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Research Article

New Oscillation Criteria for Second-Order Delay Differential Equations with Mixed Nonlinearities

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We establish new oscillation criteria for second-order delay differential equations with mixed nonlinearities of the form $(p(t)x'(t))' + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) + \sum_{i=1}^{n} q_i(t)|x(t-\tau_i)|^{\alpha_i} \operatorname{sgn} x(t-\tau_i) = e(t)$, $t \geq 0$, where $p(t)$, $p_i(t)$, $q_i(t)$, and $e(t)$ are continuous functions defined on $[0, \infty)$, and $p(t) > 0$, $p'(t) \geq 0$, and $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$. No restriction is imposed on the potentials $p_i(t)$, $q_i(t)$, and $e(t)$ to be nonnegative. These oscillation criteria extend and improve the results given in the recent papers. An interesting example illustrating the sharpness of our results is also provided.

1. Introduction

We consider the second-order delay differential equations containing mixed nonlinearities of the form

$$(p(t)x'(t))' + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) + \sum_{i=1}^{n} q_i(t)|x(t-\tau_i)|^{\alpha_i} \operatorname{sgn} x(t-\tau_i) = e(t), \quad t \geq 0. \tag{1.1}$$

In what follows we assume that $\tau_i \geq 0$, $p \in C^1[0, \infty)$, $p(t) > 0$, $p'(t) \geq 0$, $p_i, q_i, e \in C[0, \infty)$, $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$ $(n > m \geq 1)$, and

$$\int_0^{\infty} \frac{1}{p(t)} dt = \infty. \tag{1.2}$$

As usual, a solution $x(t)$ of (1.1) is called oscillatory if it is defined on some ray $[T, \infty)$ with $T \geq 0$ and has arbitrary large zeros, otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all of its extendible solutions are oscillatory.

Recently, Mustafa [1] has studied the oscillatory solutions of certain forced Emden-Fowler like equations

$$x''(t) + a(t)|x(t)|^{\lambda}\operatorname{sgn} x(t) = e(t), \quad t \geq t_0 \geq 1. \tag{1.3}$$

Sun and Wong [2], as well as Sun and Meng [3] have established oscillation criteria for the second-order equation

$$\left(p(t)x'(t)\right)' + q(t)x(t) + \sum_{i=1}^{n} q_i(t)|x(t)|^{\alpha_i}\operatorname{sgn} x(t) = e(t), \quad t \geq 0. \tag{1.4}$$

Later in [4], Li and Chen have extended (1.4) to the delay differential equation

$$\left(p(t)x'(t)\right)' + q(t)x(t-\tau) + \sum_{i=1}^{n} q_i(t)|x(t-\tau)|^{\alpha_i}\operatorname{sgn} x(t-\tau) = e(t), \quad t \geq 0. \tag{1.5}$$

As it is indicated in [2, 3], further research on the oscillation of equations of mixed type is necessary as such equations arise in mathematical modeling, for example, in the growth of bacteria population with competitive species. In this paper, we will continue in the direction to study the oscillatory properties of (1.1). We will employ the method in study of Kong in [5] and the arithmetic-geometric mean inequality (see [6]) to establish the interval oscillation criteria for the unforced (1.1) and forced (1.1), which extend and improve the known results. Our results are generalizations of the main results in [3, 4]. We also give an example to illustrate the sharpness of our main results.

2. Main Results

We need the following lemma proved in [2, 3] for our subsequent discussion.

Lemma 2.1. *For any given n-tuple $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$, there corresponds an n-tuple $\{\eta_1, \eta_2, \ldots, \eta_n\}$ such that*

$$\sum_{i=1}^{n} \alpha_i \eta_i = 1, \tag{a}$$

which also satisfies either

$$\sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1, \tag{b}$$

or

$$\sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{c}$$

For a given set of exponents $\{\alpha_i\}$ satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$, Lemma 2.1 ensures the existence of an n-tuple $\{\eta_1, \eta_2, \ldots, \eta_n\}$ such that either (a) and (b) hold or (a) and (c) hold. When $n = 2$ and $\alpha_1 > 1 > \alpha_2 > 0$, in the first case, we have

$$\eta_1 = \frac{1 - \alpha_2(1 - \eta_0)}{\alpha_1 - \alpha_2}, \qquad \eta_2 = \frac{\alpha_1(1 - \eta_0) - 1}{\alpha_1 - \alpha_2}, \tag{2.1}$$

where η_0 can be any positive number satisfying $0 < \eta_0 < (\alpha_1 - 1)/\alpha_1$. This will ensure that $0 < \eta_1, \eta_2 < 1$, and conditions (a) and (b) are satisfied. In the second case, we simply solve (a) and (c) and obtain

$$\eta_1 = \frac{1 - \alpha_2}{\alpha_1 - \alpha_2}, \qquad \eta_2 = \frac{\alpha_1 - 1}{\alpha_1 - \alpha_2}. \tag{2.2}$$

Following Philos [7], we say that a continuous function $H(t,s)$ belongs to a function class $\mathfrak{D}_{a,b}$, denoted by $H \in \mathfrak{D}_{a,b}$, if $H(b,s) > 0, H(s,a) > 0$ for $b > s > a$, and $H(t,s)$ has continuous partial derivatives $\partial H(t,s)/\partial t$ and $\partial H(t,s)/\partial s$ in $[a,b] \times [a,b]$. Set

$$h_1(t,s) = \frac{\partial H(t,s)/\partial t}{2\sqrt{H(t,s)}}, \qquad h_2(t,s) = \frac{-\partial H(t,s)/\partial s}{2\sqrt{H(t,s)}}. \tag{2.3}$$

Based on Lemma 2.1, we have the following interval criterion for oscillation of (1.1).

Theorem 2.2. *If, for any $T \geq 0$, there exist a_1, b_1, c_1, a_2, b_2 and c_2 such that $T \leq a_1 < c_1 < b_1 \leq a_2 < c_2 < b_2$,*

$$\begin{gathered} p_i(t) \geq 0, \quad t \in [a_1 - \tau_i, b_1] \cup [a_2 - \tau_i, b_2],\ i = 1,2,\ldots,n, \\ q_i(t) \geq 0, \quad t \in [a_1 - \tau_i, b_1] \cup [a_2 - \tau_i, b_2],\ i = 1,2,\ldots,n, \\ e(t) \leq 0, \quad t \in [a_1 - \tau_i, b_1], \qquad e(t) \geq 0, \quad t \in [a_2 - \tau_i, b_2], \end{gathered} \tag{2.4}$$

and there exist $H_j \in \mathfrak{D}_{a_j,b_j}$ such that

$$\begin{aligned} &\frac{1}{H_j(c_j,a_j)} \int_{a_j}^{c_j} \Big(Q_j(s) H_j(s,a_j) - p(s) h_{j1}^2(s,a_j) \Big) ds \\ &\quad + \frac{1}{H_j(b_j,c_j)} \int_{c_j}^{b_j} \Big(Q_j(s) H_j(b_j,s) - p(s) h_{j2}^2(b_j,s) \Big) ds > 0, \end{aligned} \tag{2.5}$$

for $j = 1,2$, where h_{j_1}, h_{j_2} are defined as in (2.3)*, $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying* (a) *and* (b) *in Lemma 2.1, $\eta_0 = 1 - \sum_{i=1}^{n} \eta_i$, and*

$$Q_j(t) = \sum_{i=1}^{n} p_i(t) \left(\frac{t - a_j}{t - a_j + \tau_i} \right) + \left(\eta_0^{-1} |e(t)| \right)^{\eta_0} \prod_{i=1}^{n} \left(\eta_i^{-1} q_i(t) \right)^{\eta_i} \left(\frac{t - a_j}{t - a_j + \tau_i} \right)^{\alpha_i \eta_i}, \tag{2.6}$$

then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_1 - \overline{\tau} \geq 0$, where t_1 depends on the solution $x(t)$ and $\overline{\tau} = \max\{\tau_i\}$, $i = 1, \ldots, n$. When $x(t)$ is eventually negative, the proof follows the same argument by using the interval $[a_2, b_2]$ instead of $[a_1, b_1]$. Choose $a_1, b_1 \geq t_1$ such that $p_i(t),\ q_i(t) \geq 0$, $e(t) \leq 0$ for $t \in [a_1 - \tau_i, b_1]$, and $i = 1, 2, \ldots, n$.

From (1.1), we have that $x'(t) \geq 0$ for $t \in [a_1 - \tau_i, b_1]$. If not, there exists $t_2 \in [a_1 - \tau_i, b_1]$ such that $x'(t_2) < 0$. Because

$$\left(p(t)x'(t)\right)' \leq 0, \tag{2.7}$$

we have $p(t)x'(t) \leq p(t_2)x'(t_2)$. Integrating from t_2 to t, we obtain

$$x(t) \leq x(t_2) + p(t_2)x'(t_2)\int_{t_2}^{\tau} \frac{1}{p(s)} ds. \tag{2.8}$$

Noting the assumption (1.2), we have $x(t) \leq 0$ for sufficient large t. This is a contradiction with $x(t) > 0$. From (2.7) and the conditions $p(t) > 0$, $p'(t) \geq 0$, we obtain $x''(t) \leq 0$ for $t \in [a_1 - \tau_i, b_1]$.

Employing the convexity of $x(t)$, we obtain

$$\frac{x(t-\tau_i)}{x(t)} \geq \frac{t-a_1}{t-a_1+\tau_i}, \quad t \in [a_1, b_1]. \tag{2.9}$$

Define

$$\omega(t) = -\frac{p(t)x'(t)}{x(t)}. \tag{2.10}$$

Recall the arithmetic-geometric mean inequality

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}, \quad u_i \geq 0, \tag{2.11}$$

where $\eta_0 = 1 - \sum_{i=1}^{n} \eta_i$ and $\eta_i > 0$, $i = 1, 2, \ldots, n$, are chosen according to the given $\alpha_1, \alpha_2, \ldots, \alpha_n$ as in Lemma 2.1 satisfying (a) and (b). Let

$$u_0(t) = \eta_0^{-1}|e(t)|, \qquad u_i(t) = \eta_i^{-1} q_i(t)(x(t-\tau_i))^{\alpha_i}. \tag{2.12}$$

We have

$$
\begin{aligned}
\omega'(t) &= \frac{-(p(t)x(t))'}{x(t)} + \frac{\omega^2(t)}{p(t)} \\
&= \frac{\sum_{i=1}^{n} p_i(t)x(t-\tau_i) + \sum_{i=1}^{n} q_i(t)(x(t-\tau_i))^{\alpha_i} - e(t)}{x(t)} + \frac{\omega^2(t)}{p(t)} \\
&\geq \sum_{i=1}^{n} p_i(t)\left(\frac{t-a_1}{t-a_1+\tau_i}\right) + \left(\eta_0^{-1}|e(t)|\right)^{\eta_0} \frac{\prod_{i=1}^{n}\left(\eta_i^{-1}q_i(t)\right)^{\eta_i} x^{\alpha_i\eta_i}(t-\tau_i)}{x(t)} + \frac{\omega^2(t)}{p(t)} \\
&= \sum_{i=1}^{n} p_i(t)\left(\frac{t-a_1}{t-a_1+\tau_i}\right) + \left(\eta_0^{-1}|e(t)|\right)^{\eta_0} \frac{\prod_{i=1}^{n}\left(\eta_i^{-1}q_i(t)\right)^{\eta_i} x^{\alpha_i\eta_i}(t-\tau_i)}{\prod_{i=1}^{n} x^{\alpha_i\eta_i}(t)} + \frac{\omega^2(t)}{p(t)} \\
&\geq \sum_{i=1}^{n} p_i(t)\left(\frac{t-a_1}{t-a_1+\tau_i}\right) + \left(\eta_0^{-1}|e(t)|\right)^{\eta_0} \prod_{i=1}^{n}\left(\eta_i^{-1}q_i(t)\right)^{\eta_i}\left(\frac{t-a_1}{t-a_1+\tau_i}\right)^{\alpha_i\eta_i} + \frac{\omega^2(t)}{p(t)} \\
&= Q_1(t) + \frac{\omega^2(t)}{p(t)}.
\end{aligned} \tag{2.13}
$$

Multiplying both sides of (2.13) by $H_1(b_1,t) \in \mathfrak{D}_{a_1,b_1}$ and integrating by parts, we find that

$$
-\omega(c_1)H_1(b_1,c_1) \geq \int_{c_1}^{b_1}\left(Q_1(s)H_1(b_1,s) - p(s)h_{12}^2(b_1,s)\right)ds. \tag{2.14}
$$

That is,

$$
-\omega(c_1) \geq \frac{1}{H_1(b_1,c_1)}\int_{c_1}^{b_1}\left(Q_1(s)H_1(b_1,s) - p(s)h_{12}^2(b_1,s)\right)ds. \tag{2.15}
$$

On the other hand, multiplying both sides of (2.13) by $H_1(t,a_1) \in \mathfrak{D}_{a_1,b_1}$ and integrating by parts, we can easily obtain

$$
\omega(c_1) \geq \frac{1}{H_1(c_1,a_1)}\int_{a_1}^{c_1}\left(Q_1(s)H_1(s,a_1) - p(s)h_{11}^2(s,a_1)\right)ds. \tag{2.16}
$$

Equations (2.15) and (2.16) yield

$$
\begin{aligned}
&\frac{1}{H_1(c_1,a_1)}\int_{a_1}^{c_1}\left(Q_1(s)H_1(s,a_1) - p(s)h_{11}^2(s,a_1)\right)ds \\
&\quad + \frac{1}{H_1(b_1,c_1)}\int_{c_1}^{b_1}\left(Q_1(s)H_1(b_1,s) - p(s)h_{12}^2(b_1,s)\right)ds \leq 0,
\end{aligned} \tag{2.17}
$$

which contradicts (2.5) for $j = 1$. The proof of Theorem 2.2 is complete. □

Remark 2.3. When $\tau_1 = \cdots = \tau_n = 0$, $\Sigma_{i=1}^n p_i(t) = q(t)$, the conditions $q(t) \geq 0$ for $t \in [a_1, b_1] \cup [a_2, b_2]$, $p'(t) \geq 0$ and (1.2) can be removed. Therefore, Theorem 2.2 reduces to Theorem 1 in [3].

Remark 2.4. When $\tau_1 = \cdots = \tau_n = \tau$, $\Sigma_{i=1}^n p_i(t) = q(t)$, Theorem 2.2 reduces to Theorem 1 in [4] for which the conditions $q(t) \geq 0$ for $t \in [a_1 - \tau, b_1] \cup [a_2 - \tau, b_2]$, $p'(t) \geq 0$ and (1.2) are needed. There are some mistakes in the proof of Theorem 1 in [4].

The following theorem gives an oscillation criterion for the unforced (1.1).

Theorem 2.5. *If, for any $T \geq 0$, there exist a, b, and c such that $T \leq a < c < b$, $p_i(t) \geq 0$, and $q_i(t) \geq 0$ for $t \in [a - \tau_i, b]$, $i = 1, 2, \ldots, n$, and there exists $H \in \mathfrak{D}_{a,b}$, such that*

$$\frac{1}{H(c,a)} \int_a^c \left(H(s,a)Q(s) - p(s)h_1^2(s,a)\right) ds + \frac{1}{H(b,c)} \int_c^b \left(H(b,s)Q(s) - p(s)h_2^2(b,s)\right) ds > 0, \tag{2.18}$$

where

$$Q(t) = \sum_{i=1}^n p_i(t) \left(\frac{t-a}{t-a+\tau_i}\right) + \prod_{i=1}^n \left(\eta_i^{-1} q_i(t)\right)^{\eta_i} \left(\frac{t-a}{t-a+\tau_i}\right)^{\alpha_i \eta_i}, \tag{2.19}$$

$\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying (a) *and* (c) *in Lemma 2.1, and h_1, h_2 are defined as in* (2.3)*, then the unforced* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_1 - \overline{\tau} \geq 0$, where t_1 depends on the solution $x(t)$ and $\overline{\tau} = \max\{\tau_i\}$, $i = 1, \ldots, n$. Similar to the proof in Theorem 2.2, we can obtain

$$\frac{x(t-\tau_i)}{x(t)} \geq \frac{t-a}{t-a+\tau_i}, \quad t \in [a,b]. \tag{2.20}$$

Define

$$\omega(t) = -\frac{p(t)x'(t)}{x(t)}. \tag{2.21}$$

Recall the arithmetic-geometric mean inequality

$$\sum_{i=1}^n \eta_i u_i \geq \prod_{i=1}^n u_i^{\eta_i}, \quad u_i \geq 0, \tag{2.22}$$

where $\eta_i > 0$, $i = 1, 2, \ldots, n$, are chosen according to the given $\alpha_1, \alpha_2, \ldots, \alpha_n$ as in Lemma 2.1 satisfying (a) and (c). Let

$$u_i = \eta_i^{-1} q_i(t) (x(t-\tau_i))^{\alpha_i}. \tag{2.23}$$

We can obtain

$$
\begin{aligned}
\omega'(t) &= \frac{-\left(p(t)x(t)\right)'}{x(t)} + \frac{\omega^2(t)}{p(t)} \\
&= \frac{\sum_{i=1}^{n} p_i(t)x(t-\tau_i) + \sum_{i=1}^{n} q_i(t)(x(t-\tau_i))^{\alpha_i}}{x(t)} + \frac{\omega^2(t)}{p(t)} \\
&\geq \sum_{i=1}^{n} p_i(t)\left(\frac{t-a}{t-a+\tau_i}\right) + \frac{\prod_{i=1}^{n}\left(\eta_i^{-1}q_i(t)\right)^{\eta_i} x^{\alpha_i\eta_i}(t-\tau_i)}{x(t)} + \frac{\omega^2(t)}{p(t)} \\
&= \sum_{i=1}^{n} p_i(t)\left(\frac{t-a}{t-a+\tau_i}\right) + \frac{\prod_{i=1}^{n}\left(\eta_i^{-1}q_i(t)\right)^{\eta_i} x^{\alpha_i\eta_i}(t-\tau_i)}{\prod_{i=1}^{n} x^{\alpha_i\eta_i}(t)} + \frac{\omega^2(t)}{p(t)} \\
&\geq \sum_{i=1}^{n} p_i(\mathrm{t})\left(\frac{t-a}{t-a+\tau_i}\right) + \prod_{i=1}^{n}\left(\eta_i^{-1}q_i(t)\right)^{\eta_i}\left(\frac{t-a}{t-a+\tau_i}\right)^{\alpha_i\eta_i} + \frac{\omega^2(t)}{p(t)} \\
&= Q(t) + \frac{\omega^2(t)}{p(t)}.
\end{aligned}
\tag{2.24}
$$

Multiplying both sides of (2.24) by $H(b,t) \in \mathfrak{D}_{a,b}$ and integrating by parts, we obtain

$$
\begin{aligned}
\int_c^b H(b,t)\omega'(t)dt &\geq \int_c^b H(b,t)Q(t)dt + \int_c^b H(b,t)\frac{\omega^2(t)}{p(t)}dt, \\
-H(b,c)\omega(c) &\geq \int_c^b H(b,t)Q(t)dt + \int_c^b \left(H(b,t)\frac{\omega^2(t)}{p(t)} - 2\omega(t)h_2(b,t)\sqrt{H(b,t)}\right)dt \\
&= \int_c^b \left(H(b,t)Q(t) - p(t)h_2^2(b,t)\right)dt + \int_c^b \left(\sqrt{\frac{H(b,t)}{p(t)}}\omega(t) - \sqrt{p(t)}h_2(b,t)\right)^2 dt \\
&\geq \int_c^b \left(H(b,t)Q(t) - p(t)h_2^2(b,t)\right)dt.
\end{aligned}
\tag{2.25}
$$

It follows that

$$
-\omega(c) \geq \frac{1}{H(b,c)}\int_c^b \left(H(b,t)Q(t) - p(t)h_2^2(b,t)\right)dt. \tag{2.26}
$$

On the other hand, multiplying both sides of (2.24) by $H(t,a) \in \mathfrak{D}_{a,b}$ and integrating by parts, we have

$$
\omega(c) \geq \frac{1}{H(c,a)}\int_a^c \left(H(t,a)Q(t) - p(t)h_1^2(t,a)\right)dt. \tag{2.27}
$$

Equations (2.26) and (2.27) yield

$$\frac{1}{H(c,a)}\int_a^c\Big(H(t,a)Q(t)-p(t)h_1^2(t,a)\Big)dt+\frac{1}{H(b,c)}\int_c^b\Big(H(b,t)Q(t)-p(t)h_2^2(b,t)\Big)dt<0, \tag{2.28}$$

which contradicts (2.24). The proof of Theorem 2.5 is complete. □

Remark 2.6. When $\tau_1=\cdots=\tau_n=0$, $\Sigma_{i=1}^n p_i(t)=q(t)$, the conditions $q(t)\geq 0$ for $t\in[a,b]$, $p'(t)\geq 0$ and (1.2) can be removed. Therefore, Theorem 2.5 reduces to Theorem 2 in [3].

Remark 2.7. When $\tau_1=\cdots=\tau_n=\tau$, $\Sigma_{i=1}^n p_i(t)=q(t)$, Theorem 2.5 reduces to Theorem 2 in [4] for which the conditions $q(t)\geq 0$ for $t\in[a-\tau,b]$, $p'(t)\geq 0$ and (1.2) are needed.

3. Example

In this section, we provide an example to illustrate our results.

Consider the following equation:

$$x''(t)+k\sin t\left|x\left(t-\frac{\pi}{8}\right)\right|^{\alpha_1}\operatorname{sgn} x\left(t-\frac{\pi}{8}\right)+l\cos t\left|x\left(t-\frac{\pi}{4}\right)\right|^{\alpha_2}\operatorname{sgn} x\left(t-\frac{\pi}{4}\right)=-m\cos 2t,\quad t\geq 0, \tag{3.1}$$

where k, l, m are positive constants, $\alpha_1>1$, and $0<\alpha_2<1$. Here

$$\begin{gathered} p(t)=1,\qquad p_1(t)=p_2(t)=0,\qquad q_1(t)=k\sin t,\qquad q_2(t)=l\cos t,\\ \tau_1=\frac{\pi}{8},\qquad \tau_2=\frac{\pi}{4},\qquad e(t)=-m\cos 2t. \end{gathered} \tag{3.2}$$

According to the direct computation, we have

$$Q_j(t)=k_0|\cos 2t|^{\eta_0}(\sin t)^{\eta_1}(\cos t)^{\eta_2}\left(\frac{t-a_j}{t-a_j+\tau_1}\right)^{\alpha_1\eta_1}\left(\frac{t-a_j}{t-a_j+\tau_2}\right)^{\alpha_2\eta_2},\quad j=1,2, \tag{3.3}$$

where $k_0=(\eta_0^{-1}/m)^{\eta_0}(\eta_1^{-1}/k)^{\eta_1}(\eta_2^{-1}/l)^{\eta_2}$, η_0 can be any positive number satisfying $0<\eta_0<(\alpha_1-1)/\alpha_1$, and η_1, η_2 satisfy (2.1). For any $T\geq 0$, we can choose

$$a_1=2i\pi,\qquad a_2=b_1=2i\pi+\frac{\pi}{4},\qquad b_2=2i\pi+\frac{\pi}{2},\qquad c_1=2i\pi+\frac{\pi}{8},\qquad c_2=2i\pi+\frac{3\pi}{8}, \tag{3.4}$$

for $i = 0, 1, \ldots,$ and $H_1(t,s) = H_2(t,s) = (t-s)^2$. By simple computation, we obtain $h_{j1}(t,s) = h_{j2}(t,s) = 1$, $j = 1, 2$. From Theorem 2.2, we have that (3.1) is oscillatory if

$$\begin{gathered}\int_{2i\pi}^{2i\pi+\pi/8} Q_1(s)(s-2i\pi)^2 ds + \int_{2i\pi+\pi/8}^{2i\pi+\pi/4} Q_1(s)\left(2i\pi + \frac{\pi}{4} - s\right)^2 ds > \frac{\pi}{4}, \\ \int_{2i\pi+\pi/4}^{2i\pi+3\pi/8} Q_2(s)\left(s - 2i\pi - \frac{\pi}{4}\right)^2 ds + \int_{2i\pi+3\pi/8}^{2i\pi+\pi/2} Q_2(s)\left(2i\pi + \frac{\pi}{2} - s\right)^2 ds > \frac{\pi}{4}.\end{gathered} \tag{3.5}$$

If $H_1(t,s) = H_2(t,s) = \sin^2(t-s)$, by simple computation, we obtain $h_{j1}(t,s) = h_{j2}(t,s) = \cos(t-s)$ for $j = 1, 2$. From Theorem 2.2, we have that (3.1) is oscillatory if

$$\begin{gathered}\int_{2i\pi}^{2i\pi+\pi/8} Q_1(s)(s-2i\pi)^2 ds + \int_{2i\pi+\pi/8}^{2i\pi+\pi/4} Q_1(s)\left(2i\pi + \frac{\pi}{4} - s\right)^2 ds > \frac{\pi}{16} + \frac{\sqrt{2}}{8}, \\ \int_{2i\pi+\pi/4}^{2i\pi+3\pi/8} Q_2(s)\left(s - 2i\pi - \frac{\pi}{4}\right)^2 ds + \int_{2i\pi+3\pi/8}^{2i\pi+\pi/2} Q_2(s)\left(2i\pi + \frac{\pi}{2} - s\right)^2 ds > \frac{\pi}{16} + \frac{\sqrt{2}}{8}.\end{gathered} \tag{3.6}$$

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