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Research Article

Existence and Multiplicity of Solutions for Discrete Nonlinear Two-Point Boundary Value Problems

Jianmin Guo and Caixia Guo

School of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037008, China

Correspondence should be addressed to Jianmin Guo, dtdxguojianmin@163.com

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By using Morse theory, the critical point theory, and the character of $K^{1/2}$, we consider the existence and multiplicity results of solutions to the following discrete nonlinear two-point boundary value problem $-\Delta^2 x(k-1) = f(k, x(k)), k \in \mathbb{Z}(1,T)$ subject to $x(0) = 0 = \Delta x(T)$, where T is a positive integer, $\mathbb{Z}(1,T) = \{1, 2, \ldots, T\}, \Delta$ is the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$, and $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ is continuous. In argument, Morse inequalities play an important role.

1. Introduction

In this paper, we discuss the existence and multiplicity results of solutions to the following discrete nonlinear two-point boundary value problem (BVP):

$$\begin{gathered} -\Delta^2 x(k-1) = f(k, x(k)), \quad k \in \mathbb{Z}(1,T), \\ x(0) = 0 = \Delta x(T), \end{gathered} \tag{1.1}$$

where T is a positive integer, $\mathbb{Z}(1,T) = \{1, 2, \ldots, T\}, \Delta$ is the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$, and $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ is continuous.

Recent years, there have been many papers studying the existence and multiplicity of solutions for differential equations. For example, by employing the strongly monotone operator principle and the critical point theory, F. Li et al. in [1] establish some conditions on f which are able to guarantee a class of boundary value problem on differential equation has a unique solution, at least one nonzero solution and infinitely many solutions; by using the critical point theory, and Morse theory, Yang and Zhang in [2] obtained some existence results for differential equations with parameters; there are also many authors who studied the existence results of positive solutions to boundary value problem on differential equation by employing the cone expansion or compression fixed point theorem or the critical point theory, see [3–5]; Jiang and Zhou in [6] obtained variational framework of the BVP(1.1) first by virtue of Green's function and separation of linear operator and studied the existence of a unique solution or at least one nontrivial solution by employing the strongly monotone operator principle and the critical point theory respectively.

In this paper, the main difference from the ordinary literatures is that we apply Morse theory and critical point theory to deal with problems on discrete systems. Then we establish some conditions on f which include sublinear, superlinear or asymptotical case to guarantee that BVP(1.1) has at least one solution, at least two nontrivial solutions, and infinitely many solutions.

2. Preliminary

In this section, we give some notations and lemmas.

Let $B = \{x : \mathbb{Z}(1, T+1) \to \mathbb{R},\ x(0) = 0 = \Delta x(T)\}$ is a T-dimensional Hilbert space with inner product $(x, y) = \sum_{k=1}^{T} x(k)y(k)$, and we denote the induced norm by $\|x\| = (\sum_{k=1}^{T} |x(k)|^2)^{1/2}$. $B_\rho = \{x \in B : \|x\| \leqslant \rho\}, S_\rho = \partial B_\rho$. And $J_a = \{x \in H : J(x) \leqslant a\}$ for any $J \in C(B, \mathbb{R}^1)$ and $a \in \mathbb{R}^1$. Let E be a real Banach space, and let $C^1(E, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on E.

Let G [6] be the Green's function of the linear boundary value problem

$$\begin{gathered} -\Delta^2 x(k-1) = 0, \quad k \in \mathbb{Z}(1, T), \\ x(0) = 0 = \Delta x(T), \end{gathered} \tag{2.1}$$

where

$$G(k, t) = \min\{k, t\} = \begin{cases} k, & k \leqslant t, \\ t, & t \leqslant k \end{cases} \tag{2.2}$$

then the (BVP)(1.1) has a solution if and only if the following equation:

$$x(k) = \sum_{t=1}^{T} G(k, t) f(t, x(t)), \quad k \in \mathbb{Z}(1, T) \tag{2.3}$$

has a solution in B. Define operators $K, \mathbf{f} : B \to B$ by

$$Kx(k) = \sum_{t=1}^{T} G(k, t) x(t), \quad \mathbf{f}x(k) = f(k, x(k)), \quad k \in \mathbb{Z}(1, T),\ \forall x \in B. \tag{2.4}$$

It is easy to see that a solution of (2.3) is equivalent to a solution in B of the following operator equation:

$$x = K\mathbf{f}x. \tag{2.5}$$

It is well know, that all eigenvalues of K are

$$\lambda_t = \frac{1}{4\sin^2((2t-1)\pi/(4T+2))}, \quad t = 1,2,\ldots,T, \tag{2.6}$$

which have the corresponding orthonormal eigenfunctions

$$\{e_t\}_{t=1}^{T} = \sin\frac{(2t-1)\pi}{2T+1}\left(\sum_{t=1}^{T}\sin^2\frac{(2n-1)\pi}{2T+1}\right)^{-1/2}, \quad t \in \mathbb{Z}(1,T). \tag{2.7}$$

Remark 2.1 (see [6]).

(i) $K : B \to B$ is a linear continuous operator; furthermore, K is symmetric in B.

(ii) There are $x = \sum_{k=1}^{T}(x,e_k)e_k$, $\|x\|^2 = \sum_{k=1}^{T}|x,e_k|^2$, $Kx = \sum_{k=1}^{T}\lambda_k(x,e_k)e_k$, $x \in B$.

(iii) The square root operator of K: $K^{1/2} : B \to B$ is bounded linear and symmetric,

$$K^{1/2}x = \sum_{k=1}^{T}\lambda_k^{1/2}(x,e_k)e_k, \quad x \in B. \tag{2.8}$$

(iv) $(K^{1/2}x,x) = \sum_{k=1}^{T}\lambda_k^{1/2}|(x,e_k)|^2$, $x \in B$, this implies that $K^{1/2}x \neq 0$ for all $x \in H$ with $x \neq 0$. Therefore, $K^{1/2}x_1 \neq K^{1/2}x_2$ for all $x_1, x_2 \in B$ with $x_1 \neq x_2$.

In next section, we will use the critical point theory and Morse theory to discuss the main results. Here, we state some necessary definitions and lemmas.

Definition 2.2. Let E be a real Banach space; D is an open subset of E. Suppose the functional $J : D \to \mathbb{R}$ is $C^1(E,\mathbb{R})$ on D. If $x_0 \in D$ and the Fréchet derivative $f'(x_0) = 0$, then we call that x_0 is a critical point of the functional f, and $c = f(x_0)$ is a critical value of f.

Definition 2.3. Let $f \in C^1(E,\mathbb{R})$. If any sequence $\{x_m\} \subset E$ for which $\{f(x_m)\}$ is bounded and $f'(x_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence, then we say that f satisfies Palais-Smale condition (denoted by P.S. condition for short).

Lemma 2.4 (see [7, Definition 1.16, page 13]). *Let E be a real reflexive Banach space. Suppose that the functional $J : E \to \mathbb{R}^1$ is C^1, and is bounded below and satisfies P.S. condition, then J must have a minimum in E, that is, there exists a $x^* \in E$ such that $J(x^*) = \inf_{x\in E} J(x)$, therefore x^* is a critical point of the functional J.*

Definition 2.5 (see [2]). Let $J(0) = 0$, $E = V \oplus X$ with $\dim V < \infty$ be a real Banach space. Assume that there exists $\rho > 0$ small, such that

$$\begin{aligned} J(x) &\leqslant 0, \quad x \in V, \ \|x\| \leqslant \rho, \\ J(x) &> 0, \quad x \in X, \ 0 < \|x\| \leqslant \rho \end{aligned} \tag{2.9}$$

Then J has a local linking at 0.

Definition 2.6. Let E be a real Banach space, let u be an isolated critical point of J with $J(u) = c$ and let U be a neighborhood of u, containing the unique critical point. We call $C_q(J, u) = H_q(J_c \cap U, J_c \cap U \setminus \{u\})$ the q-th critical group of J at $u, q = 0, 1, 2, \ldots$, where $H_q(\cdot, \cdot)$ stands for the q-th singular relative homology group with integer coefficients. We say that u is a homological nontrivial critical point of J if at least one of its critical groups is nontrivial.

Lemma 2.7 (see [8]). *Assume that $J \in C^1(E, \mathbb{R})$ satisfies P.S. condition and has a local linking at 0. Then $C_k(J, \theta) \ncong 0$, that is, 0 is a homological nontrivial critical point of J.*

Lemma 2.8 (see [8]). *Let M be a Finsler manifold of C^2. Assume that $f \in C^1(M, \mathbb{R})$ satisfies the P.S. condition and a is the only critical value of f in $[a, b)$. If connected component of K_a is only composed of isolated critical points, then f_a is a deformation retract of $f_b \setminus K_b$.*

Lemma 2.9 (see [2]). (i) *The operator equation*

$$x = K\mathbf{f}x \tag{2.10}$$

has a solution in B if and only if the operator equation

$$x = K^{1/2}\mathbf{f}K^{1/2}x \tag{2.11}$$

has a solution in B.

(ii) *The uniqueness of solution for these two above equations is also equivalent.*

(iii) *If* (2.11) *has a nonzero solution in B, then* (2.10) *has a nonzero solution in B. If* (2.11) *has infinitely many solutions in B, then* (2.10) *has also infinitely many solutions in B.*

Lemma 2.10 (see [9]). *Suppose that the functional*

$$J(x) = \frac{1}{2}\|x\|^2 - \sum_{k=1}^{T} F\left(k, K^{1/2}x(k)\right), \quad x \in B \tag{2.12}$$

has a critical point $x \in B$, where $F(k, x) = \int_0^x f(k, s)ds$, then (BVP)(1.1) *has a solution in B.*

Lemma 2.11 (see [8]). *Suppose $J \in C^1(B, \mathbb{R})$ satisfies P.S. and (A) conditions, then, one has*

$$M_q - M_{q-1} + \cdots + (-1)^q M^0 \geqslant \beta_q - \beta_{q-1} + \cdots + (-1)^q \beta_0, \quad q = 0, 1, 2, \ldots, \tag{2.13}$$

where $\beta_q = \beta_q(a,b) = \operatorname{rank} H_q(f_b, f_a), q = 0,1,2,\ldots$. *Furthermore, if the left of the equation is convergent, then one has that*

$$\sum_{q=0}^{\infty}(-1)^q M_q = \sum_{q=0}^{\infty}(-1)^q \beta_q. \tag{2.14}$$

(A) Suppose that there are two regular values, then J has at most finite critical points and rank of critical groups of every critical point is finite.

Lemma 2.12 (see [8, page 100, Theorem 3.2]). *Let $J \in C^2(B,\mathbb{R})$ be a functional and satisfy P.S. condition. Suppose that $J' = I - A$, where A is a compact map and p_0 is a isolated critical point of J, then we have*

$$\operatorname{ind}(J', p_0) = \sum_{q=0}^{\infty}(-1)^q \operatorname{rank} C_q(J, p_0). \tag{2.15}$$

3. Mail Results

Lemma 3.1. *Suppose f satisfies one of the following conditions:*

(a) $(H_1)\limsup_{|x|\to\infty} F(k,x)/x^2 < a$ *for* $k \in \mathbb{Z}(1,T)$, *where* $a \in [0, 2\sin^2\pi/(4T+2))$,

(b) $(H_2)\lim_{|x|\to\infty} F(k,x)/x^2 = a$ *for* $k \in \mathbb{Z}(1,T)$, *where* $a \in [0, 2\sin^2(\pi/(4T+2))$; *and* $(H_3)\lim_{|x|\to\infty}(F(k,x) - ax^2) = -\infty$ *for* $k \in \mathbb{Z}(1,T)$, *where* $a \in [0, 2\sin^2(\pi/(4T+2))$,

(c) (H_2) *and* $(H_4)\lim_{|x|\to\infty}(f(k,x)x - 2F(k,x)) = +\infty$, *uniformly for* $k \in \mathbb{Z}(1,T)$, *where* $a \in [0, 2\sin^2(\pi/(4T+2))$, *then one has*

(i) *J is coercive on B, that is $J(x) \to +\infty$ as $\|x\| \to \infty$, (ii) J satisfies P.S. condition.*

Proof. (i) Let (a) hold. It follows from (H_1) that there is a constant $c > 0$ such that $F(k,x) \leqslant ax^2 + c, x \in \mathbb{R}, k \in \mathbb{Z}(1,T)$. Therefore,

$$\begin{aligned}
J(x) &= \frac{1}{2}\|x\|^2 - \sum_{k=1}^{T} F\left(k, K^{1/2}x(k)\right) \\
&\geqslant \frac{1}{2}\|x\|^2 - \left(a\left|K^{1/2}x\right|^2 + c\right) \\
&= \frac{1}{2}\|x\|^2 - a(Kx,x) - c \\
&\geqslant \frac{1}{2}\|x\|^2 - \frac{a}{4\sin^2((\pi/4T)+2)}\|x\|^2 - c \\
&= \frac{1}{2}\left(1 - \frac{a}{2\sin^2((\pi/4T)+2)}\right)\|x\|^2 - c \longrightarrow +\infty, \quad \|x\| \longrightarrow \infty.
\end{aligned} \tag{3.1}$$

□

Let (b) or (c) hold. By the condition (H_2), write $f(k,x) = 2ax + g(k,x), F(k,x) = ax^2 + G(k,x)$. If (H_3) holds, then $\lim_{|x|\to\infty} G(k,x) = -\infty$; if (H_4) holds, then $\lim_{|x|\to\infty} G(k,x)/x^2 = 0, \lim_{|x|\to\infty}(g(k,x)x - 2G(k,x)) = +\infty$. It follows that for every $M > 0$, there is $R_M > 0$ such that

$$g(k,x)x - 2G(k,x) \geqslant M, \quad |x| \geqslant R_M. \tag{3.2}$$

Integrating the equality

$$\frac{d}{dx}\frac{G(k,x)}{x^2} = \frac{g(k,x)x - 2G(k,x)}{x^3} \geqslant \frac{M}{x^3} \tag{3.3}$$

over the interval $[x,y] \subset [R_M, +\infty)$, we have

$$\frac{G(k,y)}{y^2} - \frac{G(k,x)}{x^2} \geqslant -\frac{M}{2}\left(\frac{1}{y^2} - \frac{1}{x^2}\right). \tag{3.4}$$

Letting $y \to +\infty$, we see that $G(k,x) \leqslant -M/2, k \in \mathbb{Z}(1,T), x \geqslant R_M$. In a similar way, we have $G(k,x) \leqslant -M/2, k \in \mathbb{Z}(1,T), x \leqslant -R_M$. Hence, $\lim_{|x|\to\infty} G(k,x) = -\infty$.

Let $\{x_n\} \subset B$ be such that $\|x_n\| \to \infty$ as $n \to \infty$ and $J(x_n) \leqslant C$ for some constant $C \in \mathbb{R}$. Taking $y_n = x_n/\|x_n\|$, then $y_n, y_n \rightharpoonup y_0, y_0 \in B$ and $\|y_0\| \le 1$. So, $K^{1/2}y_n \to K^{1/2}y_0$ in B.

$$\begin{aligned}
&\frac{C^*}{\|y_n\|^2} \\
&\geq \frac{J(y_n)}{\|y_n\|^2} \\
&= \frac{1}{2} - \frac{1}{\|x_n\|^2}\sum_{k=1}^{T} F\left(k, K^{1/2}x_n(k)\right) \\
&= \frac{1}{2} - \frac{1}{\|x_n\|^2}\sum_{k=1}^{T}\left(a\left(K^{1/2}x_n(k)\right)^2 + G\left(k, K^{1/2}x_n(k)\right)\right) \\
&= \frac{1}{2} - a\left(K^{1/2}\frac{x_n}{\|x_n\|}\right)^2 - \frac{1}{\|x_n\|^2}\sum_{k=1}^{T} G\left(k, K^{1/2}x_n(k)\right) \\
&= \frac{1}{2} - 2\sin^2\frac{\pi}{4T+2}\left(K^{1/2}y_n\right)^2 - \frac{1}{\|x_n\|^2}\sum_{k=1}^{T} G\left(k, K^{1/2}x_n(k)\right) \\
&\geq \frac{1}{2} - 2\sin^2\frac{\pi}{4T+2}\left(K^{1/2}y_n\right)^2 - \frac{C}{\|x_n\|^2}.
\end{aligned} \tag{3.5}$$

Then,

$$(Ky_0, y_0) = \sum_{k=1}^{T}\left(K^{1/2}y_0(k)\right)^2 = \lim_{n\longrightarrow\infty}\sum_{k=1}^{T}\left(K^{1/2}y_n(k)\ \right)^2 = \lim_{n\longrightarrow\infty}\left(K^{1/2}y_n\right)^2 \geq \frac{1}{4\sin^2((\pi/4T)+2)}. \tag{3.6}$$

Hence $|K^{1/2}y_n(k)| \neq 0$ and $|K^{1/2}x_n(k)| = |K^{1/2}(\|x_n\|y_n(k))| = \|x_n\||K^{1/2}y_n(k)| \to \infty$. And $G(k, K^{1/2}x_n(k)) \to -\infty$, as $n \to \infty$ for $k \in \mathbb{Z}(1,T)$. Therefore,

$$\begin{aligned} C^* \geq J(x_n) &= \frac{1}{2}\|x_n\|^2 - a\left(K^{1/2}x_n\right)^2 - \sum_{k=1}^{T}G\left(k, K^{1/2}x_n(k)\right) \\ &\geq \frac{1}{2}\left(\|x_n\|^2 - a(Kx_n, x_n)\right) - \sum_{k=1}^{T}G\left(k, K^{1/2}x_n(k)\right) \\ &\geq -\sum_{k=1}^{T}G\left(k, K^{1/2}x_n(k)\right) \longrightarrow +\infty, \quad n \longrightarrow \infty. \end{aligned} \tag{3.7}$$

This is impossible, so J is coercive on B.

(ii) Let $\{x_n\} \in B, \{J(x_n)\}$ is bounded and $J'(x_n) \to \theta, n \to \infty$. By (i), $\{x_n\}$ is bounded on B. Clearly, $\{x_n\}$ possesses a convergent subsequence. Then, the P.S. condition is satisfied.

Lemma 3.2. *Suppose that*

(H_5) there exist $\delta, a, b \in (0, +\infty)$ and an integer $m \in \mathbb{Z}(1,T)$, which satisfy

$$2\sin^2\frac{(2m-1)\pi}{4T+1} \leqslant b \leqslant a \leqslant 2\sin^2\frac{(2m+1)\pi}{4T+1}, \tag{3.8}$$

such that $by^2 \leq F(k,y) \leq ay^2, |y| \leq \delta, k \in \mathbb{Z}(1,T)$.

Then the functional J has a local linking with respect to $B = V_1 \oplus V_2$, where $V_1 = \mathrm{span}\{e_1, e_2, \ldots, e_m\}, V_2 = V_1^{\perp}$.

Proof. Let $V_1 = \mathrm{span}\{e_1, e_2, \ldots, e_m\}, V_2 = V_1^{\perp}$. If $x \in V_1$, then $(Kx, x) \geq \lambda_m\|x\|^2$; if $x \in V_2$, then $(Kx, x) \leq \lambda_{m+1}\|x\|^2$.

By Remark 2.1, we can assume that for the given $\delta > 0$, there exists a $\rho = \delta/L > 0$ such that $x \in V_1, \|x\| \leq \rho \Rightarrow |K^{1/2}x| \leq L\|x\| \leq \delta$, and thus, by (H_5), we have

$$\begin{aligned}
J(x) &= \frac{1}{2}\|x\|^2 - \sum_{k=1}^{T} F\left(K^{1/2}x(k)\right) \\
&\leq \frac{1}{2}\|x\|^2 - \sum_{k=1}^{T} b\left(K^{1/2}x(k)\right)^2 \\
&= \frac{1}{2}\|x\|^2 - b\left(K^{1/2}x, K^{1/2}x\right) \\
&= \frac{1}{2}\|x\|^2 - b(Kx, x) \\
&\leq \left(\frac{1}{2} - \frac{b}{4\sin^2((2m-1)(\pi/4T)+1)}\right)\|x\|^2 \leq 0, \quad x \in V_1,\ \|x\| \leq \rho.
\end{aligned} \tag{3.9}$$

For $x \in V_2$, consider the above ρ. By Remark 2.1, we still have $\|x\| \leq \rho \Rightarrow |K^{1/2}x| \leq L\|x\| \leq \delta$; thus by (H_5), we have

$$\begin{aligned}
J(x) &= \frac{1}{2}\|x\|^2 - \sum_{k=1}^{T} F\left(K^{1/2}x(k)\right) \\
&\geq \frac{1}{2}\|x\|^2 - \sum_{k=1}^{T} A(K^{1/2}x)^2 \\
&= \frac{1}{2}\|x\|^2 - a\left(K^{1/2}x, K^{1/2}x\right) \\
&= \frac{1}{2}\|x\|^2 - a(Kx, x) \\
&\geq \left(\frac{1}{2} - \frac{a}{4\sin^2((2m+1)(\pi/4T)+1)}\right)\|x\|^2 \geq 0, \quad x \in V_2,\ \|x\| \leq \rho.
\end{aligned} \tag{3.10}$$

This implies that J has a local linking at 0 with respect to $B = V_1 \oplus V_2$. □

Theorem 3.3. *If condition (H_1) holds, then the BVP(1.1) has at least a solution.*

Proof. We will verify that the functional $J(x)$ defined in Lemma 2.10 has a critical point $x \in B$. By [6], we know that $J : B \to \mathbb{R}$ is C^1 functional. And by Lemma 3.1, it is easy to know that J is bounded below and satisfies P.S. condition. It follows from Lemma 2.4 that J has a critical point in $x \in B$. □

Theorem 3.4. *If conditions (H_2) and (H_3) hold, then the BVP(1.1) has at least a solution.*

Theorem 3.5. *If conditions (H_2) and (H_4) hold, then the BVP(1.1) has at least a solution. The proof of Theorem 3.4 and Theorem 3.5 is similar to the proof of Theorem 3.3.*

Theorem 3.6. *If conditions* (H_1) *and* (H_5) *hold, then the BVP*(1.1) *has at least two nontrivial solutions.*

Proof. By Lemma 3.1, we know that J satisfies P.S. condition, and it follows from Lemmas 2.7 and 3.2 that $x = \theta$ is a homological nontrivial critical point of J, then $C_k(J,0) \not\cong 0$. If $\inf_{x\in B} J(x) \geq 0$, then $J(x) = \inf_{x\in V_1} J(x) = 0, x \in V_1, \|x\| < \rho$, which implies that all $x \in V_1$ with $\|x\| < \rho$ are solutions of the BVP(1.1). If $\inf_{x\in B} J(x) < 0$, that is, 0 is not a minimizer of J, by Lemma 3.1, it is easy to know that J is bounded below and satisfies P.S. condition. Then by Lemma 2.4, J has a minimizer $x_0 \to B$, that is, a critical point and $x_0 \neq 0$; Without loss of generality, we may suppose that the minimizer x_0 is unique. Let $c = J(x_0)$, then Morse index of x_0 is

$$\begin{aligned} C_q(J, x_0) &= H_q(J_c \cap U_{x_0}, (J_c \setminus \{x_0\}) \cap U_{x_0}) \\ &= \begin{cases} G, & q = 0, \\ 0, & q \neq 0. \end{cases} \end{aligned} \tag{3.11}$$

We suppose, J only have two critical points with θ and x_0, where $c_1 = c$, $c_2 = J(\theta)$, $c_1 < c_2$. Let $b > \max\{c_1, c_2\}, \varepsilon > 0$ is small enough such that $c_1 + \varepsilon < c_2 - \varepsilon, c_2 + \varepsilon < b$.

If $\chi(X, Y) = \sum_{q=0}^{\infty} (-1)^q H_q(X, Y)$ is finite, then $\chi(X, Y)$ satisfies additivity. So,

$$\chi(J_b, J_{c_1-\varepsilon}) = \chi(J_b, J_{c_2-\varepsilon}) + \chi(J_{c_2-\varepsilon}, J_{c_1-\varepsilon}). \tag{3.12}$$

It follows from the second deformation lemma, exactness, and excision that $H_q(J_b, J_{c_2-\varepsilon}) \cong H_q(J_{c_2+\varepsilon}, J_{c_2-\varepsilon}) \cong H_q(J_{c_2+\varepsilon}, J_{c_2} \setminus \{\theta\}) \cong H_q(J_{c_2}, J_{c_2} \setminus \{\theta\}) \cong H_q(J_{c_2} \cap U_\theta, (J_{c_2} \setminus \{\theta\}) \cap U_\theta)$. Thus,

$$\begin{aligned} H_q(J_b, J_{c_2-\varepsilon}) &= H_q(J_{c_2} \cap U_\theta, (J_{c_2} \setminus \{\theta\}) \cap U_\theta) \\ &= C_q(J, \theta) \\ &= \begin{cases} G, & q = m, \\ 0, & q \neq m. \end{cases} \end{aligned} \tag{3.13}$$

Hence, $\chi(J_b, J_{c_2-\varepsilon}) = (-1)^m$.

By the same way,

$$\begin{aligned} H_q(J_{c_2+\varepsilon}, J_{c_1-\varepsilon}) &\cong H_q(J_{c_1+\varepsilon}, J_{c_1-\varepsilon}) \\ &\cong H_q(J_{c_1}, J_{c_1-\varepsilon}) \\ &\cong H_q(J_{c_1}, J_{c_1} \setminus \{x_0\}) \\ &\cong C_q(J, x_0) \\ &= \begin{cases} G, & q = 0, \\ 0, & q \neq 0. \end{cases} \end{aligned} \tag{3.14}$$

So, $\chi(J_{c_2+\varepsilon}, J_{c_1-\varepsilon}) = 1$. But

$$\chi(J_b, J_{c_1-\varepsilon}) = \sum_{q=0}^{\infty}(-1)^q H_q(J_b, J_{c_1-\varepsilon}) = \sum_{q=0}^{\infty}(-1)^q H_q(J_b) = 1. \tag{3.15}$$

Then, $(-1)^m + 1 \neq 1$ is a contradict with (3.12).

Thus, J has at least three critical points, that is, the BVP (1.1) has at least two nontrivial solutions. □

Theorem 3.7. *If condition* (H_2), (H_3), *and* (H_5) *hold, then the BVP*(1.1) *has at least two nontrivial solutions.*

Theorem 3.8. *If condition* (H_2), (H_4), *and* (H_5) *hold, then the BVP*(1.1) *has at least two nontrivial solutions.*

The proof of Theorems 3.5, and 3.6 is similar to the proof of Theorem 3.6.

Theorem 3.9. *Suppose that*

(H_6) there exists $m \in \mathbb{Z}(1,T)$, such that

$$4\sin^2\frac{(2m-1)\pi}{4T+1} < f'_x(k,0) < 4\sin^2\frac{(2m+1)\pi}{4T+1}, \quad k \in \mathbb{Z}(1,T), \tag{3.16}$$

(H_7) there exist $\mu \in (0,1/2)$ and $R > 0$ such that $0 < F(k,x) = \int_0^x f(k,s)ds \leqslant \mu x f(k,x)$ for all $k \in \mathbb{Z}(1,T)$ and $x \in \mathbb{R}$ with $|x| \geqslant R$.

Then the BVP(1.1) *has at least one nontrivial solution.*

Proof. Obviously, $dJ(\theta) = \theta$ and $d^2J(\theta) = I - \lambda K^{1/2}\mathbf{f}'_x(\theta)K^{1/2} := I - \Psi$, where $\Psi = K^{1/2}\mathbf{f}'_x(\theta)K^{1/2}$. It follows from ($H_6$) that there exists a sufficiently small $\varepsilon_0 > 0$ such that

$$a = 4\sin^2\frac{(2m-1)\pi}{4T+1} + \varepsilon_0 \leqslant f'_x(k,0) \leqslant 4\sin^2\frac{(2m+1)\pi}{4T+1} - \varepsilon_0 = b, \quad k \in \mathbb{Z}(1,T). \tag{3.17}$$

Since $(\Psi y, y) = (f'_x(\cdot,0)K^{1/2}y, K^{1/2}y)$ for all $y \in B$, $a(Ky,y) \leqslant (\Psi y, y) \leqslant b(Ky,y)$ for $y \in B$. So, for every $m \in \mathbb{Z}(1,T)$,

$$a\inf_{E_{m-1}}\sup_{y\in E_{m-1}^{\perp}, \|y\|=1}(Ky,y) \leqslant \inf_{E_{m-1}}\sup_{y\in E_{m-1}^{\perp}, \|y\|=1}(\Psi y,y) \leqslant b\inf_{E_{m-1}}\sup_{y\in E_{m-1}^{\perp}, \|y\|=1}(Ky,y), \tag{3.18}$$

where E_{m-1} is $(m-1)$-dimension subspace of B. Therefore, $4a\sin^2((2m-1)\pi/(4T+1)\ \mu_k \leqslant 4b\sin^2((2m-1)(\pi/4T)+1)$, where m-th eigenvalues of Ψ are μ_m. If there exists m_0 such that $\mu_{m_0}=1$, then $4a\sin^2((2m_0-1)(\pi/4T)+1) \leqslant 1 \leqslant 4b\sin^2((2m_0-1)(\pi/4T)+1)$. Thus, $1/4\sin^2((2m-1)\pi/(4T+1))+\varepsilon_0 \leqslant 1/4\,a\sin^2((2m_0-1)\pi/(4T+1)) \leqslant 1/4\sin^2((2m+1)\pi/(4T+1))-\varepsilon_0$. It is not true. So, 1 is not eigenvalues of Ψ. Thus θ is a nondegenerate critical point of J and Morse index number is n. Then, $C_q(J,\theta)=\delta_{qn}G$.

Asumme that J is no other critical points, then Morse index number of (H,J_a) is

$$M_n=1,\quad M_q=0,\quad q\neq n,\quad \sum_{q=0}^{\infty}(-1)^q M_q=(-1)^n. \tag{3.19}$$

It follows from (H_7) that there exists $C_1,C_2>0$ such that $F(k,x)\geqslant C_1|x|^{1/\mu}-C_2$, $(k,x)\in\mathbb{Z}(1,T)\times\mathbb{R}^1$. So,

$$J(\tau x)=\frac{1}{2}\tau^2\|x\|^2-\sum_{t=1}^{T}F\left(k,\tau K^{1/2}x(k)\right)\leqslant\frac{1}{2}\tau^2\|x\|^2-C_1\tau^{1/\mu}\left\|K^{1/2}x\right\|_{1/\mu}^{1/\mu}+C_2,\quad x\in B. \tag{3.20}$$

This implies $\lim_{\tau\to+\infty}J(\tau x)=-\infty$ for all $x\in S^\infty$. So, there exists $\gamma(x)>0$ such that $J(\gamma(x)x)=a$ for all $x\in S^\infty$. Then, we will prove that there exists $A>0$ such that $a<-A$, and if $J(\tau x)\leqslant a$, then $d/d\tau[J(\tau x)]<0$.

Indeed, let $A=C_3$, then for all $x\in S^\infty$,

$$\begin{aligned}
\frac{d}{d\tau}J(\tau x)&=\left(J'(\tau x),x\right)\\
&=\tau-\sum_{t=1}^{T}K^{1/2}x(k)f\left(k,\tau K^{1/2}x(k)\right)\\
&\leqslant\tau-\frac{1}{\tau\mu}\sum_{t=1}^{T}F\left(k,\tau K^{1/2}x(k)\right)+\frac{C_3}{\tau\mu}\\
&\leqslant\frac{1}{\tau\mu}\left(\frac{1}{2}\tau^2-\sum_{t=1}^{T}F\left(k,\tau K^{1/2}x(k)\right)\right)+\frac{C_3}{\tau\mu}\\
&=\frac{1}{\tau\mu}\left(J(\tau x)+C_3\right)<0.
\end{aligned} \tag{3.21}$$

So, $\gamma(x)$ satisfies $J(\gamma(x)x)=a$ and for all $x\in S^\infty$ is unique. It follows from implicit function theorem that we have $\gamma\in C(S^\infty,\mathbb{R}_+^1)$, where $\mathbb{R}_+^1=[0,+\infty)$.

At last, $J(\theta)=0$ and there exists $\varepsilon_0>0$ such that $B_{\varepsilon_0}\cap J_a=\emptyset$ and $\gamma(x)\geqslant\varepsilon_0$ for all $x\in S^\infty$. We define a deformation retract $\eta:[0,1]\times(B\setminus B_{\varepsilon_0})\to B\setminus B_{\varepsilon_0}$ by

$$\eta(s,x)=\begin{cases}(1-s)x+s\gamma\left(\dfrac{x}{\|x\|}\right)\dfrac{x}{\|x\|}, & x\in(B\setminus B_{\varepsilon_0})\setminus J_a,\\[2ex] x, & x\in J_a.\end{cases} \tag{3.22}$$

This proves that $B \setminus B_{\varepsilon_0} \simeq J_a$. The,n $S^\infty \simeq J_a$. Therefore, $H_q(H, J_a) \simeq H_q(H, S^\infty) \simeq 0$. Since S^∞ is contractible, we have Betti number $\beta_q = 0, q \geqslant 0$. Thus,

$$\sum_{q=0}^{\infty} (-1)^q \beta_q = 0. \tag{3.23}$$

Then by (3.19) and (3.23), we have $\sum_{q=0}^{\infty} (-1)^q M_q \neq \sum_{q=0}^{\infty} (-1)^q \beta_q$. Thus, according to the Morse inequality, J has at least one nontrivial critical point. □

Theorem 3.10. *Suppose that condition* (H_7) *is satisfied and that*

(H_8) *$f(k,x)$ is odd in x, that is, $f(k,x) = -f(k,-x)$ for all $(k,x) \in [0,1] \times \mathbb{R}^1$.*

Then the BVP(1.1) *has infinitely many solutions.*

Proof. First, it follows from (H_7) that J satisfies P.S. condition in B [6].

It follows from (H_8) that J is even. Asumme that J has only finitely many critical points $\{x_1, -x_1, x_2, -x_2, \ldots, x_m, -x_m\}$. We choose two constant numbers $a < 0$ and $b > 0$, such that they satisfy $a < \min\{J(x_1), J(x_2), \ldots, J(x_m), -C_4\}$ and $b > \min\{J(x_1), J(x_2), \ldots, J(x_m)\}$.

On the one hand, it follows from Theorem 3.9 that $B \setminus B_{\varepsilon_0} \simeq J_a$. Then, $S^\infty \simeq J_a$. Since S^∞ is contractible, we can deduce that Betti number $\beta_q = 0, q \geqslant 0$.

Therefore,

$$\sum_{q=0}^{\infty} (-1)^q \beta_q = 0. \tag{3.24}$$

On the other hand, we choose enough small $r > 0$ such that $B(\theta, r), B(y_i, r), B(-y_i, r), i = 1, 2, \ldots, m$ is mutually disjoint. So,

$$M_q = M_q(a,b) = \sum_{i=1}^{m} \left[\operatorname{rank} C_q(J, y_i) + \operatorname{rank} C_q(J, -y_i)\right] + \operatorname{rank} C_q(J, \theta), \quad q \geqslant 0. \tag{3.25}$$

It follows from Borsuk theorem that

$$\begin{aligned}
\sum_{q=0}^{\infty} (-1)^q M_q &= \sum_{i=1}^{m} \left[\sum_{q=0}^{\infty} (-1)^q \left(\operatorname{rank} C_q(J, y_i) + \operatorname{rank} C_q(J, -y_i)\right)\right] + \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(J, \theta) \\
&= \sum_{i=1}^{m} \left[\deg(I - A, B(y_i, r), \theta) + \deg(I - A, B(-y_i, r), \theta)\right] + \deg(I - A, B(\theta, r), \theta) \\
&= \deg\left(I - A, \bigcup_{i=1}^{m} (B(y_i, r) \cup B(-y_i, r)) \cup B(\theta, r), \theta\right),
\end{aligned} \tag{3.26}$$

that is

$$\sum_{q=0}^{\infty}(-1)^q M_q = \text{an odd number}. \tag{3.27}$$

By (3.24) and (3.27), we obtain that

$$\sum_{q=0}^{\infty}(-1)^q M_q \neq \sum_{q=0}^{\infty}(-1)^q \beta_q. \tag{3.28}$$

It is a contradiction. Thus, J has infinitely many critical points in B. □

Example 3.11. Consider the (BVP)

$$\begin{gathered} -\Delta^2 x(k-1) = \frac{\cos x}{1+\sin x}, \quad k \in \mathbb{Z}(1,T), \\ x(0) = 0 = \Delta x(T), \end{gathered} \tag{3.29}$$

Here, $f(k,x) = \cos x/(1+\sin x), F(k,x) = \ln(1+\sin x)$. By simple calculation, we have $\lim_{|x|\to\infty}(F(k,x)/x^2) < 2(\sin^2)(\pi/(4T+2))$. Thus, the condition ($H_1$) holds. Then by Theroem 3.3., the (BVP) has at least one solution.

Example 3.12. Consider the (BVP) of Example 3.11, here

$$\begin{gathered} f(k,x) = \begin{cases} 4x\sin^2\dfrac{2\pi}{4T+1}, & -1 < x < 1,\ k \in \mathbb{Z}(1,T); \\ x\sin^2\dfrac{\pi}{4T+k+2} + 132, & x \geqslant 1,\ k \in \mathbb{Z}(1,T); \\ x\sin^2\dfrac{\pi}{4T+k+2} - 132, & x \leqslant -1,\ k \in \mathbb{Z}(1,T), \end{cases} \\ F(k,x) = \begin{cases} 2x^2\sin^2\dfrac{2\pi}{4T+1}, & -1 < x < 1,\ k \in \mathbb{Z}(1,T); \\ 2x^2\sin^2\dfrac{\pi}{4T+k+2} + 132x - 66, & x \geqslant 1,\ k \in \mathbb{Z}(1,T); \\ 2x^2\sin^2\dfrac{\pi}{4T+k+2} - 132x - 66, & x \leqslant -1,\ k \in \mathbb{Z}(1,T). \end{cases} \end{gathered} \tag{3.30}$$

By simple calculation, we have

$$\lim_{|x|\longrightarrow\infty}\sup\frac{F(k,x)}{x^2} = 2\sin^2\frac{\pi}{4T+k+2} < 2\sin^2\frac{\pi}{4T+2}, \quad k \in \mathbb{Z}(1,T), \tag{3.31}$$

and there exists $m = 1$ and $\delta = 1 > 0$ such that

$$2\sin^2\frac{2\pi - 1}{4T+1}x^2 \leqslant F(k,x) \leqslant 2\sin^2\frac{2\pi + 1}{4T+1}x^2, \quad |x| \leqslant 1. \tag{3.32}$$

Let $a = 2\sin^2((2\pi+1)/(4T+1))$, $b = 2\sin^2((2\pi-1)/(4T+1))$ then a, b satisfy

$$2\sin^2\frac{\pi}{4T+1} \leqslant b \leqslant a \leqslant 2\sin^2\frac{3\pi}{4T+1}. \tag{3.33}$$

Thus, the conditions (H_1) and (H_5) hold. Hence, by Theorem 3.6, the (BVP) has at least two nontrivial solutions.

Example 3.13. Consider the (BVP) of Example 3.11, where

$$f(k,x) = \frac{\ln(1+x)}{1+x} + 10x^5 \quad (k,x) \in \mathbb{Z}(1,T) \times \mathbb{R}^1. \tag{3.34}$$

It is obvious that $f(k,x) \in C^1(\mathbb{Z}(1,T) \times \mathbb{R}^1, \mathbb{R}^1)$ is odd with respect to x. On the other hand,

$$F(k,x) = \int_0^x f(k,s)ds = \frac{1}{2}\ln^2(1+x) + \frac{5}{3}x^6, \quad k \in \mathbb{Z}(1,T), \tag{3.35}$$

and we have

$$\lim_{|x|\longrightarrow\infty} \frac{F(k,x)}{xf(k,x)} = \frac{1}{6}. \tag{3.36}$$

It is easy to prove that condition (H_7) holds. Hence, by Theorem 3.10, the (BVP) has infinitely many solutions.

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