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Research Article

Oscillation of Certain Second-Order Sub-Half-Linear Neutral Impulsive Differential Equations

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By introducing auxiliary functions, we investigate the oscillation of a class of second-order sub-half-linear neutral impulsive differential equations of the form $[r(t)\phi_\beta(z'(t))]' + p(t)\phi_\alpha(x(\sigma(t))) = 0$, $t \neq \theta_k$, $\Delta\phi_\beta(z'(t))|_{t=\theta_k} + q_k\phi_\alpha(x(\sigma(\theta_k))) = 0$, $\Delta x(t)|_{t=\theta_k} = 0$, where $\beta > \alpha > 0$, $z(t) = x(t) + \lambda(t)x(\tau(t))$. Several oscillation criteria for the above equation are established in both the case $0 \le \lambda(t) \le 1$ and the case $-1 < -\mu \le \lambda(t) \le 0$, which generalize and complement some existing results in the literature. Two examples are also given to illustrate the effect of impulses on the oscillatory behavior of solutions to the equation.

1. Introduction

Impulsive differential equations appear as a natural description of observed evolution phenomena of several real-world problems involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulates systems [1–5]. In recent years, impulsive differential equations have received a lot of attention.

We are here concerned with the following second-order sub-half-linear neutral impulsive differential equation:

$$\begin{gathered}
\left[r(t)\phi_\beta\left(z'(t)\right)\right]' + p(t)\phi_\alpha(x(\sigma(t))) = 0, \quad t \neq \theta_k, \\
\Delta\phi_\beta\left(z'(t)\right)|_{t=\theta_k} + q_k\phi_\alpha(x(\sigma(\theta_k))) = 0, \\
\Delta x(t)|_{t=\theta_k} = 0,
\end{gathered} \tag{1.1}$$

where $\beta > \alpha > 0$, $z(t) = x(t) + \lambda(t)x(\tau(t))$, $t \geq t_0$ and $\theta_k \geq t_0$ for some $t_0 \in R$, $\{\theta_k\}_{k=1}^{\infty}$ is a strictly increasing unbounded sequence of real numbers, $\phi_\gamma(u) = |u|^{\gamma-1}u$ for $\gamma > 0$, and

$$\Delta u(t)|_{t=\theta_k} = u(\theta_k^+) - u(\theta_k^-), \quad u(\theta_k^\pm) = \lim_{t \to \theta_k^\pm} u(t). \tag{1.2}$$

Let $PLC(J, R)$ denote the set of all real-valued functions $u(t)$ defined on $J \subset [t_0, \infty)$ such that $u(t)$ is continuous for all $t \in J$ except possibly at $t = \theta_k$ where $u(\theta_k^\pm)$ exists and $u(\theta_k) := u(\theta_k^-)$.

We assume throughout this paper that

(a) $r(t) \in C^1([t_0, \infty), R)$, $r(t) > 0$ and $\int_{t_0}^{\infty} [r(t)]^{-1/\beta} dt = \infty$;

(b) $\lambda(t) \in C^2([t_0, \infty), R)$, $0 \leq \lambda(t) \leq 1$ or $-1 < -\mu \leq \lambda(t) \leq 0$;

(c) $p(t) \in PLC([t_0, \infty), R)$, $p(t) \geq 0$;

(d) q_k is a sequence of nonnegative real numbers;

(e) $\tau(t), \sigma(t) \in C([t_0, \infty), R,\ 0 \leq \tau(t), \sigma(t) \leq t,\ \lim_{t\to\infty}\tau(t) = \infty$, and $\lim_{t\to\infty}\sigma(t) = \infty$.

By a solution of (1.1) we mean a function $x(t)$ defined on $[T_x, \infty)$ with $T_x \geq t_0$ such that $x, z', z'' \in PLC([t_0, \infty), R)$ and x satisfies (1.1). It is tacitly assumed that such solutions exist. Note the assumption $\Delta x(t)|_{t=\theta_k} = 0$; we have that each solution of (1.1) is continuous on $[t_0, \infty)$. As usual, a nontrivial solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if its every nontrivial solution is oscillatory.

Compared to equations without impulses, little has been known about the oscillation problem for impulsive differential equations due to difficulties caused by impulsive perturbations [6–17].

When $\beta = 1$, $r(t) \equiv 1$, and $\lambda(t) \equiv 0$, (1.1) reduces to the following sublinear impulsive delay equation:

$$\begin{gathered} x''(t) + p(t)\phi_\alpha(x(\sigma(t))) = 0, \quad t \neq \theta_k, \\ \Delta x'(t)|_{t=\theta_k} + q_k \phi_\alpha(x(\sigma(\theta_k))) = 0, \\ \Delta x(t)|_{t=\theta_k} = 0, \end{gathered} \tag{1.3}$$

which has received a lot of attention in the literature. However, for the general sub-half-linear neutral equation (1.1) under the impulse condition given in this paper, little has been known about the oscillation of (1.1) to the best of our knowledge, especially for the case when $-1 < -\mu \leq \lambda(t) \leq 0$.

The main objective of this paper is to establish oscillation criteria for the sub-half-linear impulsive differential equation (1.1) in both the case $0 \leq \lambda(t) \leq 1$ and the case $-1 < -\mu \leq \lambda(t) \leq 0$. By introducing an auxiliary function $g \in C^1[t_0, \infty)$ and a function $H(t, s)$ defined below, we establish some new oscillation criteria for (1.1) which complement the oscillation theory of impulsive differential equations. Examples are also given to show the effect of impulses on oscillation of solutions of (1.1).

2. Main Results

Theorem 2.1. *Let* $0 \leq \lambda(t) \leq 1$. *If there exists a positive function* $g \in C^1[t_0, \infty)$ *such that*

$$\frac{g'(t)}{g(t)} \leq -\frac{\alpha}{\beta}\frac{r'(t)}{r(t)}, \tag{2.1}$$

$$\int^{\infty}[1-\lambda(\sigma(t))]^{\alpha}\left[R_{\beta}(t)\right]^{\alpha}g(t)p(t)dt + \sum^{\infty}[1-\lambda(\sigma(\theta_k))]^{\alpha}g(\theta_k)r(\theta_k)\left[R_{\beta}(\theta_k)\right]^{\alpha}q_k = \infty, \tag{2.2}$$

where $R_{\beta}(t) = r^{1/\beta}(t)\int_{t_0}^{\sigma(t)}[r(s)]^{-1/\beta}ds$, *then* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. The case $x(t)$ being eventually negative can be similarly discussed. From (1.1), we have that

$$z(t) > 0, \quad \left[r(t)\phi_{\beta}(z'(t))\right]' \leq 0, \quad t \geq t_1,\ t \neq \theta_k. \tag{2.3}$$

Based on the impulsive condition $\Delta\phi_{\beta}(z'(t))|_{t=\theta_k} \leq 0$, we can deduce that $r(t)\phi_{\beta}(z'(t))$ is nonincreasing on $[t_1, \infty)$. We may claim that $z'(t) > 0$ holds eventually. Otherwise, there exists $t_* \geq t_1$ such that $z'(t_*) < 0$. Noting that $z(t)$ is continuous on $[t_*, \infty)$, we have that

$$\begin{aligned} z(t) &= z(t_*) + \int_{t_*}^{t} r^{-1/\beta}(s)r^{1/\beta}(s)z'(s)ds \\ &\leq z(t_*) + r^{1/\beta}(t_*)z'(t_*)\int_{t_*}^{t} r^{-1/\beta}(s)ds, \quad t \geq t_*, \end{aligned} \tag{2.4}$$

which implies that $z(t)$ is eventually negative since $\int^{\infty} r^{-1/\beta}(s)ds = \infty$. This is a contradiction. Without loss of generality, say $z'(t) > 0$ for $t \geq t_1$. Choose sufficiently large $t_2 \geq t_1$ such that $\tau(t) \geq t_1$ for $t \geq t_2$, and

$$\int_{t_1}^{t} r^{-1/\beta}(s)ds \geq \frac{1}{2}\int_{t_0}^{t} r^{-1/\beta}(s)ds, \quad t \geq t_2, \tag{2.5}$$

which is always possible because $\int^{\infty} r^{-1/\beta}(s)ds = \infty$. Thus, we have

$$\begin{aligned} z(t) &\geq r^{1/\beta}(t)z'(t)\int_{t_1}^{t} r^{-1/\beta}(s)ds \\ &\geq \frac{1}{2}r^{1/\beta}(t)z'(t)\int_{t_0}^{t} r^{-1/\beta}(s)ds, \quad t \geq t_2. \end{aligned} \tag{2.6}$$

By choosing t_3 sufficiently large such that $\sigma(t) \geq t_2$ for $t \geq t_3$ and using (2.6) and the nonincreasing character of $r^{1/\beta}(t)z'(t)$, we have

$$z(\sigma(t)) \geq \frac{1}{2}r^{1/\beta}(t)z'(t)\int_{t_0}^{\sigma(t)} r^{-1/\beta}(s)ds = \frac{R_\beta(t)}{2}z'(t), \quad t \geq t_3. \tag{2.7}$$

Since $z'(t) > 0$ for $t \geq t_1$ and $z(t)$ is continuous, we have

$$x(t) = z(t) - \lambda(t)x(\tau(t)) \geq z(t) - \lambda(t)z(\tau(t)) \geq [1-\lambda(t)]z(t),\ t \geq t_2. \tag{2.8}$$

By (1.1), (2.7), and (2.8), we get

$$\left[r(t)\phi_\beta\left(z'(t)\right)\right]' + 2^{-\alpha}p(t)[1-\lambda(\sigma(t))]^\alpha\left[R_\beta(t)\right]^\alpha\left[z'(t)\right]^\alpha \leq 0, \quad t \geq t_3,\ t \neq \theta_k, \tag{2.9}$$

which implies

$$g(t)\left[r(t)\phi_\beta\left(z'(t)\right)\right]' + 2^{-\alpha}g(t)p(t)[1-\lambda(\sigma(t))]^\alpha\left[R_\beta(t)\right]^\alpha\left[z'(t)\right]^\alpha \leq 0, \quad t \geq t_3,\ t \neq \theta_k. \tag{2.10}$$

From (2.1), we get

$$\begin{aligned}
\frac{g(t)\left[r(t)\phi_\beta(z'(t))\right]'}{[z'(t)]^\alpha} &= \frac{\beta g(t)r(t)[z'(t)]^{\beta-1}z'' + g(t)r'(t)[z'(t)]^\beta}{[z'(t)]^\alpha} \\
&\geq \frac{\beta g(t)r(t)[z'(t)]^{\beta-1}z'' + \left(\beta/(\beta-\alpha)\right)\left[g(t)r(t)\right]'[z'(t)]^\beta}{[z'(t)]^\alpha} \\
&= \frac{\beta}{\beta-\alpha}\left[g(t)r(t)(z'(t))^{\beta-\alpha}\right]'.
\end{aligned} \tag{2.11}$$

Multiplying (2.10) by $((\beta-\alpha)/\beta)[z'(t)]^{-\alpha}$, we get

$$\left[g(t)r(t)\left(z'(t)\right)^{\beta-\alpha}\right]' + 2^{-\alpha}g(t)p(t)[1-\lambda(\sigma(t))]^\alpha\left[R_\beta(t)\right]^\alpha \leq 0, \quad t \geq t_3,\ t \neq \theta_k. \tag{2.12}$$

Integrating (2.12) from t_3 to t, we have that

$$\begin{aligned}
&\sum_{t_3 \leq \theta_k < t} g(\theta_k)r(\theta_k)\left\{\left[z'(\theta_k)\right]^{\beta-\alpha} - \left[z'(\theta_k^+)\right]^{\beta-\alpha}\right\} + g(t)r(t)\left[z'(t)\right]^{\beta-\alpha} - g(t_3)r(t_3)\left[z'(t_3)\right]^{\beta-\alpha} \\
&\quad + \frac{(\beta-\alpha)2^{-\alpha}}{\beta}\int_{t_3}^{t}[1-\lambda(\sigma(s))]^\alpha\left[R_\beta(s)\right]^\alpha g(s)p(s)ds \leq 0, \quad t \geq t_3,
\end{aligned} \tag{2.13}$$

which implies that

$$\begin{aligned}
&\sum_{t_3\le\theta_k<t} g(\theta_k)r(\theta_k)\left\{\left[z'(\theta_k)\right]^{\beta-\alpha}-\left[z'(\theta_k^+)\right]^{\beta-\alpha}\right\}\\
&\qquad+\frac{(\beta-\alpha)2^{-\alpha}}{\beta}\int_{t_3}^{t}[1-\lambda(\sigma(s))]^{\alpha}\left[R_\beta(s)\right]^{\alpha}g(s)p(s)ds\le g(t_3)r(t_3)\left[z'(t_3)\right]^{\beta-\alpha}.
\end{aligned}\tag{2.14}$$

On the other hand, by the given impulsive condition, we get

$$\begin{aligned}
\left[z'(\theta_k)\right]^{\beta-\alpha}-\left[z'(\theta_k^+)\right]^{\beta-\alpha}&=\left[z'(\theta_k)\right]^{\beta-\alpha}-\left\{\left[z'(\theta_k)\right]^{\beta}-q_k[x(\sigma(\theta_k))]^{\alpha}\right\}^{(\beta-\alpha)/\beta}\\
&=\left[z'(\theta_k)\right]^{\beta-\alpha}\left[1-(1-u_k)^{(\beta-\alpha)/\beta}\right],
\end{aligned}\tag{2.15}$$

where

$$u_k=q_k\frac{[x(\sigma(\theta_k))]^{\alpha}}{\left[z'(\theta_k)\right]^{\beta}}.\tag{2.16}$$

Note that $0<(\beta-\alpha)/\beta<1$, $1-(1-u_k)^{(\beta-\alpha)/\beta}\ge((\beta-\alpha)/\beta)u_k$ for $1\ge u_k\ge 0$. Consequently, we see from (2.7), (2.8), and (2.15) that

$$\begin{aligned}
\left[z'(\theta_k)\right]^{\beta-\alpha}-\left[z'(\theta_k^+)\right]^{\beta-\alpha}&\ge\frac{\beta-\alpha}{\beta}\frac{q_k[x(\sigma(\theta_k))]^{\alpha}}{\left[z'(\theta_k)\right]^{\alpha}}\\
&\ge\frac{(\beta-\alpha)2^{-\alpha}}{\beta}q_k[1-\lambda(\sigma(\theta_k))]^{\alpha}\left[R_\beta(\theta_k)\right]^{\alpha}.
\end{aligned}\tag{2.17}$$

Substituting (2.17) into (2.14) yields

$$\int^{\infty}[1-\lambda(\sigma(t))]^{\alpha}\left[R_\beta(t)\right]^{\alpha}p(t)dt+\sum^{\infty}[1-\lambda(\sigma(\theta_k))]^{\alpha}g(\theta_k)r(\theta_k)\left[R_\beta(\theta_k)\right]^{\alpha}q_k<\infty,\tag{2.18}$$

which contradicts (2.2). This completes the proof. □

Theorem 2.2. *Let $-1<-\mu\le\lambda(t)\le 0$. If there exists a positive function $g\in C^1[t_0,\infty)$ such that* (2.1) *holds and*

$$\int^{\infty}\left[R_\beta(t)\right]^{\alpha}g(t)p(t)dt+\sum^{\infty}g(\theta_k)r(\theta_k)\left[R_\beta(\theta_k)\right]^{\alpha}q_k=\infty,\tag{2.19}$$

where $R_\beta(t)$ is defined as in Theorem 2.1, then every solution of (1.1) *is either oscillatory or tends to zero.*

Proof. Suppose to the contrary that there is a solution $x(t)$ of (1.1) which is neither oscillatory nor tends to zero. Without loss of generality, we may let $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Thus, $r(t)\phi_\beta(z'(t))$ is nonincreasing for $t \geq t_1$. As a result, $z'(t)$ and $z(t)$ are eventually of constant sign. Now, we consider the following two cases: (i) $z(t) > 0$ eventually; (ii) $z(t) < 0$ eventually. For the case (i), similar to the analysis as in the proof of Theorem 2.1, we have $z'(t) > 0$ eventually and (2.6) holds. Notice that $x(t) = z(t) - p(t)x(\tau(t)) \geq z(t)$ because $p(t) \leq 0$; from (1.1) and (2.6), we get

$$\left[r(t)\phi_\beta(z'(t))\right]' + 2^{-\alpha}\left[R_\beta(t)\right]^\alpha p(t)\left[z'(t)\right]^\alpha \leq 0. \tag{2.20}$$

Following the similar arguments as in the proof of Theorem 2.1, we can get a contradiction with (2.19).

For the case (ii), assume that $z(t) < 0$ for $t \geq t_2 \geq t_1$. It must now hold that $\tau(t) < t$ for $t \geq t_2$. Let us consider two cases: (a) $x(t)$ is unbounded; (b) $x(t)$ is bounded. If $x(t)$ is unbounded, then we have

$$x(t) = z(t) - p(t)x(\tau(t)) < -p(t)x(\tau(t)) < x(\tau(t)), \quad t \geq t_2. \tag{2.21}$$

On the other hand, there exists a sequence $\{T_n\}$ satisfying $\lim_{n\to\infty}T_n = \infty$, $\lim_{n\to\infty}x(t) = \infty$, and $\max_{T_1\leq t\leq T_n}x(t) = x(T_n)$. Let t_n be sufficiently large such that $T_n > t_2$ and $\tau(T_n) > T_1$. Then, we have $\max_{\tau(T_n)\leq t\leq T_n}x(t) = x(T_n)$ which contradicts (2.21). If $x(t)$ is bounded, then we can prove that $\lim_{t\to\infty}x(t) = 0$. In fact,

$$\begin{aligned} 0 \geq \limsup_{t\to\infty} z(t) &= \limsup_{t\to\infty}\left[x(t) + p(t)x(\tau(t))\right] \\ &\geq \limsup_{t\to\infty} x(t) + \limsup_{t\to\infty} p(t)x(\tau(t)) \\ &\geq \limsup_{t\to\infty} x(t) - \mu \limsup_{t\to\infty} x(\tau(t)) \\ &\geq (1-\mu)\limsup_{t\to\infty} x(t), \end{aligned} \tag{2.22}$$

which implies that $\lim_{t\to\infty}x(t) = 0$ since $1 - \mu > 0$. This is a contradiction. The proof of Theorem 2.2 is complete. □

When there is no impulse, (1.1) reduces to

$$\left[r(t)\phi_\beta(z'(t))\right]' + p(t)\phi_\alpha(x(\sigma(t))) = 0, \quad t \geq t_0. \tag{2.23}$$

The following oscillation results for (2.23) are immediate.

Corollary 2.3. *Let $0 \leq \lambda(t) \leq 1$. If there exists a positive function $g \in C^1[t_0,\infty)$ such that* (2.1) *holds and*

$$\int^\infty [1-\lambda(\sigma(t))]^\alpha \left[R_\beta(t)\right]^\alpha g(t)p(t)dt = \infty, \tag{2.24}$$

where $R_\beta(t)$ is the same as in Theorem 2.1, then (2.23) *is oscillatory.*

Corollary 2.4. *Let $-1 < -\mu \leq \lambda(t) \leq 0$. If there exists a positive function $g \in C^1[t_0,\infty)$ such that* (2.1) *holds and*

$$\int^{\infty} \left[R_\beta(t)\right]^{\alpha} g(t)p(t)dt = \infty, \tag{2.25}$$

where $R_\beta(t)$ is the same as in Theorem 2.1, then every solution of (2.23) *is either oscillatory or tends to zero.*

Next, we introduce the function defined in [18] to further study oscillation of (1.1). Say that $H(t,s)$ defined on $D = \{(t,s) : t \geq s \geq t_0\}$ belongs to the function class $\mathcal{X}$ if $\partial H/\partial s \in L_{\text{loc}}(D,R)$, $H(t,t) = 0$, $H(t,s) \geq 0$, and $(\partial H/\partial s)(t,s) \leq 0$ for $(t,s) \in D$.

Theorem 2.5. *Let $0 \leq \lambda(t) \leq 1$. If there exist a positive function $g \in C^1[t_0,\infty)$ and $H \in \mathcal{X}$ such that* (2.1) *holds and*

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)[1-\lambda(\sigma(s))]^{\alpha}\left[R_\beta(s)\right]^{\alpha} g(s)p(s)ds\\&\quad+\frac{1}{H(t,t_0)}\sum_{t_0\leq\theta_k<t} H(t,\theta_k)[1-\lambda(\sigma(\theta_k))]^{\alpha} g(\theta_k)r(\theta_k)\left[R_\beta(\theta_k)\right]^{\alpha} q_k = \infty,\end{aligned} \tag{2.26}$$

where $R_\beta(t)$ is defined as in Theorem 2.1, then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Similar to the proof of Theorem 2.1, we have that (2.12) holds. Multiplying $H(t,s)$ on both sides of (2.12) and integrating it from t_3 to t, we get

$$\begin{aligned}&\sum_{t_3\leq\theta_k<t} H(t,\theta_k)g(\theta_k)r(\theta_k)\left\{\left[z'(\theta_k)\right]^{\beta-\alpha}-\left[z'(\theta_k^+)\right]^{\beta-\alpha}\right\}\\&\quad-\int_{t_3}^{t}\frac{\partial H(t,s)}{\partial s} g(s)r(s)\left[z'(s)\right]^{\beta-\alpha}ds - H(t,t_3)g(t_3)r(t_3)\left[z'(t_3)\right]^{\beta-\alpha}\\&\quad+\frac{(\beta-\alpha)2^{-\alpha}}{\beta}\int_{t_3}^{t} H(t,s)[1-\lambda(\sigma(s))]^{\alpha}\left[R_\beta(s)\right]^{\alpha} g(s)p(s)ds \leq 0, \quad t \geq t_3,\end{aligned} \tag{2.27}$$

which implies that

$$\begin{aligned}&\sum_{t_3\leq\theta_k<t} H(t,\theta_k)g(\theta_k)r(\theta_k)\left\{\left[z'(\theta_k)\right]^{\beta-\alpha}-\left[z'(\theta_k^+)\right]^{\beta-\alpha}\right\}\\&\qquad+\frac{(\beta-\alpha)2^{-\alpha}}{\beta}\int_{t_3}^{t} H(t,s)[1-\lambda(\sigma(s))]^{\alpha}\left[R_\beta(s)\right]^{\alpha} g(s)p(s)ds\\&\quad\leq H(t,t_3)g(t_3)r(t_3)\left[z'(t_3)\right]^{\beta-\alpha}.\end{aligned} \tag{2.28}$$

Therefore,

$$\begin{aligned}
&\sum_{t_0\le\theta_k<t} H(t,\theta_k)g(\theta_k)r(\theta_k)\left\{\left[z'(\theta_k)\right]^{\beta-\alpha}-\left[z'(\theta_k^+)\right]^{\beta-\alpha}\right\}\\
&\qquad+\frac{(\beta-\alpha)2^{-\alpha}}{\beta}\int_{t_0}^{t} H(t,s)[1-\lambda(\sigma(s))]^{\alpha}\left[R_\beta(s)\right]^{\alpha}g(s)p(s)ds\\
&\quad\le H(t,t_0)g(t_3)r(t_3)\left[z'(t_3)\right]^{\beta-\alpha}\\
&\qquad+\sum_{t_0\le\theta_k<t_3} H(t,\theta_k)g(\theta_k)r(\theta_k)\left\{\left[z'(\theta_k)\right]^{\beta-\alpha}-\left[z'(\theta_k^+)\right]^{\beta-\alpha}\right\}\\
&\qquad+\frac{(\beta-\alpha)2^{-\alpha}}{\beta}\int_{t_0}^{t_3} H(t,s)[1-\lambda(\sigma(s))]^{\alpha}\left[R_\beta(s)\right]^{\alpha}g(s)p(s)ds.
\end{aligned}\tag{2.29}$$

Proceeding as in the proof of Theorem 2.1, we get a contradiction with (2.26). This completes the proof. □

For the case $-1\le\mu\le\lambda(t)\le 0$, we have the following oscillation result. Since the proof is similar to that of Theorem 2.2, we omit it here.

Theorem 2.6. *Let $-1\le\mu\le\lambda(t)\le 0$. If there exist a positive function $g\in C^1[t_0,\infty)$ and $H\in\mathcal{X}$ such that* (2.1) *holds and*

$$\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t} H(t,s)\left[R_\beta(s)\right]^{\alpha}g(s)p(s)ds+\sum_{t_0\le\theta_k<t} H(t,\theta_k)g(\theta_k)r(\theta_k)\left[R_\beta(\theta_k)\right]^{\alpha}q_k\right\}=\infty,\tag{2.30}$$

where $R_\beta(t)$ is defined as in Theorem 2.1, then every solution of (1.1) *is either oscillatory or tends to zero.*

The following two corollaries for (2.23) are immediate.

Corollary 2.7. *Let $0\le\lambda(t)\le 1$. If there exist a positive function $g\in C^1[t_0,\infty)$ and $H\in\mathcal{X}$ such that* (2.1) *holds and*

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)[1-\lambda(\sigma(s))]^{\alpha}\left[R_\beta(s)\right]^{\alpha}g(s)p(s)ds=\infty,\tag{2.31}$$

where $R_\beta(t)$ is defined as in Theorem 2.1, then (2.23) *is oscillatory.*

Corollary 2.8. *Let $-1\le\mu\le\lambda(t)\le 0$. If there exist a positive function $g\in C^1[t_0,\infty)$ and $H\in\mathcal{X}$ such that* (2.1) *holds and*

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\left[R_\beta(s)\right]^{\alpha}g(s)p(s)ds=\infty,\tag{2.32}$$

where $R_\beta(t)$ is defined as in Theorem 2.1, then every solution of (2.23) *is either oscillatory or tends to zero.*

3. Examples

We now present two examples to illustrate the effect of impulses on oscillation of solutions of (1.1).

Example 3.1. Consider the following impulsive delay differential equation:

$$\begin{gathered}\left[\frac{(|z'(t)|z'(t))}{t}\right]' + t^{-2}|x(t-1)|^{-1/2}x(t-1) = 0, \qquad t \neq k, \\ \Delta\left(\left|z'(t)\right|z'(t)\right)|_{t=k} + k^{-1/2}|x(k-1)|^{-1/2}x(k-1) = 0, \\ \Delta x(t)|_{t=k} = 0,\end{gathered} \tag{3.1}$$

where $z(t) = x(t) + \lambda x(t-1)$, λ is a constant, $t \geq 2$, and $k \geq 2$. We see that $\tau(t) = \sigma(t) = t-1$, $r(t) = 1/t$, $\beta = 2$, $\alpha = 1/2$, $p(t) = t^{-2}$, $q_k = k^{-1/2}$, and $\theta_k = k$. Let $g(t) = 1$. A straightforward computation yields $R_\beta(t) = (2/3)t^{-1/2}[(t-1)^{3/2} - 2^{3/2}]$. Therefore, when $0 < \lambda < 1$, it is not difficult to verify that (2.1) and (2.2) hold. Thus, (3.1) is oscillatory by Theorem 2.1. However, when there is no impulse in (3.1), Corollary 2.3 cannot guarantee the oscillation of (3.1) since condition (2.24) is invalid for this case. Therefore, the impulsive perturbations may greatly affect the oscillation of (3.1). If $-1 < \lambda < 0$, then we have that every solution of (3.1) is either oscillatory or tends to zero by Theorem 2.2. Such behavior of solutions of (3.1) is determined by the impulsive perturbations to a great extent, since Corollary 2.4 fails to apply for this case.

Example 3.2. Consider the following impulsive delay differential equation:

$$\begin{gathered}\left[t\left|z'(t)\right|z'(t)\right]' + t^{-2}x(t-1) = 0, \quad t \neq k, \\ \Delta\left(\left|z'(t)\right|z'(t)\right)|_{t=k} + k^{-2}x(k-1) = 0, \\ \Delta x(t)|_{t=k} = 0,\end{gathered} \tag{3.2}$$

where $z(t) = x(t) + \lambda x(t-1)$, λ is a constant, $t \geq 2$, and $k \geq 2$. We see that $\tau(t) = \sigma(t) = t-1$, $r(t) = t$, $\beta = 2$, $\alpha = 1$, $p(t) = t^{-2}$, $q_k = k^{-2}$, and $\theta_k = k$. Let $g(t) = t^{-1/2}$. It is not difficult to verify that (2.1) and (2.2) hold if $0 < \lambda < 1$, which implies that (3.2) is oscillatory by Theorem 2.1. We also can verify that and (2.1) and (2.19) hold if $-1 < \lambda < 0$. Thus, by Theorem 2.2, every solution of (3.2) is either oscillatory or tends to zero. However, Corollaries (1.1) and (2.1) do not apply for this case. Therefore, the impulsive perturbations play a key role in the oscillation problem of (3.2).

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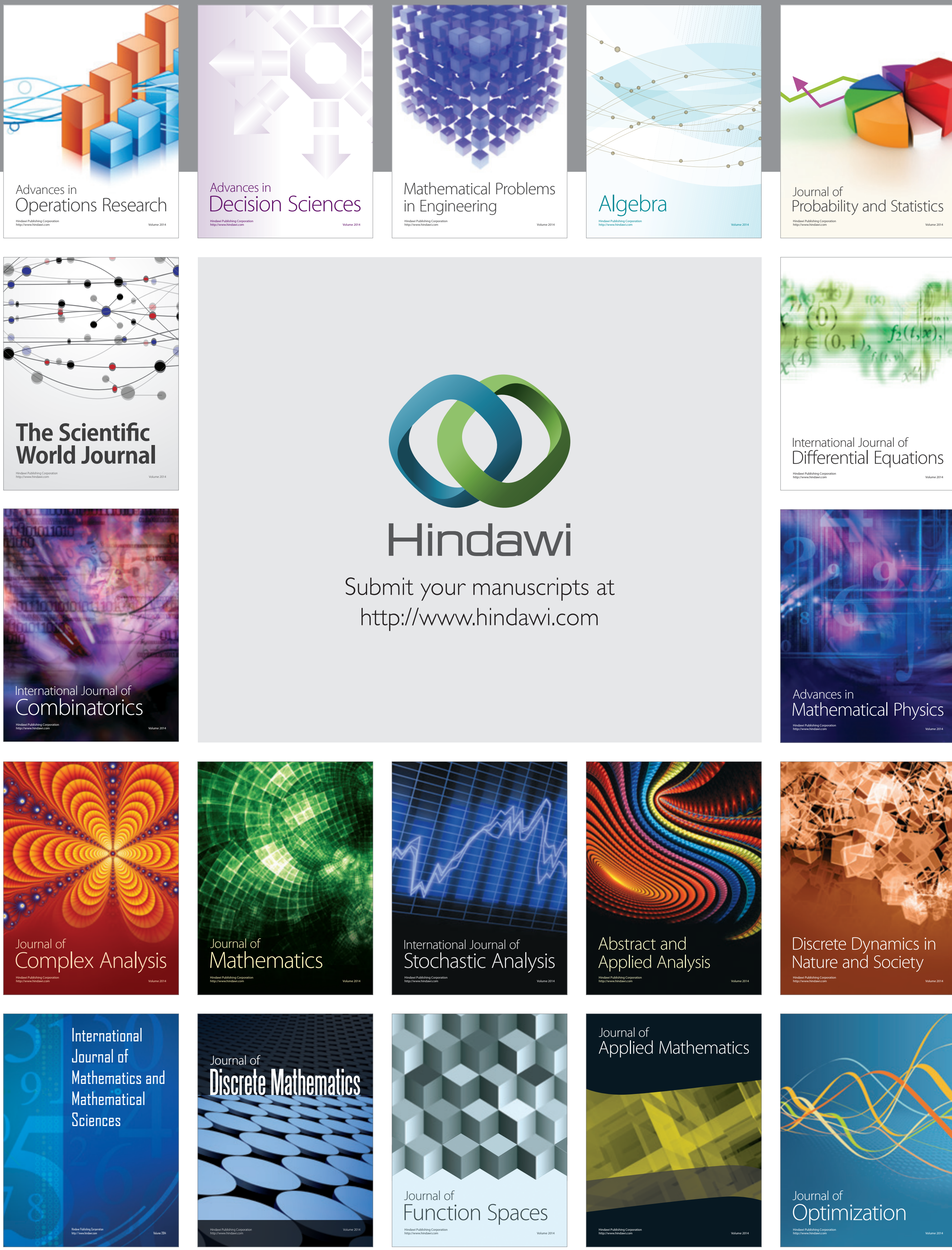
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