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Research Article

Global Behavior of a Higher-Order Difference Equation

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This paper is concerned with the global behavior of higher-order difference equation of the form $y_{n+1} = (y_n \exp(\beta(1 - 2\sum_{i=0}^{k} a_i y_{n-i})))/(1 - y_n + y_n \exp(\beta(1 - 2\sum_{i=0}^{k} a_i y_{n-i})))$, $n = 0, 1, 2\ldots$, $y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, 1)$. Under some certain assumptions, it is proved that the positive equilibrium is globally asymptotical stable.

1. Introduction and Preliminaries

Nonlinear difference equations of order greater than one are of paramount importance in applications where the $(n + 1)$th generation of the system depends on the previous k generations. Such equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, and economics [1–8]. The global character of difference equations is a most important topic and there have been many recent investigations and interest in the topic [1, 5–7, 9–14]. In particular, many researchers have paid attention to the global attractivity and convergence of the kth-order recursive sequence [2, 10, 14–19] and several approaches have been developed for finding the global character of difference equations; see [2, 6, 7, 9–15, 17, 19–21]. Moreover, we refer to [3, 4, 6, 7, 16] and the references therein for the oscillation and nonoscillation of difference equations. However, a large number of the literatures concerned with the *rational* difference equations and it is not enough to understand the global dynamics of a general difference equations, particularly irrational difference equation.

In this paper we study the global behavior of higher-order difference equation of some genotype selection model:

$$y_{n+1} = \frac{y_n \exp\left(\beta\left(1 - 2\sum_{i=0}^{k} a_i y_{n-i}\right)\right)}{1 - y_n + y_n \exp\left(\beta\left(1 - 2\sum_{i=0}^{k} a_i y_{n-i}\right)\right)}, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $\beta \in (0, \infty)$, $k \in \{0, 1, 2, \ldots\}$, $a_0, a_1, \ldots, a_k \in [0, 1]$, $\sum_{i=0}^{k} a_i = 1$, and initial conditions are $y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, 1)$. When $a_0 = a_1 = \cdots = a_{k-1} = 0$, $a_k = 1$, (1.1) reduces to

$$y_{n+1} = \frac{y_n \exp(\beta(1 - 2y_{n-k}))}{1 - y_n + y_n \exp(\beta(1 - 2y_{n-k}))}, \quad n = 1, 2, \ldots, \tag{1.2}$$

which was introduced by [6] as an example of a map generation by a simple mode for frequency dependent natural selection. The local stability of positive equilibrium $\overline{y} = 1/2$ of (1.2) was investigated by [6].

We note that the appearance of y_{n-i} $(i = 0, 1, \ldots, k)$ in the selection coefficient reflects the fact that the environment at the present time depends upon the activity of the population at some time in the past and that this in turn depends upon the gene frequency at that time. The points 0, 1/2, and 1 are the only equilibrium solutions of (1.1). One can easily see that $y_n \in [0, 1]$ for all $n = 1, 2, \ldots$. If $y_N = 0$ for some $N \in \mathbb{N}$, then $y_n = 0$ for all $n \geq N$ and if $y_N = 1$ for some $N \in \mathbb{N}$, then $y_n = 1$ for all $n \geq N$, So in the following, we will restrict our attention to the difference equation:

$$y_{n+1} = \frac{y_n \exp\left(\beta\left(1 - 2\sum_{i=0}^{k} a_i y_{n-i}\right)\right)}{1 - y_n + y_n \exp\left(\beta\left(1 - 2\sum_{i=0}^{k} a_i y_{n-i}\right)\right)}, \quad y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, 1). \tag{1.3}$$

By introducing the substitution

$$y_n = \frac{x_n}{1 + x_n}, \tag{1.4}$$

then (1.3) becomes

$$x_{n+1} = x_n \exp\left(\beta\left(2\sum_{i=0}^{k} \frac{a_i}{1 + x_{n-i}} - 1\right)\right), \quad n = 1, 2, \ldots, \tag{1.5}$$

where $\beta \in (0, \infty)$, $k \in \{0, 1, 2, \ldots\}$, $\sum_{i=0}^{k} a_i = 1$, and $x_{-k}, x_{-k+1}, \ldots, x_0 \in (0, \infty)$ are arbitrary initial conditions.

In the sequel we will consider (1.5). It is clear that (1.5) has unique positive equilibrium point $\overline{x} = 1$.

In the following, we give some results which will be useful in our investigation of the behavior of solutions of (1.5), and the proof of lemmas can be found in [6].

Definition 1.1. The equilibrium point $\overline{x}$ of the equation $x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k})$, $n = 0, 1, \ldots$, is the point that satisfies the condition $\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x})$.

Definition 1.2. (a) A sequence $\{x_n\}$ is said to be oscillate about zero or simply oscillate if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise the sequence is called nonoscillatory. A sequence $\{x_n\}$ is called strictly oscillatory if for every $n_0 \geq 0$, there exist $n_1, n_2 \geq n_0$ such that $x_{n_1} x_{n_2} < 0$.

(b) A sequence $\{x_n\}$ is said to be oscillate about $\overline{x}$ if the sequence $\{x_n - \overline{x}\}$ oscillates. The sequence is called strictly oscillatory about $\overline{x}$ if the sequence $\{x_n - \overline{x}\}$ is strictly oscillatory.

Definition 1.3. Let $\overline{y}$ be an equilibrium point of equation $y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k})$; then the equilibrium point $\overline{y}$ is called

(a) locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y_k, y_{-k+1}, \ldots, y_0 \in I$ with $|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + \cdots + |y_0 - \overline{y}| < \delta$, we have $|y_n - \overline{y}| < \varepsilon$, for all $n > -1$;

(b) locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$ with $|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + \cdots + |y_0 - \overline{y}| < \gamma$, we have $\lim_{n\to\infty} y_n = \overline{y}$;

(c) a global attractor if for all $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$, we have $\lim_{n\to\infty} y_n = \overline{y}$.

(d) globally asymptotically stable if $\overline{y}$ is locally stable and $\overline{y}$ is a global attractor.

Lemma 1.4. *Assume that β is a positive real number and k is a nonnegative integer; then the following statements are true.*

(a) *If $k = 0$, then every solution of* (1.2) *oscillates about $\overline{y} = 1/2$ if and only if $\beta > 2$.*

(b) *If $k \geq 1$, then every solution of* (1.2) *oscillates about $\overline{y} = 1/2$ if and only if*

$$\beta > 2\frac{k^k}{(k+1)^{k+1}}. \tag{1.6}$$

Lemma 1.5. *The linear difference equation*

$$x_{n+k} + \sum_{i=1}^{k} p_i x_{n+k-i} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.7}$$

where $p_1, p_2, \ldots, p_m \in \mathbb{R}$, is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1. \tag{1.8}$$

Lemma 1.6. *The linear difference equation*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0, \quad n = 1, 2, \ldots, \tag{1.9}$$

where $p_1, \ldots, p_m \in (0, \infty)$ *and* $k_1, k_2, \ldots, k_m$ *are positive integers, is asymptotically stable provided that*

$$\sum_{i=1}^{m} k_i p_i < 1. \tag{1.10}$$

Lemma 1.7. *Consider the difference equation*

$$x_{n+1} = x_n f(x_n, x_{n-k_1}, \ldots, x_{n-k_r}); \tag{1.11}$$

one assumes that the function f *satisfies the following hypotheses.*

(H_1) $f \in C[[0,\infty) \times [0,\infty)^r, (0,\infty)]$, *and* $g \in C[[0,\infty)^{r+1}, (0,\infty)]$, *where*

$$\begin{gathered} g(u_0, u_1, \ldots, u_r) = u_0 f(u_0, u_1, \ldots, u_r) \quad \text{for } u_0 \in (0,\infty),\ u_1, u_2, \ldots, u_r \in [0,\infty), \\ g(0, u_1, \ldots, u_r) = \lim_{u_o \to 0_+} g(u_0, u_1, \ldots, u_r). \end{gathered} \tag{1.12}$$

(H_2) $f(u_0, u_1, \ldots, u_r)$ *is nonincreasing in* $u_1, \ldots, u_r$.

(H_3) *The equation*

$$f(x, x, \ldots, x) = 1 \tag{1.13}$$

has a unique positive solution $\overline{x}$.

(H_4) *Either the function* $f(u_0, u_1, \ldots, u_r)$ *does not depend on* u_0 *or for every* $x > 0$ *and* $u \geq 0$

$$\left[f(x, u, \ldots, u) - f(\overline{x}, u, \ldots, u)\right](x - \overline{x}) \leq 0 \tag{1.14}$$

with

$$\left[f(x, u, \ldots, u) - f(\overline{x}, u, \ldots, u)\right](x - \overline{x}) < 0 \quad \text{for } x \neq \overline{x}. \tag{1.15}$$

(H_5) *The function* $f(u_0, u_1, \ldots, u_r)$ *does not depend on* u_0 *or that for every* $x, y \in (0,\infty)$ *with* $x \neq \overline{x}$

$$\left[x f(x, y, \ldots, y) - \overline{x} f(\overline{x}, y, \ldots, y)\right] > 0. \tag{1.16}$$

Furthermore assume that the function $F(x)$ *is given by*

$$F(x) = \overline{x} f(\overline{x}, x, \ldots, x)^{k+1} \tag{1.17}$$

and has no periodic orbits of prime period 2; then $\overline{x}$ *is a global attractor of all positive solutions of* (1.11).

Lemma 1.8. *Let $F \in C[[0,\infty),(0,\infty)]$ be a nonincreasing function, and let $\overline{x}$ denote the (unique) fixed point of F. Then the following statements are equivalent:*

(a) *$\overline{x}$ is only fixed point of F^2 in $(0,\infty)$;*

(b) *$F^2(x) > x$ for $0 < x < \overline{x}$.*

2. Main Results

In this section, we will investigate the asymptotic stability and global behavior of the positive equilibrium point of (1.5).

Theorem 2.1. *Assume that β is a positive real number and k is a nonnegative integer; then the following statements are true.*

(a) *If $k = 0$, then every solution of (1.5) oscillates about $\overline{x} = 1$ if and only if $\beta > 2$.*

(b) *If $k \geq 1$, $0 < \beta < \dfrac{4}{k(k+1)}$, $a_0 = 0$, then $\overline{x} = 1$ is an asymptotically stable solution of (1.5).*

(c) *If $k \geq 1$, $0 < \beta \leq 2$, $1/2 < a_0 \leq 1$, then $\overline{x} = 1$ is an asymptotically stable solution of (1.5).*

Proof. When $k = 0$, then $a_0 = 1$, $a_1 = a_2 = \cdots = a_k = 0$, (1.5) becomes (1.2), and case (a) follows from Lemma 1.4(a).

When $k \neq 0$, the linearized equation of (1.5) about equilibrium point $\overline{x} = 1$ is

$$z_{n+1} - \left(1 - \frac{\beta a_0}{2}\right) z_n + \sum_{i=1}^{k} \frac{\beta a_i}{2} z_{n-i} = 0. \tag{2.1}$$

If $a_0 = 0$, by applying the Lemma 1.6, (2.1) is asymptotically stable provided that

$$\sum_{i=1}^{k} i \frac{\beta a_i}{2} < 1, \tag{2.2}$$

so for

$$0 < \beta < \frac{4}{k(k+1)}, \tag{2.3}$$

and $\overline{x} = 1$ is an asymptotically stable solution of (1.5). The proof of (b) is completed.

If $1/2 < a_0 \leq 1$, by applying Lemma 1.5, (2.1) is asymptotically stable provided that

$$\left|\left(1 - \frac{\beta a_0}{2}\right)\right| + \sum_{i=1}^{k} \left|\frac{\beta a_i}{2}\right| < 1, \tag{2.4}$$

so for $0 < \beta < 2$, and $\overline{x} = 1$ is an asymptotically stable solution of (1.5). The proof of (c) is completed. □

Theorem 2.2. *Assume that β is a positive real number and k is a positive integer. Then the equilibrium $\overline{x} = 1$ of (1.5) is globally asymptotically stable, when one of the following three cases holds:*

(a) $1 \le k \le 7,\ 0 < \beta < 1/2(k+1),\ a_0 = 0$;

(b) $k \ge 8,\ 0 < \beta < 4/k(k+1),\ a_0 = 0$;

(c) $k \ge 1,\ 0 < \beta < 2c_0/(k+1),\ 1/2 < a_0 < 1$, *where c_0 is a constant with $1/2 < c_0 < 1$.*

Proof. If one of the three conditions (a), (b), (c) is satisfied, by applying Theorem 2.1, $\overline{x} = 1$ is an asymptotically stable solution of (1.5). To complete the proof it remains to show that $\overline{x}$ is a global attractor of all positive solutions of (1.5). We will employ Lemma 1.7; set

$$f(u_0, u_1, \dots, u_k) = \exp\left[\beta\left(2\sum_{i=0}^{k}\frac{a_i}{1+u_i} - 1\right)\right]; \tag{2.5}$$

clearly equation $x_{n+1} = x_n f(x_n, x_{n-1}, \dots, x_{n-k})$ satisfies the hypotheses (H_1)–(H_4) of Lemma 1.7. Set

$$h(x, y) = x f(x, y, \dots, y) = x \exp\left(\beta\left(\frac{1-y-2a_0}{1+y} + \frac{2a_0}{1+x}\right)\right). \tag{2.6}$$

If $0 \le a_0 \le 1,\ x \ne \overline{x},\ 0 < \beta < 2$, then

$$\begin{aligned}\frac{\partial h}{\partial x} &= \exp\left(\beta\left(\frac{1-y-2a_0}{1+y} + \frac{2a_0}{1+x}\right)\right)\left(1 - \frac{2\beta a_0 x}{(1+x)^2}\right)\\ &= \exp\left(\beta\left(\frac{1-y-2a_0}{1+y} + \frac{2a_0}{1+x}\right)\right)\left(\frac{x^2 + (2-2\beta a_0)x + 1}{(1+x)^2}\right) > 0;\end{aligned} \tag{2.7}$$

hence, the function $f(u_0, u_1, \dots, u_r)$ does not depend on u_0 or that for every $x, y \in (0, \infty)$ with $x \ne \overline{x}$,

$$[x f(x, y, \dots, y) - \overline{x} f(\overline{x}, y, \dots, y)] > 0. \tag{2.8}$$

Furthermore assume that the function $F(x)$ is given by

$$F(x) = \overline{x} f(\overline{x}, x, \dots, x)^{k+1} = \exp\left((k+1)\beta\left(\frac{2(1-a_0)}{1+x} + a_0 - 1\right)\right). \tag{2.9}$$

Set

$$\Phi(x) = x - F(F(x)); \tag{2.10}$$

then

$$\begin{aligned}\Phi'(x) &= 1 - F'(F(x))F'(x)\\ &= 1 - \left[\exp\left((k+1)\beta\left(\frac{2(1-a_0)}{1+x} + a_0 - 1\right)\right)(k+1)\beta\frac{2(1-a_0)}{(1+x)^2}\right]\Bigg|_{F(x)}\\ &\quad\times \exp\left((k+1)\beta\left(\frac{2(1-a_0)}{1+x} + a_0 - 1\right)\right)(k+1)\beta\frac{2(1-a_0)}{(1+x)^2}.\end{aligned} \tag{2.11}$$

When $0 < \beta(k+1)(1-a_0) < c_0,\ 0 < x < 1$, then $1 < F(x) < \exp((k+1)\beta(1-a_0)) < e^{c_0}$. Hence,

$$\begin{aligned}\Phi'(x) &= 1 - \left[\exp\left((k+1)\beta\left(\frac{2(1-a_0)}{1+x} + a_0 - 1\right)\right)(k+1)\beta\frac{2(1-a_0)}{(1+x)^2}\right]\Bigg|_{F(x)}\\ &\quad\times \left(\exp\left((k+1)\beta\left(\frac{2(1-a_0)}{1+x} + a_0 - 1\right)\right)(k+1)\beta\frac{2(1-a_0)}{(1+x)^2}\right.\\ &\geq 1 - \left[\exp\left((k+1)\beta\left(\frac{2(1-a_0)}{1+x} + a_0 - 1\right)\right)(k+1)\beta\frac{2(1-a_0)}{(1+x)^2}\right]\Bigg|_{F(x)=1}\\ &\quad\times \exp\left((k+1)\beta\left(\frac{2(1-a_0)}{1+x} + a_0 - 1\right)\right)(k+1)\beta\frac{2(1-a_0)}{(1+x)^2}\Bigg|_{x=0}\\ &\geq 1 - (k+1)^2\beta^2(1-a_0)^2 e^{c_0} \geq 1 - c_0^2 e^{c_0} = 0,\end{aligned} \tag{2.12}$$

where c_0 is a constant with $1 - c_0^2 e^{c_0} = 0$, and clearly $1/2 < c_0 < 1$. So $\Phi(x)$ is an increasing function, hence

$$\Phi(x) = x - F^2(x) < \Phi(1) = 0, \tag{2.13}$$

and so, $F^2(x) > x$ for $0 < x < 1$, by Lemma 1.8, we know that $\overline{x}$ is only fixed point of F^2 in $(0, \infty)$, and by Lemma 1.7, $\overline{x}$ is a global attractor of all positive solutions of (1.5). The proof is complete. □

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