* $\omega$*.*

According to Theorem 4.2, we first prove that there exists a bounded solution of (1.7) and then construct an adaptive Lyapunov functional for (1.7).

We denote by $\Omega$ the set of all solutions $X(n) = (x_1(n), x_2(n), u_1(n), u_2(n))$ of system (1.7) satisfying $x_{i*} \leq x_i(n) \leq x_i^*, u_{i*} \leq u_i(n) \leq u_i^*$ $(i = 1,2)$ for all $n \in Z^+$.

**Lemma 4.3.** *Assume that (H) and the conditions of Lemma 4.1 hold, then* $\Omega \neq \emptyset$*.*

*Proof.* It is now possible to show by an inductive argument that system (1.7) leads to

$$x_i(n) = x_i(0)\exp\sum_{l=0}^{n-1}\left\{r_i(l)\left[\frac{K_i(l)+\alpha_i(l)x_j(l)}{1+x_j(l)} - x_i(l) - b_i(l)u_i(l)\right]\right\},$$
$$u_i(n) = u_i(0) - \sum_{l=0}^{n-1}\{a_i(l)u_i(l) - c_i(l)x_i(l)\}, \tag{4.4}$$

for $i, j = 1,2, i \neq j$. From Lemma 4.1, for any solution $X(n) = (x_1(n), x_2(n), u_1(n), u_2(n))$ of (1.7) with initial condition (1.9) satisfies (4.2). Hence, for any $\varepsilon > 0$, there exist $\mathrm{n}_0$, if $n_0$ is sufficiently large, we have

$$x_{i*} - \varepsilon \leq x_i(n) \leq x_i^* + \varepsilon, \quad u_{i*} - \varepsilon \leq u_i(n) \leq u_i^* + \varepsilon, \quad \forall n \geq n_0,\ i = 1,2. \tag{4.5}$$

Let $\{\tau_\alpha\}$ be any integer-valued sequence such that $\tau_\alpha \to \infty$ as $\alpha \to \infty$, we claim that there exists a subsequence of $\{\tau_\alpha\}$, we still denote by $\{\tau_\alpha\}$, such that

$$x_i(n+\tau_\alpha) \longrightarrow x_i^*(n) \tag{4.6}$$

uniformly in $n$ on any finite subset $B$ of $Z$ as $\alpha \to \infty$, where $B = \{a_1, a_2, \ldots, a_m\}, a_h \in Z(h = 1,2,\ldots,m)$ and $m$ is a finite number.

In fact, for any finite subset $B \subset Z$, when $\alpha$ is large enough, $\tau_\alpha + a_h > n_0, h = 1,2,\ldots,m$. So

$$x_{i_*} - \varepsilon \leq x_i(n+\tau_\alpha) \leq x_i^* + \varepsilon, \quad u_{i_*} - \varepsilon \leq u_i(n+\tau_\alpha) \leq u_i^* + \varepsilon. \tag{4.7}$$

That is, $\{x_i(n+\tau_\alpha)\}, \{u_i(n+\tau_\alpha)\}$ are uniformly bounded for large enough $\alpha$.

Now, for $a_1 \in B$, we can choose a subsequence $\{\tau_\alpha^{(1)}\}$ of $\{\tau_\alpha\}$ such that $\{x_i(a_1+\tau_\alpha^{(1)})\}$, $\{u_i(a_1+\tau_\alpha^{(1)})\}$ uniformly converges on $Z^+$ for $\alpha$ large enough.

Similarly, for $a_2 \in B$, we can choose a subsequence $\{\tau_\alpha^{(2)}\}$ of $\{\tau_\alpha^{(1)}\}$ such that $\{x_i(a_2 + \tau_\alpha^{(2)})\}$, $\{u_i(a_2+\tau_\alpha^{(2))}\}$ uniformly converges on $Z^+$ for $\alpha$ large enough.

Repeating this procedure, for $a_m \in B$, we obtain a subsequence $\{\tau_\alpha^{(m)}\}$ of $\{\tau_\alpha^{(m-1)}\}$ such that $\{x_i(a_m+\tau_\alpha^{(m)})\}$, $\{u_i(a_m+\tau_\alpha^{(m)})\}$ uniformly converges on $Z^+$ for $\alpha$ large enough.

Now pick the sequence $\{\tau_\alpha^{(m)}\}$ which is a subsequence of $\{\tau_\alpha\}$, we still denote it as $\{\tau_\alpha\}$, then for all $n \in B$, we have $x_i(n+\tau_\alpha) \to x_i^*(n)$, $u_i(n+\tau_\alpha) \to u_i^*(n)$ uniformly in $n \in B$ as $\alpha \to \infty$.

By the arbitrariness of $B$, the conclusion is valid.

Since $\{r_i(n)\}, \{K_i(n)\}, \{\alpha_i(n)\}, \{a_i(n)\}, \{b_i(n)\}$, and $\{c_i(n)\}$ are almost periodic sequences, for above sequence $\{\tau_\alpha\}$, $\tau_\alpha \to \infty$ as $\alpha \to \infty$, there exists a subsequence still denote by $\{\tau_\alpha\}$ (if necessary, we take subsequence), such that

$$\begin{gathered} r_i(n+\tau_\alpha) \longrightarrow r_i(n), \quad K_i(n+\tau_\alpha) \longrightarrow K_i(n), \quad \alpha_i(n+\tau_\alpha) \longrightarrow \alpha_i(n), \\ a_i(n+\tau_\alpha) \longrightarrow a_i(n), \quad b_i(n+\tau_\alpha) \longrightarrow b_i(n), \quad c_i(n+\tau_\alpha) \longrightarrow c_i(n), \end{gathered} \tag{4.8}$$

as $\alpha \to \infty$ uniformly on $Z^+$.

For any $\sigma \in Z$, we can assume that $\tau_\alpha + \sigma \ge n_0$ for $\alpha$ large enough. Let $n \ge 0$ and $n \in Z^+$, by an inductive argument of (1.7) from $\tau_\alpha + \sigma$ to $n + \tau_\alpha + \sigma$ leads to

$$\begin{aligned} x_i(n+\tau_\alpha+\sigma) &= x_i(\tau_\alpha+\sigma)\exp\sum_{l=\tau_\alpha+\sigma}^{n+\tau_\alpha+\sigma-1}\left\{r_i(l)\left[\frac{K_i(l)+\alpha_i(l)x_j(l)}{1+x_j(l)} - x_i(l) - b_i(l)u_i(l)\right]\right\}, \\ u_i(n+\tau_\alpha+\sigma) &= u_i(\tau_\alpha+\sigma) - \sum_{l=\tau_\alpha+\sigma}^{n+\tau_\alpha+\sigma-1}\{a_i(l)u_i(l) + c_i(l)x_i(l)\}. \end{aligned} \tag{4.9}$$

Then, for $i, j = 1, 2, i \neq j$, we have

$$\begin{aligned} x_i(n+\tau_\alpha+\sigma) &= x_i(\tau_\alpha+\sigma)\exp\sum_{l=\sigma}^{n+\sigma-1}\left\{r_i(l+\tau_\alpha)\left[\frac{K_i(l+\tau_\alpha)+\alpha_i(l+\tau_\alpha)x_j(l+\tau_\alpha)}{1+x_j(l+\tau_\alpha)}\right.\right. \\ &\qquad\left.\left. -x_i(l+\tau_\alpha) - b_i(l+\tau_\alpha)u_i(l+\tau_\alpha)\right]\right\}, \\ u_i(n+\tau_\alpha+\sigma) &= u_i(\tau_\alpha+\sigma) - \sum_{l=\sigma}^{n+\sigma-1}\{a_i(l+\tau_\alpha)u_i(l+\tau_\alpha) - c_i(l+\tau_\alpha)x_i(l+\tau_\alpha)\}. \end{aligned} \tag{4.10}$$

Let $\alpha \to \infty$, for any $n \ge 0$,

$$\begin{aligned} x_i^*(n+\sigma) &= x_i^*(\sigma)\exp\sum_{l=\sigma}^{n+\sigma-1}\left\{r_i(l)\left[\frac{K_i(l)+\alpha_i(l)x_j^*(l)}{1+x_j^*(l)} - x_i^*(l) - b_i(l)u_i(l)\right]\right\}, \\ u_i^*(n+\sigma) &= u_i^*(\sigma) - \sum_{l=\sigma}^{n+\sigma-1}\{a_i(l)u_i^*(l) - c_i(l)x_i^*(l)\}. \end{aligned} \tag{4.11}$$

By the arbitrariness of $\sigma$, $X^*(n) = (x_1^*(n), x_2^*(n), u_1^*(n), u_2^*(n))$ is a solution of model (1.7) on $Z^+$. It is clear that $0 < x_{i*} \le x_i^*(n) \le x_i^*, 0 < u_{i*} \le u_i^*(n) \le u_i^*$, for all $n \in Z^+, i = 1, 2$. So $\Omega \neq \emptyset$. Lemma 4.3 is valid. □

**Theorem 4.4.** *Suppose that the conditions of Lemma 4.3 are satisfied, moreover,* $0 < \beta < 1$, *where*

$$\beta = \min\left\{r_{ij}, r_{ij}^*\right\},$$
$$r_{ij} = 2r_i^l x_{i*} - \left(r_i^u x_j^* + r_i^{u2} x_i^* x_j^*\right)\left(\alpha_i^u - K_i^l\right) - r_i^{u2} x_i^{*2} - c_i^{u2} x_i^{*2} - b_i^u x_i^* r_i^{u2} - r_i^u b_i^u - c_i^u x_i^*\left(1 - a_i^l\right)$$
$$- r_j^{l2} x_i^{*2}\left(\alpha_j^u - k_j^l\right)^2 - \left(r_j^u x_i^* + r_j^{u2} x_j^* x_i^*\right)\left(\alpha_j^u - k_j^l\right) - b_j^u x_i^* r_j^{u2}\left(\alpha_j^u - k_j^l\right),$$
$$r_{ij}{}^* = -b_i^u x_j^* r_i^{u2}\left(\alpha_i^u - K_i^l\right) - r_i^{u2} b_i^{u2} - a_i^{u2} - 2a_i^u - b_i^u x_i^* r_i^{u2} - r_i^u b_i^u - c_i^u x_i^*\left(1 - a_i^l\right), \tag{4.12}$$

$i, j = 1,2, i \neq j$, *then there exists a unique uniformly asymptotically stable almost periodic solution* $X(n) = (x_1(n), x_2(n), u_1(n), u_2(n))$ *of* (1.7) *which is bounded by* $\Omega$ *for all* $n \in Z^+$.

*Proof.* Let $p_1(n) = \ln x_1(n), p_2(n) = \ln x_2(n)$. From (1.7), we have

$$p_i(n+1) = p_i(n) + r_i(n)\left[\frac{K_i(n) + \alpha_i(n)\exp p_j(n)}{1 + \exp p_j(n)} - \exp p_i(n) - b_i(n)u_i(n)\right],$$
$$\Delta u_i(n) = -a_i(n)u_i(n) + c_i(n)\exp p_i(n), \tag{4.13}$$

where $i, j = 1,2, i \neq j$. From Lemma 4.3, we know that system (4.13) has bounded solution $Y(n) = (p_1(n), p_2(n), u_1(n), u_2(n))$ satisfying

$$\ln x_{i*} \le p_i(n) \le \ln x_i^*, \quad u_{i*} \le u_i(n) \le u_i^*, \quad i = 1,2, \ n \in Z^+. \tag{4.14}$$

Hence, $|p_i(n)| \le A_i, |u_i(n)| \le B_i$, where $A_i = \max\{|\ln x_{i*}|, |\ln x_i^*|\}, B_i = \max\{u_{i*}, u_i^*\}, i = 1,2$.

For $(X, U) \in R^{2+2}$, we define the norm $\|(X, U)\| = \sum_{i=1}^2 |x_i| + \sum_{i=1}^2 |u_i|$.

Consider the product system of system (4.13)

$$p_i(n+1) = p_i(n) + r_i(n)\left[\frac{K_i(n) + \alpha_i(n)\exp p_j(n)}{1 + \exp p_j(n)} - \exp p_i(n) - b_i(n)u_i(n)\right],$$
$$\Delta u_i(n) = -a_i(n)u_i(n) + c_i(n)\exp p_i(n),$$
$$q_i(n+1) = q_i(n) + r_i(n)\left[\frac{K_i(n) + \alpha_i(n)\exp q_j(n)}{1 + \exp q_j(n)} - \exp q_i(n) - b_i(n)\omega_i(n)\right],$$
$$\Delta \omega_i(n) = -a_i(n)\omega_i(n) + c_i(n)\exp p_i(n). \tag{4.15}$$

Suppose $Z = (p_1(n), p_2(n), u_1(n), u_2(n))$, $W = (q_1(n), q_2(n), \omega_1(n), \omega_2(n))$ are any two solutions of system (4.15) defined on $Z^+ \times S^* \times S^*$, then $\|Z\| \le B, \|W\| \le B$, where

$$B = \sum_{i=1}^{2} \{A_i + B_i\},$$
$$S^* = \left\{ (p_1(n), p_2(n), u_1(n), u_2(n)) \mid \ln x_{i*} \le p_i(n) \le \ln x_i^*, u_{i*} \le u_i(n) \le u_i^*, i = 1, 2, n \in Z^+ \right\}. \tag{4.16}$$

Consider a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows:

$$V(n, Z, W) = \sum_{i=1}^{2} \left\{ (p_i(n) - q_i(n))^2 + (u_i(n) - \omega_i(n))^2 \right\}. \tag{4.17}$$

It is easy to see that the norm $\|Z - W\| = \sum_{i=1}^{2} \{|p_i(n) - q_i(n)| + |u_i(n) - \omega_i(n)|\}$ and the norm $\|Z - W\|_* = \{\sum_{i=1}^{2} \{(p_i(n) - q_i(n))^2 + (u_i(n) - \omega_i(n))^2\}\}^{1/2}$ are equivalent that is, there exist two constants $C_1 > 0, C_2 > 0$, such that

$$C_1 \|Z - W\| \le \|Z - W\|_* \le C_2 \|Z - W\|, \tag{4.18}$$

then

$$(C_1 \|Z - W\|)^2 \le V(n, Z, W) \le (C_2 \|Z - W\|)^2. \tag{4.19}$$

Let $a \in C(R^+, R^+)$, $a(x) = C_1{}^2 x^2$, $b \in C(R^+, R^+), b(x) = C_2{}^2 x^2$, thus condition (i) in Theorem 4.2 is satisfied.

In addition,

$$\begin{aligned}
&\left| V(n, Z, W) - V\left(n, \widetilde{Z}, \widetilde{W}\right) \right| \\
&= \left| \sum_{i=1}^{2} \left\{ (p_i(n) - q_i(n))^2 + (u_i(n) - \omega_i(n))^2 \right\} - \sum_{i=1}^{2} \left\{ (\tilde{p}_i(n) - \tilde{q}_i(n))^2 + (\tilde{u}_i(n) - \tilde{\omega}_i(n))^2 \right\} \right| \\
&\le \sum_{i=1}^{2} \left| (p_i(n) - q_i(n))^2 - (\tilde{p}_i(n) - \tilde{q}_i(n))^2 \right| + \sum_{i=1}^{2} \left| (u_i(n) - \omega_i(n))^2 - (\tilde{u}_i(n) - \tilde{\omega}_i(n))^2 \right| \\
&= \sum_{i=1}^{2} \left\{ \left| (p_i(n) - q_i(n)) + (\tilde{p}_i(n) - \tilde{q}_i(n)) \right| \left| (p_i(n) - q_i(n)) - (\tilde{p}_i(n) - \tilde{q}_i(n)) \right| \right\} \\
&\quad + \sum_{i=1}^{2} \{ |(u_i(n) - \omega_i(n)) + (\tilde{u}_i(n) - \tilde{\omega}_i(n))| |(u_i(n) - \omega_i(n)) - (\tilde{u}_i(n) - \tilde{\omega}_i(n))| \} \\
&\le \sum_{i=1}^{2} \left\{ \left( |p_i(n)| + |q_i(n)| + |\tilde{p}_i(n)| + |\tilde{q}_i(n)| \right) \left( |p_i(n) - \tilde{p}_i(n)| + |q_i(n) - \tilde{q}_i(n)| \right) \right\}
\end{aligned}$$

$$+\sum_{i=1}^{2}\{(|u_i(n)|+|\omega_i(n)|+|\tilde{u}_i(n)|+|\tilde{\omega}_i(n)|)(|u_i(n)-\tilde{u}_i(n)|+|\omega_i(n)-\tilde{\omega}_i(n)|)\}$$

$$\le L\left\{\sum_{i=1}^{2}\{|p_i(n)-\tilde{p}_i(n)|+|u_i(n)-\tilde{u}_i(n)|\}+\sum_{i=1}^{2}\{|q_i(n)-\tilde{q}_i(n)|+|\omega_i(n)-\tilde{\omega}_i(n)|\}\right\}$$

$$=L\left\{\left\|Z-\tilde{Z}\right\|+\left\|W-\widetilde{W}\right\|\right\}, \tag{4.20}$$

where $L=4\max\{A_i,B_i\}(i=1,2)$. Hence the condition (ii) of Theorem 4.2 is satisfied. Finally, calculate the $\Delta V$ of $V(n)$ along the solutions of (4.15), we can obtain

$$\begin{aligned}
\Delta V_{(4.15)}(n) &= V(n+1)-V(n)\\
&=\sum_{i=1}^{2}\left\{(p_i(n+1)-q_i(n+1))^2+(u_i(n+1)-\omega_i(n+1))^2\right\}\\
&\quad-\sum_{i=1}^{2}\left\{(p_i(n)-q_i(n))^2+(u_i(n)-\omega_i(n))^2\right\}\\
&=\sum_{i=1}^{2}\left\{(p_i(n+1)-q_i(n+1))^2-(p_i(n)-q_i(n))^2\right.\\
&\quad\left.+(u_i(n+1)-\omega_i(n+1))^2-(u_i(n)-\omega_i(n))^2\right\}\\
&=\sum_{i=1}^{2}\left\{\left[(p_i(n)-q_i(n))+r_i(n)\frac{(\alpha_i(n)-K_i(n))(e^{p_j(n)}-e^{q_j(n)})}{(1+e^{p_j(n)})(1+e^{q_j(n)})}\right.\right.\\
&\quad\left.-r_i(n)\left(e^{p_i(n)}-e^{q_i(n)}\right)-r_i(n)b_i(n)(u_i(n)-\omega_i(n))\right]^2\\
&\quad-(p_i(n)-q_i(n))^2+\left[(1-a_i(n))(u_i(n)-\omega_i(n))+c_i(n)\left(e^{p_i(n)}-e^{q_i(n)}\right)\right]^2\\
&\quad\left.-(u_i(n)-\omega_i(n))^2\right\}\\
&=\sum_{i=1}^{2}\left\{r_i^2(n)\frac{(\alpha_i(n)-K_i(n))^2(e^{p_j(n)}-e^{q_j(n)})^2}{(1+e^{p_j(n)})^2(1+e^{q_j(n)})^2}\right.\\
&\quad+2r_i(n)\frac{(\alpha_i(n)-K_i(n))(e^{p_j(n)}-e^{q_j(n)})(p_i(n)-q_i(n))}{(1+e^{p_j(n)})(1+e^{q_j(n)})}\\
&\quad+r_i^2(n)\left(e^{p_i(n)}-e^{q_i(n)}\right)^2+r_i^2(n)b_i^2(n)(u_i(n)-\omega_i(n))^2\\
&\quad+2b_i(n)r_i^2(n)\left(e^{p_i(n)}-e^{q_i(n)}\right)(u_i(n)-\omega_i(n))-2r_i(n)\\
&\quad\times(p_i(n)-q_i(n))\left(e^{p_i(n)}-e^{q_i(n)}\right)
\end{aligned}$$

$$
\begin{aligned}
&- 2b_i(n)r_i(n)\left(p_i(n) - q_i(n)\right)\left(u_i(n) - \omega_i(n)\right)\\
&- 2r_i^2(n)\frac{(\alpha_i(n) - K_i(n))\left(e^{p_j(n)} - e^{q_j(n)}\right)\left(e^{p_i(n)} - e^{q_i(n)}\right)}{\left(1 + e^{p_j(n)}\right)\left(1 + e^{q_j(n)}\right)}\\
&- 2b_i(n)r_i^2(n)\frac{(\alpha_i(n) - K_i(n))\left(e^{p_j(n)} - e^{q_j(n)}\right)(u_i(n) - \omega_i(n))}{\left(1 + e^{p_j(n)}\right)\left(1 + e^{q_j(n)}\right)}\\
&- (1 - a_i(n))^2(u_i(n) - \omega_i(n))^2 + c_i^2(n)\left(e^{p_i(n)} - e^{q_i(n)}\right)^2\\
&\left.+2c_i(n)(1 - a_i(n))(u_i(n) - \omega_i(n))\left(e^{p_i(n)} - e^{q_i(n)}\right) - (u_i(n) - \omega_i(n))^2\right\}.
\end{aligned}
\tag{4.21}
$$

Using the mean value theorem, we get

$$
e^{p_i(n)} - e^{q_i(n)} = \xi_i(n)\left(p_i(n) - q_i(n)\right), \quad i = 1, 2,
\tag{4.22}
$$

where $\xi_i(n)$ lies between $e^{p_i(n)}$ and $e^{q_i(n)}$, $i = 1, 2$. From (4.21), (4.22), we have

$$
\begin{aligned}
\Delta V_{(4.15)}(n) = \sum_{i=1}^{2}\Bigg\{ & r_i^2(n)\xi_j^2(n)(\alpha_i(n) - K_i(n))^2\frac{\left(p_j(n) - q_j(n)\right)^2}{\left(1 + e^{p_j(n)}\right)^2\left(1 + e^{q_j(n)}\right)^2}\\
&+ 2r_i(n)\xi_j(n)(\alpha_i(n) - K_i(n))\frac{\left(p_i(n) - q_i(n)\right)\left(p_j(n) - q_j(n)\right)}{\left(1 + e^{p_j(n)}\right)\left(1 + e^{q_j(n)}\right)}\\
&+ r_i^2(n)\xi_i^2(n)\left(p_i(n) - q_i(n)\right)^2 + r_i^2(n)b_i^2(n)(u_i(n) - \omega_i(n))^2\\
&+ 2b_i(n)\xi_i(n)r_i^2(n)\left(p_i(n) - q_i(n)\right)\left(u_i(n) - \omega_i(n)\right)\\
&- 2r_i(n)\xi_i(n)\left(p_i(n) - q_i(n)\right)^2\\
&- 2b_i(n)r_i(n)\left(p_i(n) - q_i(n)\right)\left(u_i(n) - \omega_i(n)\right)\\
&- r_i^2(n)\xi_i(n)\xi_j(n)(\alpha_i(n) - K_i(n))\frac{\left(p_j(n) - q_j(n)\right)\left(p_i(n) - q_i(n)\right)}{\left(1 + e^{p_j(n)}\right)\left(1 + e^{q_j(n)}\right)}\\
&- 2b_i(n)\xi_j(n)r_i^2(n)(\alpha_i(n) - K_i(n))\frac{\left(u_i(n) - \omega_i(n)\right)\left(p_j(n) - q_j(n)\right)}{\left(1 + e^{p_j(n)}\right)\left(1 + e^{q_j(n)}\right)}\\
&+ (1 - a_i(n))^2(u_i(n) - \omega_i(n))^2 + c_i^2(n)\xi_i^2(n)\left(p_i(n) - q_i(n)\right)^2\\
&+ 2c_i(n)(1 - a_i(n))(u_i(n) - \omega_i(n))\xi_i(n)\\
&\times\left(p_i(n) - q_i(n)\right) - (u_i(n) - \omega_i(n))^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{2}\Bigg\{ r_i^2(n)\xi_j^2(n)(\alpha_i(n)-K_i(n))^2\frac{\left(p_j(n)-q_j(n)\right)^2}{\left(1+e^{p_j(n)}\right)^2\left(1+e^{q_j(n)}\right)^2} \\
&\quad + 2\Big[r_i(n)\xi_j(n)(\alpha_i(n)-K_i(n)) - r_i^2(n)\xi_i(n)\xi_j(n)(\alpha_i(n)-K_i(n))\Big] \\
&\quad \times \frac{\left(p_i(n)-q_i(n)\right)\left(p_j(n)-q_j(n)\right)}{\left(1+e^{p_j(n)}\right)\left(1+e^{q_j(n)}\right)} \\
&\quad - 2b_i(n)\xi_j(n)r_i^2(n)(\alpha_i(n)-K_i(n))\frac{\left(u_i(n)-\omega_i(n)\right)\left(p_j(n)-q_j(n)\right)}{\left(1+e^{p_j(n)}\right)\left(1+e^{q_j(n)}\right)} \\
&\quad + \Big[r_i^2(n)\xi_i^2(n) - 2r_i(n)\xi_i(n) + c_i^2(n)\xi_i^2(n)\Big]\left(p_i(n)-q_i(n)\right)^2 \\
&\quad + \Big[r_i^2(n)b_i^2(n) + (1-a_i(n))^2 - 1\Big](u_i(n)-\omega_i(n))^2 \\
&\quad + 2\Big[b_i(n)\xi_i(n)r_i^2(n) - b_i(n)r_i(n) + c_i(n)\xi_i(n)(1-a_i(n))\Big] \\
&\quad \times (u_i(n)-\omega_i(n))\left(p_i(n)-q_i(n)\right)\Bigg\} \\
&\le \sum_{i=1}^{2}\Bigg\{ r_i^2(n)\xi_j^2(n)(\alpha_i(n)-K_i(n))^2\frac{\left(p_j(n)-q_j(n)\right)^2}{\left(1+e^{p_j(n)}\right)^2\left(1+e^{q_j(n)}\right)^2} \\
&\quad + 2\Bigg|\Big[r_i(n)\xi_j(n)(\alpha_i(n)-K_i(n)) \\
&\qquad - r_i^2(n)\xi_i(n)\xi_j(n)(\alpha_i(n)-K_i(n))\Big]\frac{\left(p_i(n)-q_i(n)\right)\left(p_j(n)-q_j(n)\right)}{\left(1+e^{p_j(n)}\right)\left(1+e^{q_j(n)}\right)}\Bigg| \\
&\quad + 2\left|b_i(n)\xi_j(n)r_i^2(n)(\alpha_i(n)-K_i(n))\frac{\left(u_i(n)-\omega_i(n)\right)\left(p_j(n)-q_j(n)\right)}{\left(1+e^{p_j(n)}\right)\left(1+e^{q_j(n)}\right)}\right| \\
&\quad + \left|\Big[r_i^2(n)\xi_i^2(n) - 2r_i(n)\xi_i(n) + c_i^2(n)\xi_i^2(n)\Big]\left(p_i(n)-q_i(n)\right)^2\right| \\
&\quad + \left|\Big[r_i^2(n)b_i^2(n) + (1-a_i(n))^2 - 1\Big](u_i(n)-\omega_i(n))^2\right| \\
&\quad + 2\Bigg|\Big[b_i(n)\xi_i(n)r_i^2(n) - b_i(n)r_i(n) + c_i(n)\xi_i(n)(1-a_i(n))\Big] \\
&\qquad \times (u_i(n)-\omega_i(n))\left(p_i(n)-q_i(n)\right)\Bigg|\Bigg\},
\end{aligned}
\tag{4.23}
$$

we get

$$
\Delta V_{(4.15)}(n) \le \sum_{i=1}^{2}\left\{V_{1ij}+V_{2ij}+V_{3ij}+V_{4ij}+V_{5ij}+V_{6ij}\right\}, \quad j=1,2, \tag{4.24}
$$

where

$$
\begin{aligned}
V_{1ij} &= r_i^2(n)\xi_j^2(n)(\alpha_i(n)-K_i(n))^2\frac{(p_j(n)-q_j(n))^2}{\left(1+e^{p_j(n)}\right)^2\left(1+e^{q_j(n)}\right)^2}\\
&\le r_i^{u2}\left(\alpha_i^u-K_i^l\right)^2 x_j^{*2}(p_j(n)-q_j(n))^2,\\
V_{2ij} &= 2\Bigg|\left[r_i(n)\xi_j(n)(\alpha_i(n)-K_i(n))-r_i^2(n)\xi_i(n)\xi_j(n)(\alpha_i(n)-K_i(n))\right]\\
&\qquad\times\frac{(p_i(n)-q_i(n))(p_j(n)-q_j(n))}{\left(1+e^{p_j(n)}\right)\left(1+e^{q_j(n)}\right)}\Bigg|\\
&\le\left[\left(r_i^u x_j^*+r_i^{u2}x_i^*x_j^*\right)\left(\alpha_i^u-K_i^l\right)\right]\left[(p_i(n)-q_i(n))^2+(p_j(n)-q_j(n))^2\right],\\
V_{3ij} &= 2\left|b_i(n)\xi_j(n)r_i^2(n)(\alpha_i(n)-K_i(n))\frac{(u_i(n)-\omega_i(n))(p_j(n)-q_j(n))}{\left(1+e^{p_j(n)}\right)\left(1+e^{q_j(n)}\right)}\right|\\
&\le b_i^u x_j^* r_i^{u2}\left(\alpha_i^u-K_i^l\right)\left[(u_i(n)-\omega_i(n))^2+(p_j(n)-q_j(n))^2\right],\\
V_{4ij} &= \left|\left[r_i^2(n)\xi_i^2(n)-2r_i(n)\xi_i(n)+c_i^2(n)\xi_i^2(n)\right](p_i(n)-q_i(n))^2\right|\\
&\le\left[r_i^{u2}x_i^{*2}-2r_i^l x_{i*}+c_i^{u2}x_i^{*2}\right](p_i(n)-q_i(n))^2,\\
V_{5ij} &= \left|\left[r_i^2(n)b_i^2(n)+(1-a_i(n))^2-1\right](u_i(n)-\omega_i(n))^2\right|\\
&\le\left(r_i^{u2}b_i^{u2}+a_i^{u2}+2a_i^u\right)(u_i(n)-\omega_i(n))^2,\\
V_{6ij} &= 2\Big|\left[b_i(n)\xi_i(n)r_i^2(n)-b_i(n)r_i(n)+c_i(n)\xi_i(n)(1-a_i(n))\right]\\
&\qquad\times(u_i(n)-\omega_i(n))(p_i(n)-q_i(n))\Big|\\
&\le\left[b_i^u x_i^* r_i^{u2}+r_i^u b_i^u+c_i^u x_i^*\left(1-a_i^l\right),(u_i(n)-\omega_i(n))^2+(p_i(n)-q_i(n))^2\right].
\end{aligned}
\tag{4.25}
$$

Hence,

$$
\begin{aligned}
\Delta V_{(4.15)}(n) \le \sum_{i=1}^{2}\Bigg\{&\Big[\left(r_i^u x_j^*+r_i^{u2}x_i^*x_j^*\right)\left(\alpha_i^u-K_i^l\right)+r_i^{u2}x_i^{*2}-2r_i^l x_{i*}\\
&+c_i^{u2}x_i^{*2}+b_i^u x_i^* r_i^{u2}+r_i^u b_i^u+c_i^u x_i^*\left(1-a_i^l\right)\Big](p_i(n)-q_i(n))^2\\
&+\Big[r_i^{u2}\left(\alpha_i^u-K_i^l\right)^2 x_j^{*2}+\left(r_i^u x_j^*+r_i^{u2}x_i^*x_j^*\right)\left(\alpha_i^u-K_i^l\right)\\
&+b_i^u x_j^* r_i^{u2}\left(\alpha_i^u-K_i^l\right)\Big](p_j(n)-q_j(n))^2\\
&\quad+\Big[b_i^u x_j^* r_i^{u2}\left(\alpha_i^u-K_i^l\right)+r_i^{u2}b_i^{u2}+a_i^{u2}+2a_i^u+b_i^u x_i^* r_i^{u2}\\
&+r_i^u b_i^u+c_i^u x_i^*\left(1-a_i^l\right)\Big](u_i(n)-\omega_i(n))^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2}\Big\{\Big[2r_i^l x_{i*} - \left(r_i^u x_j^* + r_i^{u2} x_i^* x_j^*\right)\left(\alpha_i^u - K_i^l\right) - r_i^{u2} x_i^{*2} \\
&\qquad - c_i^{u2} x_i^{*2} - b_i^u x_i^* r_i^{u2} - r_i^u b_i^u + c_i^u x_i^*\left(1 - a_i^l\right) \\
&\qquad - r_j^{u2}\left(\alpha_j^u - k_j^l\right)^2 x_i^{*2} - \left(r_j^u x_i^* + r_j^{u2} x_i^* x_j^*\right)\left(\alpha_j^u - k_j^l\right) \\
&\qquad - b_j^u x_i^* r_j^{u2}\left(\alpha_j^u - k_j^l\right)\Big](p_i(n) - q_i(n))^2 \\
&\qquad + \Big[-b_i^u x_j^* r_i^{u2}\left(\alpha_i^u - K_i^l\right) - r_i^{u2} b_i^{u2} - a_i^{u2} - 2a_i^u - b_i^u x_i^* r_i^{u2} \\
&\qquad\quad - r_i^u b_i^u - c_i^u x_i^*\left(1 - a_i^l\right)\Big](u_i(n) - \omega_i(n))^2\Big\} \\
&\le -\sum_{i=1}^{2}\left\{r_{ij}(p_i(n) - q_i(n))^2 + r_{ij}^*(u_i(n) - \omega_i(n))^2\right\} \\
&\le -\beta\sum_{i=1}^{2}\left\{(p_i(n) - q_i(n))^2 + (u_i(n) - \omega_i(n))^2\right\} \\
&= -\beta V(n),
\end{aligned}
\tag{4.26}
$$

where $\beta = \{r_{ij}, r_{ij}^*\}, i, j = 1, 2, i \neq j$. That is, there exists a positive constant $0 < \beta < 1$ such that $\Delta V_{(4.15)}(n) \le -\beta V(n)$. From $0 < \beta < 1$, condition (iii) of Theorem 4.2 is satisfied. So, from Theorem 4.2, there exists a uniqueness uniformly asymptotically stable almost periodic solution $X(n) = (p_1(n), p_2(n), u_1(n), u_2(n))$ of (4.13) which is bounded by $S^*$ for all $n \in Z^+$. Which means that there exists a uniqueness uniformly asymptotically stable almost periodic solution $X(n) = (x_1(n), x_2(n), u_1(n), u_2(n))$ of (1.7) which is bounded by $\Omega$ for all n $\in Z^+$. This completed the proof. □

## 5. An Example

In this section, we give an example to illustrate that our results are feasible.

In system (1.7), if we take $i = 1, 2$ and

$$
\begin{gathered}
r_1(n) = 1.1 + 0.4\cos(n), \quad r_2(n) = 0.8 + 0.2\sin(n), \quad K_1(n) = 6 + \cos(n), \\
K_2(n) = 5 + \cos(n), \quad \alpha_1(n) = 1.2 + \sin(n), \quad \alpha_2(n) = 1.6 + \cos(n), \\
b_1(n) = 0.5 + 0.1\sin(n), \quad b_2(n) = 0.3 + 0.1\cos(n), \quad a_1(n) = 0.9 + 0.2\sin(n), \\
a_2(n) = 0.6 + 0.3\sin(n), \quad c_1(n) = 0.5 + 0.3\sin(n), \quad c_2(n) = 0.6 + 0.2\cos(n).
\end{gathered}
\tag{5.1}
$$

Then it is easy to see that $\{r_i(n)\}, \{K_i(n)\}, \{\alpha_i(n)\}, \{a_i(n)\}, \{b_i(n)\}$, and $\{c_i(n)\}$ for $i = 1, 2$ are bounded nonnegative almost periodic sequences. By calculation, we get

$$x_1^* = 4.4062, \quad u_1^* = 5.0357, \quad K_1^l - b_1^u u_1^* = 1.9775 > 0, \quad x_{1_*} = 0.3612, \quad u_{1_*} = 0.0903,$$
$$x_2^* = 2.6000, \quad u_2^* = 6.9333, \quad K_2^l - b_2^u u_2^* = 1.2267 > 0, \quad x_{2_*} = 0.4997, \quad u_{2_*} = 0.2498, \tag{5.2}$$
$$r_{12} = 0.6313, \quad r_{21} = 0.3583, \quad r_{12}^* = 0.6380, \quad r_{21}^* = 0.1025, \quad 0 < \beta = 0.1025 < 1.$$

Then we can see that all conditions of Theorem 4.4 hold. According to Theorem 4.4, system (1.7) has a unique uniformly asymptotically stable almost periodic solution which is bounded by $\Omega$ for all $n \in Z^+$.

## 6. Conclusions

In this paper, we consider a discrete mutualism model with feedback controls. Assuming that the coefficients in the system are almost periodic sequences, first, we establish a persistence result for the model under consideration, then based on the persistence result we obtain the existence and uniqueness of the almost periodic solution of the system which is uniformly asymptotically stable. Finally, an example is given to illustrate the feasibility of our results.

## Acknowledgment

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 10971183.

## References

[1] C. L. Wolin and L. R. Lawlor, "Models of facultative mutualism: density effects," *The American Naturalist*, vol. 114, pp. 843–862, 1984.
[2] B. S. Goh, "Stability in models of mutualism," *American Naturalist*, vol. 113, pp. 216–275, 1979.
[3] Z. B. Zhang, "Mutualism or cooperation among competitors promotes coexistence and competitive ability," *Ecological Modelling*, vol. 164, pp. 271–282, 2003.
[4] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, vol. 74 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dodrecht, The Netherlands, 1992.
[5] J. H. Vandermeer and D. H. Boucher, "Varieties of mutualistic interaction in population models," *Journal of Theoretical Biology*, vol. 74, no. 4, pp. 549–558, 1978.
[6] D. H. Boucher, S. James, and K. H. Keeler, "The ecology of mutualism," *Annual Review of Ecology and Systematics*, vol. 13, pp. 315–347, 1982.
[7] A. M. Dean, "A simple model of mutualism," *American Naturalist*, vol. 121, no. 3, pp. 409–417, 1983.
[8] L. Wolm and L. R. Lawlor, "Models of facultative mutualism Density effects," *American Naturalist*, vol. 144, pp. 843–862, 1984.
[9] D. H. Boucher, *The Biology of Mutualism Ecology and Evolution*, Croom Helm, London, UK, 1985.
[10] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, vol. 228 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.
[11] J. D. Murray, *Mathematical Biology*, vol. 19 of *Biomathematics*, Springer, Berlin, Germany, 1989.
[12] M. Fan and K. Wang, "Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system," *Mathematical and Computer Modelling*, vol. 35, no. 9-10, pp. 951–961, 2002.
[13] X. Chen and C. Fengde, "Stable periodic solution of a discrete periodic Lotka-Volterra competition system with a feedback control," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 1446–1454, 2006.
[14] F. Q. Yin and Y. K. Li, "Positive periodic solutions of a single species model with feedback regulation and distributed time delay," *Applied Mathematics and Computation*, vol. 153, no. 2, pp. 475–484, 2004.

[15] G. Fan, Y. Li, and M. Qin, "The existence of positive periodic solutions for periodic feedback control systems with delays," *Zeitschrift fur Angewandte Mathematik und Mechanik*, vol. 84, no. 6, pp. 425–430, 2004.

[16] L. F. Nie, J. G. Peng, and Z. D. Teng, "Permanence and stability in multi-species nonautonomous Lotka-Volterra competitive systems with delays and feedback controls," *Mathematical and Computer Modelling*, vol. 49, pp. 295–306, 2009.

[17] H. X. Hu, Z. D. Teng, and H. J. Jiang, "Permanence of the nonautonomous competitive systems with infinite delay and feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 4, pp. 2420–2433, 2009.

[18] X. X. Chen and F. D. Chen, "Almost-periodic solutions of a delay population equation with feedback control," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 4, pp. 559–571, 2006.

[19] X. X. Chen, "Almost-periodic solutions of a delay population equation with feedback control," *Nonlinear Analysis: Real World Applications*, vol. 8, pp. 62–72, 2007.

[20] K. Gopalsamy, "Global asymptotic stability in an almost periodic Lotka-Volterra system," *Journal of the Australian Mathematical Society. Series B*, vol. 27, pp. 346–360, 1986.

[21] C. Wang and J. Shi, "Positive almost periodic solutions of a class of Lotka-Volterra type competitive system with delays and feedback controls," *Applied Mathematics and Computation*, vol. 193, no. 1, pp. 240–252, 2007.

[22] W. Qi and B. Dai, "Almost periodic solution for n-species Lotka-Volterra competitive system with delay and feedback controls," *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 133–146, 2008.

[23] D. Cheban and C. Mammana, "Invariant manifolds, global attractors and almost periodic solutions of nonautonomous difference equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 56, no. 4, pp. 465–484, 2004.

[24] S. N. Zhang, "Existence of almost periodic solution for difference systems," *Annals of Differential Equations*, vol. 16, no. 2, pp. 184–206, 2000.