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Research Article

Topological Entropy and Special α-Limit Points of Graph Maps

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Let G a graph and $f : G \to G$ be a continuous map. Denote by $h(f)$, $R(f)$, and $\mathrm{SA}(f)$ the topological entropy, the set of recurrent points, and the set of special α-limit points of f, respectively. In this paper, we show that $h(f) > 0$ if and only if $\mathrm{SA}(f) - R(f) \neq \emptyset$.

1. Introduction

Let (X, d) be a metric space. For any $Y \subset X$, denote by $\mathring{Y}$, ∂Y, and $\overline{Y}$ the interior, the boundary, and the closure of Y in X, respectively. For any $y \in X$ and any $r > 0$, write $B(y, r) = \{x \in X : d(x, y) < r\}$. Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Denote by $C^0(X)$ the set of all continuous maps from X to X. For any $f \in C^0(X)$, let f^0 be the identity map of X and $f^n = f \circ f^{n-1}$ the composition map of f and f^{n-1}. A point $x \in X$ is called a periodic point of f with period n if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < n$. The orbit of x under f is the set $O(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}_+\}$. Write $\omega(x, f) = \bigcap_{i=1}^{\infty} \overline{O(f^i(x), f)}$, called the ω-limit set of x under f. In fact, $y \in \omega(x, f)$ if and only if there exists a sequence of positive integers $n_1 < n_2 < n_3 < \cdots$ such that $\lim_{i\to\infty} f^{n_i}(x) = y$. x is called a recurrent point of f if $x \in \omega(x, f)$. x is called a special α-limit point of f if there exist a sequence of positive integers $\{n_i\}_{i=1}^{\infty}$ and a sequence of points $\{y_i\}_{i=0}^{\infty}$ such that $f^{n_i}(y_i) = y_{i-1}$ for any $i \in \mathbb{N}$ and $\lim_{i\to\infty} y_i = x$. Denote by $P(f)$, $R(f)$, and $\mathrm{SA}(f)$ the sets of periodic points, recurrent points, and special α-limit points of f, respectively. From the definitions it is easy to see that $P(f) \subset \mathrm{SA}(f)$ and $P(f) \subset R(f)$. Let $h(f)$ denote the topological entropy of f, for the definition see [1, Chapter VIII].

A metric space X is called an arc (resp., an open arc, a circle) if it is homeomorphic to the interval $[0, 1]$ (resp., the open interval $(0, 1)$, the unit circle S^1). Let A be an arc and

$h : [0,1] \to A$ a homeomorphism. The points $h(0)$ and $h(1)$ are called the endpoints of A, and we write $\mathrm{End}(A) = \{h(0), h(1)\}$. A compact connected metric space G is called a graph if there are finitely many arcs $A_1, \ldots, A_n$ $(n \geq 1)$ in G such that $G = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \mathrm{End}(A_i) \cap \mathrm{End}(A_j)$ for all $1 \leq i < j \leq n$. A graph T is called a tree if it contains no circle. A continuous map from a graph (resp., a tree, an interval) to itself is called a graph map (resp., a tree map, an interval map).

Let G be a given graph. Take a metric d on G such that, for any $x \in G$ and any $r > 0$, the open ball $B(x,r) \equiv \{y \in G : d(y,x) < r\}$ is always connected. For any finite set S, let $|S|$ denote the number of elements of S. For any $x \in G$, write $\mathrm{val}(x) = \lim_{r \to +0} |\partial B(x,r)|$, which is called the valence of x. x is called a branching point (resp., an endpoint) of G if $\mathrm{val}(x) > 2$ (resp., $\mathrm{val}(x) = 1$). Denote by $\mathrm{End}(G)$ and $\mathrm{Br}(G)$ the sets of endpoints and branching points of G, respectively. Take a finite subset $V(G)$ of G containing $\mathrm{End}(G) \cup \mathrm{Br}(G)$ such that, for any connected component E of $G - V(G)$, the closure $\overline{E}$ is an arc. Such a subset $V(G)$ is called the set of vertexes of G, and the closure of every connected component of $G - V(G)$ is called an edge. For any edge I of G and any $a, b \in I$, we denote by $[a,b]_I$ (or simply $[a,b]$ if there is no confusion) the smallest connected closed subset of I containing $\{a,b\}$, which is called a closed interval of G. So, a closed interval is always a subset of an edge. Write $(a,b] = [b,a) = [a,b] - \{a\}$ and $(a,b) = (a,b] - \{b\}$. Let G be a graph and $J, K \subset G$ closed intervals, and $f \in C^0(G)$. We write $f(J) \sqsupset K$ if there exists a closed subinterval $L \subset J$ such that $f(L) = K$.

In the study of dynamical systems, recurrent points, topological entropy, and special α-limit points play an important role. For interval maps, Hero [2] obtained the following result.

Theorem A (see [2, Corollary]). *Let I be a compact interval and $f \in C^0(I)$. Then the following are equivalent:*

(1) *some point y that is not recurrent is a special α-limit point;*

(2) *some periodic point has period that is not a power of two.*

It is known [1, Chapter VIII, Proposition 34] that $h(f) > 0$ if and only if some periodic point of f has period that is not a power of two for interval map f.

In [3], Llibre and Misiurewicz studied the topological entropy of a graph map and obtained the following theorem.

Theorem B (see [3, Theorems 1 and 2]). *Let G be a graph and $f \in C^0(G)$. Then $h(f) > 0$ if and only if there exist $n \in \mathbb{N}$ and closed intervals $L, J, K \subset G$ with $J, K \subset L$ and $|K \cap J| \leq 1$ such that $f^n(J) \sqsupset L$ and $f^n(K) \sqsupset L$.*

Recently, there has been a lot of work on the dynamics of graph maps (see [4–13]). In this paper, we will study the topological entropy and special α-limit points of graph maps. Our main result is the following theorem.

Theorem 1.1. *Let G be a graph and $f \in C^0(G)$. Then $h(f) > 0$ if and only if $SA(f) - R(f) \neq \emptyset$.*

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To do this, we need the following lemmas.

Lemma 2.1 (see [11, Theorem 1]). *Let G be a graph and $f \in C^0(G)$. If $x \in SA(f)$, then there exist a sequence of positive integers $n_1 \leq n_2 \leq n_3 \leq \cdots$ and a sequence of points $\{y_i\}_{i=0}^{\infty}$ with $y_0 = x$ such that $f^{n_i}(y_i) = y_{i-1}$ for any $i \in \mathbb{N}$ and $\lim_{i\to\infty} y_i = x$.*

Remark 2.2. The main idea of the proof of Theorem 1 in [11] is similar to the one of Main Theorem in [2].

Lemma 2.3. *Let G be a graph and $f \in C^0(G)$. Then $SA(f) \subset f(SA(f))$.*

Proof. Let $x \in \mathrm{SA}(f)$. Then there exist a sequence of points $\{x_i\}_{i=0}^{\infty}$ and a sequence of positive integers $2 \leq m_1 \leq m_2 \leq \cdots$ such that $f^{m_i}(x_i) = x_{i-1}$ for every $i \in \mathbb{N}$ and $\lim_{i\to\infty} x_i = x$. Write $y_i = f^{m_i-1}(x_i)$ for $i \in \mathbb{N}$. Let $y_{k_0} = y_1, y_{k_1}, y_{k_2}, \ldots, y_{k_i}, \ldots$ be a convergence subsequence of $\{y_i\}_{i=1}^{\infty}$, and let $\lim_{i\to\infty} y_{k_i} = y$. Then

$$f(y) = \lim_{i\to\infty} f(y_{k_i}) = \lim_{i\to\infty} f^{m_{k_i}}(x_{k_i}) = \lim_{i\to\infty} x_{k_i-1} = x. \tag{2.1}$$

Write

$$\mu_i = \begin{cases} m_{k_1-1} + \cdots + m_1, & \text{if } i = 1, \\ m_{k_i-1} + m_{k_i-2} + \cdots + m_{k_{i-1}}, & \text{if } i \geq 2. \end{cases} \tag{2.2}$$

Then $f^{\mu_i}(y_{k_i}) = f^{\mu_i + m_{k_i}-1}(x_{k_i}) = f^{m_{k_{i-1}}-1}(x_{k_{i-1}}) = y_{k_{i-1}}$ for any $i \in \mathbb{N}$, which implies that $y \in \mathrm{SA}(f)$ and $\mathrm{SA}(f) \subset f(\mathrm{SA}(f))$. The proof is completed. □

Lemma 2.4 (see [3, Lemma 2.4]). *Let G be a graph and $f \in C^0(G)$. Suppose that J and $L = [a,b]$ are intervals of G. If there exist $x \in (a,b)$ and $y \notin (a,b)$ such that $\{x,y\} \subset f(J)$, then $f(J) \sqsupset [a,x]$ or $f(J) \sqsupset [x,b]$.*

Theorem 2.5. *Let G be a graph and $f \in C^0(G)$. Then $h(f) > 0$ if and only if $SA(f) - R(f) \neq \emptyset$.*

Proof Necessity

If $\mathrm{SA}(f) - R(f) \neq \emptyset$, then take a point $w_0 \in \mathrm{SA}(f) - R(f)$. By Lemma 2.3 and $f(R(f)) = R(f)$, for every $i = 1, 2, \ldots$, there exists a point $w_i \in \mathrm{SA}(f) - R(f)$ such that $f(w_i) = w_{i-1}$. Note that $w_0, w_1, w_2, \ldots$ are mutually different. Since the numbers of vertexes and edges of G are finite, there exists an edge I of G such that $I \cap \{w_0, w_1, w_2, \ldots\}$ is an infinite set. We can choose integers $1 < i_1 < i_2 < \cdots$ such that $\{w_{i_k} : k \in \mathbb{N}\} \subset I$ and $w_{i_k} \in (w_{i_1}, w_{i_{k+1}})$ for every $k \geq 2$. Take points $\{y, x, z\} \subset \mathring{I} \cap (\mathrm{SA}(f) - R(f))$ with $x \in (y,z)$ such that $f^m(y) = x$ and $f^n(x) = z$ for some $m, n \in \mathbb{N}$. Without loss of generality we may assume that $I = [0,1]$ and $0 < y < x < z < 1$. Since $y \in \mathrm{SA}(f) - R(f)$, we can take points $\{y_i : i \in \mathbb{N}\} \subset (0,1)$ and positive integers $m + n < m_1 < m_2 < m_3 < \cdots$ satisfying the following conditions:

(1) the sequence $(y_1, y_2, y_3, \ldots)$ is strictly monotonic with $f^{m_i}(y_i) = y_{i-1}$ for any $i \in \mathbb{N}$ and $y_0 = y$ (see Lemma 2.1) and $\lim_{i\to\infty} y_i = y$;

(2) $m_i > m_1 + m_2 + \cdots + m_{i-1}$ for any $i \geq 2$.

Let $x_i = f^m(y_i)$ and $z_i = f^n(x_i)$ for any $i \in \mathbb{Z}_+$. Then $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} z_i = z$. Noting that $x, z \in \mathrm{SA}(f) - R(f)$, we can assume that $\{x_i, z_i : i \in \mathbb{N}\} \subset (0,1)$, and there exists $\varepsilon > 0$ such that the following conditions hold:

(3) $f^i(x) \notin [x-\varepsilon, x+\varepsilon]$ for any $i \in \mathbb{N}$;

(4) the sequences $(x_1, x_2, x_3, \ldots)$ and $(z_1, z_2, z_3, \ldots)$ are strictly monotonic, and $\{x_i : i \in \mathbb{N}\} \subset [x-\varepsilon, x+\varepsilon] \subset (y, z)$.

In the following we may consider only the case that $(x_1, x_2, x_3, \ldots)$ is strictly decreasing since the other case that $(x_1, x_2, x_3, \ldots)$ is strictly increasing is similar.

Write $\mu_i = m_i + m_{i-1} + \cdots + m_1$ for any $i \in \mathbb{N}$. Put $I_i = [x_i, x_{i-1}]$ and $A_i = f^{\mu_{i-1}}(I_i)$ for any $i \ge 2$. Then A_i is a connected set, and

$$\left\{ f^{\mu_{i-1}}(x_{i-1}),\ f^{\mu_{i-1}}(x_i) \right\} = \left\{ x, f^{\mu_{i-1}}(x_i) \right\} \subset A_i. \tag{2.3}$$

Noting that $f^{m_i}(f^{\mu_{i-1}}(x_i)) = f^{\mu_i}(x_i) = x$, we have $x \in f^{m_i}(A_i) \cap A_i$. Write $S_i = \bigcup_{j=0}^{\infty} f^{jm_i}(A_i)$. Then S_i is a connected set containing x and $f^{m_i}(S_i) \subset S_i$ for every $i \ge 2$.

Since $f^{m_i}(x_{i-1}) = f^{m_i - \mu_{i-1}}(x)$ and $f^{m_i}(x_i) = x_{i-1}$ for any $i \ge 2$, by Lemma 2.4 it follows that $f^{m_i}(I_i) \sqsupset [x-\varepsilon, x_{i-1}]$ or $f^{m_i}(I_i) \sqsupset [x_{i-1}, x+\varepsilon]$. There are two cases to consider.

Case 1. There exist $2 \le \alpha < \beta < \lambda$ such that $f^{m_i}(I_i) \sqsupset [x-\varepsilon, x_{i-1}]$ for every $i \in \{\alpha, \beta, \lambda\}$.

Subcase 1.1. There exists $\lambda \le \tau$ such that $S_\tau \not\subset (0,1)$. Then $S_\tau \cap \{y_\alpha, z_{\alpha+1}\} \neq \emptyset$, and there exist $r \ge \mu_{\tau-1}$ and $u \in I_\tau$ such that $f^r(u) \in \{y_\alpha, z_{\alpha+1}\}$, from which and $m_{\alpha+1} > m+n$ it follows

$$f^{m+r}(u) = f^m(y_\alpha) = x_\alpha \quad \text{or} \quad f^{m_{\alpha+1}-n+r}(u) = f^{m_{\alpha+1}-n}(z_{\alpha+1}) = x_\alpha. \tag{2.4}$$

Noting $f^{m+r}(x_{\tau-1}) = f^{m+r-\mu_{\tau-1}}(x)$ and $f^{m_{\alpha+1}-n+r}(x_{\tau-1}) = f^{m_{\alpha+1}-n+r-\mu_{\tau-1}}(x)$, we have

$$\left\{ f^{m+r-\mu_{\tau-1}}(x), f^{m_{\alpha+1}-n+r-\mu_{\tau-1}}(x) \right\} \cap [x-\varepsilon, x+\varepsilon] = \emptyset. \tag{2.5}$$

There exists $s \in \{m+r, m_{\alpha+1}-n+r\}$ such that $f^s(I_\tau) \sqsupset I_\beta \cup I_\lambda$ or $f^s(I_\tau) \sqsupset I_\alpha$, which implies

$$f^{s+m_\lambda}(I_\lambda) \sqsupset f^s(I_\tau) \sqsupset I_\beta \cup I_\lambda \quad \text{or} \quad f^{s+m_\alpha+m_\lambda}(I_\lambda) \sqsupset f^{s+m_\alpha}(I_\tau) \sqsupset f^{m_\alpha}(I_\alpha) \sqsupset I_\beta \cup I_\lambda. \tag{2.6}$$

On the other hand, $f^{m_\beta}(I_\beta) \sqsupset I_\beta \cup I_\lambda$. Thus we can obtain $f^l(I_\lambda) \sqsupset I_\beta \cup I_\lambda$ and $f^l(I_\beta) \sqsupset I_\beta \cup I_\lambda$ for some $l \in \{(s+m_\lambda)m_\beta,\ (s+m_\alpha+m_\lambda)m_\beta\}$. By Theorem B it follows that $h(f) > 0$.

Subcase 1.2. $S_i \subset (0,1)$ for all $i \ge \lambda$, and there exists $\tau \ge \lambda$ such that $x < \sup S_\tau$. Then we can take $j \ge \tau$ such that $[x, x_j] \subset S_\tau$. Thus there exist $r \ge \mu_{\tau-1}$ and $u \in I_\tau$ such that $f^r(u) = x_j$, which implies $f^{r+m_j+\cdots+m_{\alpha+1}}(u) = x_\alpha$. Write $s = r + m_j + \cdots + m_{\alpha+1}$. Then $f^s(I_\tau) \sqsupset I_\beta \cup I_\lambda$ or $f^s(I_\tau) \sqsupset I_\alpha$ since $f^s(x_{\tau-1}) = f^{s-\mu_{\tau-1}}(x) \notin [x-\varepsilon, x+\varepsilon]$, which implies

$$f^{s+m_\lambda}(I_\lambda) \sqsupset f^s(I_\tau) \sqsupset I_\beta \cup I_\lambda \quad \text{or} \quad f^{s+m_\alpha+m_\lambda}(I_\lambda) \sqsupset f^{s+m_\alpha}(I_\tau) \sqsupset f^{m_\alpha}(I_\alpha) \sqsupset I_\beta \cup I_\lambda. \tag{2.7}$$

On the other hand, $f^{m_\beta}(I_\beta) \sqsupset I_\beta \cup I_\lambda$. Thus we can obtain $f^l(I_\lambda) \sqsupset I_\beta \cup I_\lambda$ and $f^l(I_\beta) \sqsupset I_\beta \cup I_\lambda$ for some $l \in \{(s+m_\lambda)m_\beta,\ (s+m_\alpha+m_\lambda)m_\beta\}$. By Theorem B it follows that $h(f) > 0$.

Subcase 1.3. One has $S_i \subset (0,1)$ and $x = \sup S_i$ for all $i \geq \lambda$.

If $f^{m_r}(x) < f^{2m_r}(x) < x$ for some $r \geq \lambda$, then there exist $j \geq r+2$ and $u \in I_r$ such that $f^{\mu_r}(u) = f^{2m_r}(x_j)$ since $\lim_{i\to\infty} f^{2m_r}(x_i) = f^{2m_r}(x)$ and $\{f^{m_r}(x), x\} \subset f^{\mu_r}(I_r)$, which implies $f^{\mu_r+m_j+m_{j-1}+\cdots+m_{\alpha+1}-2m_r}(u) = x_\alpha$. Using arguments similar to ones developed in the proof of Subcase 1.2, we can obtain $f^l(I_\lambda) \sqsupset I_\beta \cup I_\lambda$ and $f^l(I_\beta) \sqsupset I_\beta \cup I_\lambda$ for some $l \in \mathbb{N}$. By Theorem B it follows that $h(f) > 0$. Now we assume $f^{2m_r}(x) \leq f^{m_r}(x) < x$ for all $r \geq \lambda$. Note $f^{\mu_{r-1}}(x_r) \notin O(f^{m_r}, x)$ since $x \notin R(f)$.

If $f^{2m_r}(x) \leq f^{m_r}(x) < f^{\mu_{r-1}}(x_r) < x$ for some $r \geq \lambda$, then $f^{m_r}([f^{m_r}(x), f^{\mu_{r-1}}(x_r)]) \sqsupset [f^{m_r}(x), x]$ and $f^{m_r}([f^{\mu_{r-1}}(x_r), x]) \sqsupset [f^{m_r}(x), x]$. By Theorem B it follows that $h(f) > 0$.

If $f^{\mu_{r-1}}(x_r) < f^{m_r}(x)$ for some $r \geq \lambda$, then there exist $j \geq r+2$ and $u \in I_r$ such that $f^{\mu_{r-1}}(u) = f^{m_r}(x_j)$ since $\lim_{i\to\infty} f^{m_r}(x_i) = f^{m_r}(x)$ and $\{f^{\mu_{r-1}}(x_r), x\} \subset f^{\mu_{r-1}}(I_r)$, which implies $f^{\mu_{r-1}+m_j+m_{j-1}+\cdots+m_{\alpha+1}-m_r}(u) = x_\alpha$. Using arguments similar to ones developed in the proof of Subcase 1.2, we can obtain $f^l(I_\lambda) \sqsupset I_\beta \cup I_\lambda$ and $f^l(I_\beta) \sqsupset I_\beta \cup I_\lambda$ for some $l \in \mathbb{N}$. By Theorem B it follows that $h(f) > 0$.

Case 2. There exists $\kappa \geq 2$ such that $f^{m_i}(I_i) \sqsupset [x_{i-1}, x+\varepsilon]$ for all $i \geq \kappa$.

Subcase 2.1. There exist $\kappa \leq \alpha < \beta$ such that $S_i \not\subset (0,1)$ for every $i \in \{\alpha, \beta\}$. Then $S_\beta \cap \{y_\beta, z_{\beta+1}\} \neq \emptyset$ and $S_\alpha \cap \{y_\beta, z_{\beta+1}\} \neq \emptyset$. Thus there exist $r \geq \mu_{\beta-1}$ and $u \in I_\beta$ such that $f^r(u) \in \{y_\beta, z_{\beta+1}\}$, from which it follows that $f^{m+r}(u) = x_\beta$ or $f^{m_{\beta+1}-n+r}(u) = x_\beta$. Since $f^{m+r}(x_{\beta-1}) = f^{m+r-\mu_{\beta-1}}(x)$, $f^{m_{\beta+1}-n+r}(x_{\beta-1}) = f^{m_{\beta+1}-n+r-\mu_{\beta-1}}(x)$, and

$$\left\{f^{m+r-\mu_{\beta-1}}(x), f^{m_{\beta+1}-n+r-\mu_{\beta-1}}(x)\right\} \cap [x-\varepsilon, x+\varepsilon] = \emptyset, \tag{2.8}$$

there exists $s \in \{m+r, m_{\beta+1}-n+r\}$ such that $f^s(I_\beta) \sqsupset I_\beta \cup I_\alpha$ or $f^s(I_\beta) \sqsupset I_{\beta+1}$, which implies $f^s(I_\beta) \sqsupset I_\beta \cup I_\alpha$ or $f^{s+m_{\beta+1}}(I_\beta) \sqsupset f^{m_{\beta+1}}(I_{\beta+1}) \sqsupset I_\beta \cup I_\alpha$. In similar fashion, we can show $f^t(I_\alpha) \sqsupset I_\beta \cup I_\alpha$ for some $t \in \mathbb{N}$. Thus we get $f^l(I_\beta) \sqsupset I_\beta \cup I_\alpha$ and $f^l(I_\alpha) \sqsupset I_\beta \cup I_\alpha$ for some $l \in \{st, (s+m_{\beta+1})t\}$. It follows from Theorem B that $h(f) > 0$.

Subcase 2.2. There exists $\vartheta \geq \kappa$ such that $S_i \subset (0,1)$ for all $i \geq \vartheta$ and there exists $\tau \geq \lambda \geq \vartheta$ such that $x < \sup S_\tau$ and $x < \sup S_\lambda$. Take $j \geq \tau+2$ such that $S_i \supset [x, x_j]$ for $i \in \{\lambda, \tau\}$. Then there exist $r_1 \geq \mu_{\tau-1}$, $r_2 \geq \mu_{\lambda-1}$, and $u \in I_\tau$, $v \in I_\lambda$ such that $f^{r_1}(u) = f^{r_2}(v) = x_j$. Using arguments similar to ones developed in the proof of Subcase 2.1, we can obtain $f^l(I_\lambda) \sqsupset I_\tau \cup I_\lambda$ and $f^l(I_\tau) \sqsupset I_\tau \cup I_\lambda$ for some $l \in \mathbb{N}$. By Theorem B it follows that $h(f) > 0$.

Subcase 2.3. There exists $\vartheta \geq \kappa$ such that $S_i \subset (0,1)$ and $x = \sup S_i$ for all $i \geq \vartheta$.

If there exist $\tau > \lambda \geq \vartheta$ such that $f^{m_i}(x) < f^{2m_i}(x) < x$ for $i \in \{\tau, \lambda\}$, then there exist $j \geq \tau+2$, $u \in I_\tau$, and $v \in I_\lambda$ such that $f^{\mu_\tau}(u) = f^{2m_\tau}(x_j)$ and $f^{\mu_\lambda}(v) = f^{2m_\lambda}(x_j)$, which implies $f^{\mu_\tau+m_j+m_{j-1}+\cdots+m_{\tau+1}-2m_\tau}(u) = x_\tau$ and $f^{\mu_\lambda+m_j+m_{j-1}+\cdots+m_{\tau+1}-2m_\lambda}(v) = x_\tau$. Using arguments similar to ones developed in the proof of Subcase 2.1, we can obtain $f^l(I_\lambda) \sqsupset I_\tau \cup I_\lambda$ and $f^l(I_\tau) \sqsupset I_\tau \cup I_\lambda$ for some $l \in \mathbb{N}$. By Theorem B it follows that $h(f) > 0$. Now we assume that there exists $\theta \geq \vartheta$ such that $f^{2m_i}(x) \leq f^{m_i}(x) < x$ for all $i \geq \theta$.

If $f^{\mu_{i-1}}(x_i) < f^{m_i}(x) < x$ for all $i \geq \theta$, then using arguments similar to ones developed in the above proof, we can obtain $h(f) > 0$.

If $f^{2m_r}(x) \leq f^{m_r}(x) < f^{\mu_{r-1}}(x_r) < x$ for some $r \geq \theta$, then $f^{m_r}([f^{m_r}(x), f^{\mu_{r-1}}(x_r)]) \sqsupset [f^{m_r}(x), x]$ and $f^{m_r}([f^{\mu_{r-1}}(x_r), x]) \sqsupset [f^{m_r}(x), x]$. By Theorem B it follows $h(f) > 0$.

Sufficiency

If $h(f) > 0$, then it follows from Theorem B that there exist $n \in \mathbb{N}$ and closed intervals $L, J, K \subset G$ with $J, K \subset L$ and $|K \cap J| \le 1$ such that $f^n(J) = L$ and $f^n(K) = L$. Without loss of generality we may assume that $L = [0,1]$ and $J = [a,b]$ and $K = [c,d]$ with $0 \le a < b \le c < d \le 1$ such that $f^n([a,b]) = [0,1]$ and $f^n([c,d]) = [0,1]$. By [1, Chapter II, Lemma 2] we can choose $u, v, w \in [0,1]$ with $u < v < w$ such that one of the following statements holds:

(i) $f^n(u) = f^n(w) = u$, $f^n(v) = w$, $f^n(x) > u$ for $u < x < w$ and $x < f^n(x) < w$ for $u < x < v$.

(ii) $f^n(u) = f^n(w) = w$, $f^n(v) = u$, $f^n(x) < w$ for $u < x < w$ and $u < f^n(x) < x$ for $v < x < w$.

We may consider only case (i) since case (ii) is similar. We claim that, for any $x \in (v, w)$ and any $0 < \varepsilon < w - x$, there exist $y \in [w - \varepsilon, w)$ and $s \in \mathbb{N}$ such that $f^{sn}(y) = x$. In fact, we can choose $u < \cdots < x_i < x_{i-1} < \cdots < x_1 \le v < x_0 = x$ such that $\lim_{i \to \infty} x_i = u$ and $f^n(x_i) = x_{i-1}$ for any $i \in \mathbb{N}$. Thus there exists some $x_N \in f^n([w - \varepsilon, w))$. That is, we can choose $y \in [w - \varepsilon, w)$ satisfying $f^n(y) = x_N$, which implies $f^{(N+1)n}(y) = x$. The claim is proven.

By the above claim we can choose a sequence of positive integers $\{s_i\}_{i=1}^{\infty}$ and a sequence of points $v < y_0 < y_1 < y_2 < \cdots < w$ such that $f^{ns_i}(y_i) = y_{i-1}$ for any $i \in \mathbb{N}$ and $\lim_{i \to \infty} y_i = w$. Note that $f^n(w) = f^n(u) = u$; then $w \in \mathrm{SA}(f^n) - R(f^n) \subset \mathrm{SA}(f) - R(f)$. The proof is completed.

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References

[1] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, vol. 1513 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1992.

[2] M. W. Hero, "Special α-limit points for maps of the interval," *Proceedings of the American Mathematical Society*, vol. 116, no. 4, pp. 1015–1022, 1992.

[3] J. Llibre and M. Misiurewicz, "Horseshoes, entropy and periods for graph maps," *Topology*, vol. 32, no. 3, pp. 649–664, 1993.

[4] A. M. Blokh, "On transitive mappings of one-dimensional branched manifolds," in *Differential-Difference Equations and Problems of Mathematical Physics (Russian)*, pp. 3–9, Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, Russia, 1984.

[5] A. M. Blokh, "Dynamical systems on one-dimensional branched manifolds. I," *Theory of Functions, Functional Analysis and Applications*, no. 46, pp. 8–18, 1986 (Russian), translation in *Journal of Soviet Mathematics*, vol. 48, no. 5, pp. 500–508, 1990.

[6] A. M. Blokh, "Dynamical systems on one-dimensional branched manifolds. II," *Theory of Functions, Functional Analysis and Applications*, no. 47, pp. 67–77, 1987 (Russian), translation in *Journal of Soviet Mathematics*, vol. 48, no. 6, pp. 668–674, 1990.

[7] A. M. Blokh, "Dynamical systems on one-dimensional branched manifolds. III," *Theory of Functions, Functional Analysis and Applications*, no. 48, pp. 32–46, 1987 (Russian), translation in *Journal of Soviet Mathematics*, vol. 49, no. 2, pp. 875–883, 1990.

[8] J.-H. Mai and S. Shao, "The structure of graph maps without periodic points," *Topology and Its Applications*, vol. 154, no. 14, pp. 2714–2728, 2007.

[9] J. Mai and T. Sun, "The ω-limit set of a graph map," *Topology and Its Applications*, vol. 154, no. 11, pp. 2306–2311, 2007.

[10] J.-H. Mai and T.-X. Sun, “Non-wandering points and the depth for graph maps,” *Science in China. Series A*, vol. 50, no. 12, pp. 1818–1824, 2007.

[11] T. X. Sun, H. J. Xi, Z. H. Chen, and Y. P. Zhang, “The attracting centre and the topological entropy of a graph map,” *Advances in Mathematics*, vol. 33, no. 5, pp. 540–546, 2004 (Chinese).

[12] X. D. Ye, “The centre and the depth of the centre of a tree map,” *Bulletin of the Australian Mathematical Society*, vol. 48, no. 2, pp. 347–350, 1993.

[13] X. Ye, “Non-wandering points and the depth of a graph map,” *Journal of the Australian Mathematical Society. Series A*, vol. 69, no. 2, pp. 143–152, 2000.


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