Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 430180, 9 pages
doi:10.1155/2011/430180

Research Article

A Note on the Strong Law of Large Numbers for Arrays of Rowwise $\tilde{\rho}$-Mixing Random Variables

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Received 19 March 2011; Accepted 12 May 2011

Academic Editor: Carlo Piccardi

Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise $\tilde{\rho}$-mixing random variables. Some strong law of large numbers for arrays of rowwise $\tilde{\rho}$-mixing random variables is studied under some simple and weak conditions.

1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. The Marcinkiewicz-Zygmund strong law of large numbers states that

$$\begin{gathered} \frac{1}{n^\alpha}\sum_{i=1}^{n}(X_i - EX_i) \longrightarrow 0 \quad \text{a.s. for } 1 \leq \alpha < 2, \\ \frac{1}{n^\alpha}\sum_{i=1}^{n}X_i \longrightarrow 0 \quad \text{a.s. for } 0 < \alpha < 1 \end{gathered} \tag{1.1}$$

if and only if $E|X|^\alpha < \infty$. In the case of independence, Hu and Taylor [1] proved the following strong law of large numbers.

Theorem 1.1. *Let $\{X_{ni} : 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise independent random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $g(t)$ be a positive, even function such that $g(|t|)/|t|^p$ is an increasing function of $|t|$ and $g(|t|)/|t|^{p+1}$ is a decreasing function of $|t|$, respectively, that is,*

$$\frac{g(|t|)}{|t|^p}\uparrow, \quad \frac{g(|t|)}{|t|^{p+1}}\downarrow, \quad \text{as } |t|\uparrow \tag{1.2}$$

for some nonnegative integer p. If $p \geq 2$ and

$$\begin{gathered}
EX_{ni} = 0, \\
\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{g(|X_{ni}|)}{g(a_n)} < \infty, \\
\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{a_n}\right)^2\right)^{2k} < \infty,
\end{gathered} \tag{1.3}$$

where k is a positive integer, then

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \quad \text{a.s. as } n \longrightarrow \infty. \tag{1.4}$$

Zhu [2] generalized and improved the result of Hu and Taylor [1] for triangular arrays of rowwise independent random variables to the case of arrays of rowwise $\tilde{\rho}$-mixing random variables as follows.

Theorem 1.2. *Let $\{X_{ni} : i \geq 1, n \geq 1\}$ be an array of rowwise $\tilde{\rho}$-mixing random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $g(t)$ be a positive, even function such that $g(|t|)/|t|$ is an increasing function of $|t|$ and $g(|t|)/|t|^p$ is a decreasing function of $|t|$, respectively, that is,*

$$\frac{g(|t|)}{|t|} \uparrow, \quad \frac{g(|t|)}{|t|^p} \downarrow, \quad \text{as } |t| \uparrow \tag{1.5}$$

for some nonnegative integer p. If $p \geq 2$ and

$$\begin{gathered}
EX_{ni} = 0, \\
\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{g(|X_{ni}|)}{g(a_n)} < \infty, \\
\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{a_n}\right)^2\right)^{v/2} < \infty,
\end{gathered} \tag{1.6}$$

where v is a positive integer, $v \geq p$, then

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j} X_{ni}\right| > \varepsilon\right) < \infty. \tag{1.7}$$

In the following, we will give the definitions of a $\tilde{\rho}$-mixing sequence and the array of rowwise $\tilde{\rho}$-mixing random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Write $\mathcal{F}_S = \sigma(X_i, i \in S \subset \mathbb{N})$. For any given σ-algebras $\mathcal{B}$, $\mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})} \frac{|EXY - EXEY|}{(\operatorname{Var} X \operatorname{Var} Y)^{1/2}}. \tag{1.8}$$

Define the $\widetilde{\rho}$-mixing coefficients by

$$\widetilde{\rho}(k) = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T) : S, T \text{ are finite subsets of } \mathbb{N} \text{ such that } \operatorname{dist}(S, T) \geq k\}, \quad k \geq 0. \tag{1.9}$$

Obviously, $0 \leq \widetilde{\rho}(k+1) \leq \widetilde{\rho}(k) \leq 1$ and $\widetilde{\rho}(0) = 1$.

Definition 1.3. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be a $\widetilde{\rho}$-mixing sequence if, there exists $k \in \mathbb{N}$ such that $\widetilde{\rho}(k) < 1$.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise $\widetilde{\rho}$-mixing if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a $\widetilde{\rho}$-mixing sequence of random variables.

The $\widetilde{\rho}$-mixing random variables were introduced by Bradley [3], and many applications have been found. $\widetilde{\rho}$-mixing is similar to ρ-mixing, but both are quite different. Many authors have studied this concept providing interesting results and applications. See, for example, Zhu [2], An and Yuan [4], Kuczmaszewska [5], Bryc and Smoleński [6], Cai [7], Gan [8], Peligrad [9, 10], Peligrad and Gut [11], Sung [12], Utev and Peligrad [13], Wu and Jiang [14], and so on. When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired.

The main purpose of this paper is to further study the strong law of large numbers for arrays of rowwise $\widetilde{\rho}$-mixing random variables. We will introduce some simple conditions to prove the strong law of large numbers. The techniques used in the paper are inspired by Zhu [2].

2. Main Results

Throughout the paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different in various places.

The proofs of the main results of this paper are based upon the following lemma.

Lemma 2.1 (Utev and Peligrad [13, Theorem 2.1]). *Let $\{X_n, n \geq 1\}$ be a $\widetilde{\rho}$-mixing sequence of random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then, there exists a positive constant C depending only on p such that*

$$E\left(\max_{1 \leq j \leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{2.1}$$

As for arrays of rowwise $\tilde{\rho}$-mixing random variables $\{X_{ni} : i \geq 1, n \geq 1\}$, we assume that the constant C from Lemma 2.1 is the same for each row throughout the paper. Our main results are as follows.

Theorem 2.2. *Let $\{X_{ni} : i \geq 1, n \geq 1\}$ be an array of rowwise $\tilde{\rho}$-mixing random variables and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. Let $\{g_n(t), n \geq 1\}$ be a sequence of positive, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ and $g_n(|t|)/|t|$ is a decreasing function of $|t|$ for every $n \geq 1$, respectively, that is,*

$$g_n(|t|) \uparrow, \quad \frac{g_n(|t|)}{|t|} \downarrow, \quad \text{as } |t| \uparrow . \tag{2.2}$$

If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E g_n(|X_{ni}|)}{g_n(a_n)} < \infty, \tag{2.3}$$

then, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} X_{ni} \right| > \varepsilon \right) < \infty. \tag{2.4}$$

Proof. For fixed $n \geq 1$, define

$$\begin{gathered} X_i^{(n)} = X_{ni} I(|X_{ni}| \leq a_n), \quad i \geq 1, \\ T_j^{(n)} = \frac{1}{a_n} \sum_{i=1}^{j} \left(X_i^{(n)} - E X_i^{(n)} \right), \quad j = 1, 2, \ldots, n. \end{gathered} \tag{2.5}$$

It is easy to check that for any $\varepsilon > 0$,

$$\left(\max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} X_{ni} \right| > \varepsilon \right) \subset \left(\max_{1\leq i\leq n} |X_{ni}| > a_n \right) \bigcup \left(\max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} X_i^{(n)} \right| > \varepsilon \right), \tag{2.6}$$

which implies that

$$\begin{aligned} P\left(\max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} X_{ni} \right| > \varepsilon \right) &\leq P\left(\max_{1\leq i\leq n} |X_{ni}| > a_n \right) + P\left(\max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} X_i^{(n)} \right| > \varepsilon \right) \\ &\leq \sum_{i=1}^{n} P(|X_{ni}| > a_n) + P\left(\max_{1\leq j\leq n} \left| T_j^{(n)} \right| > \varepsilon - \max_{1\leq j\leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} E X_i^{(n)} \right| \right). \end{aligned} \tag{2.7}$$

Firstly, we will show that

$$\max_{1\le j\le n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right| \longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{2.8}$$

Actually, by conditions $g_n(|t|)/|t| \downarrow$ as $|t|\uparrow$ and (2.3), we have that

$$\begin{aligned}\max_{1\le j\le n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right| &= \max_{1\le j\le n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_{ni}I(|X_{ni}|\le a_n)\right| \\ &\le \frac{1}{a_n}\sum_{i=1}^{n}E|X_{ni}|I(|X_{ni}|\le a_n) \\ &\le \sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)I(|X_{ni}|\le a_n)}{g_n(a_n)} \\ &\le \sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)} \longrightarrow 0, \quad \text{as } n\longrightarrow\infty,\end{aligned} \tag{2.9}$$

which implies (2.8). It follows from (2.7) and (2.8) that for n large enough,

$$P\left(\max_{1\le j\le n}\left|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\right| > \varepsilon\right) \le \sum_{i=1}^{n}P(|X_{ni}|>a_n) + P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right). \tag{2.10}$$

Hence, to prove (2.4), we only need to show that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}|>a_n)<\infty, \tag{2.11}$$

$$\sum_{n=1}^{\infty}P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right)<\infty. \tag{2.12}$$

The conditions $g_n(|t|)\uparrow$ as $|t|\uparrow$ and (2.3) yield that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}|>a_n) \le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)}<\infty, \tag{2.13}$$

which implies (2.11).

By Markov's inequality, Lemma 2.1 (for $p = 2$), C_r's inequality, $g_n(|t|)/|t| \downarrow$ as $|t| \uparrow$ and (2.3), we can get that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) &\le C\sum_{n=1}^{\infty} E\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|^2\right)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}E\left|X_i^{(n)} - EX_i^{(n)}\right|^2\\
&\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}E\left|X_i^{(n)}\right|^2\\
&= C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2 I(|X_{ni}|\le a_n)}{a_n^2}\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}| I(|X_{ni}|\le a_n)}{a_n}\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|) I(|X_{ni}|\le a_n)}{g_n(a_n)}\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(|X_{ni}|)}{g_n(a_n)} < \infty,
\end{aligned} \tag{2.14}$$

which implies (2.12). This completes the proof of the theorem. □

Corollary 2.3. *Under the conditions of Theorem 2.2,*

$$\frac{1}{a_n}\sum_{i=1}^{n}X_{ni} \longrightarrow 0 \quad \text{a.s. as } n \longrightarrow \infty. \tag{2.15}$$

Theorem 2.4. *Let $\{X_{ni} : i \ge 1, n \ge 1\}$ be an array of rowwise $\tilde{\rho}$-mixing random variables and let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers. Let $\{g_n(t), n \ge 1\}$ be a sequence of nonnegative, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for every $n \ge 1$. Assume that there exists a constant $\delta > 0$ such that $g_n(t) \ge \delta t$ for $0 < t \le 1$. If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\left(\frac{X_{ni}}{a_n}\right) < \infty, \tag{2.16}$$

then for any $\varepsilon > 0$, (2.4) holds true.

Proof. We use the same notations as that in Theorem 2.2. The proof is similar to that of Theorem 2.2.

Firstly, we will show that (2.8) holds true. In fact, by the conditions $g_n(t) \geq \delta t$ for $0 < t \leq 1$ and (2.16), we have that

$$\begin{aligned}
\max_{1\le j\le n}\left|\frac{1}{a_n}\sum_{i=1}^{j} EX_i^{(n)}\right| &= \max_{1\le j\le n}\left|\frac{1}{a_n}\sum_{i=1}^{j} EX_{ni} I(|X_{ni}| \le a_n)\right| \\
&\le \sum_{i=1}^{n} E\left(\frac{|X_{ni}|}{a_n} I(|X_{ni}| \le a_n)\right) \\
&\le \frac{1}{\delta}\sum_{i=1}^{n} E g_n\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| \le a_n) \\
&\le \frac{1}{\delta}\sum_{i=1}^{n} E g_n\left(\frac{X_{ni}}{a_n}\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{2.17}$$

which implies (2.8).

According to the proof of Theorem 2.2, we only need to prove that (2.11) and (2.12) hold true.

When $|X_{ni}| > a_n > 0$, we have $g_n(X_{ni}/a_n) \geq g_n(1) \geq \delta$, which yields that

$$P(|X_{ni}| > a_n) = EI(|X_{ni}| > a_n) \le \frac{1}{\delta} E g_n\left(\frac{X_{ni}}{a_n}\right). \tag{2.18}$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) \le \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} E g_n\left(\frac{X_{ni}}{a_n}\right) < \infty, \tag{2.19}$$

which implies (2.11).

By Markov's inequality, Lemma 2.1 (for $p = 2$), C_r's inequality, $g_n(t) \geq \delta t$ for $0 < t \leq 1$ and (2.16), we can get that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}|^2 I(|X_{ni}| \le a_n)}{a_n^2} \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E|X_{ni}| I(|X_{ni}| \le a_n)}{a_n} \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E g_n\left(\frac{X_{ni}}{a_n}\right) I(|X_{ni}| \le a_n) \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E g_n\left(\frac{X_{ni}}{a_n}\right) < \infty,
\end{aligned} \tag{2.20}$$

which implies (2.12). This completes the proof of the theorem. □

Corollary 2.5. *Under the conditions of Theorem 2.4,*

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \quad a.s. \text{ as } n \longrightarrow \infty. \tag{2.21}$$

Corollary 2.6. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise $\tilde{\rho}$-mixing random variables and let $\{a_n, n \geq 1\}$ be a positive real numbers. If there exists a constant $\beta \in (0,1]$ such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\frac{|X_{ni}|^{\beta}}{|a_n|^{\beta} + |X_{ni}|^{\beta}}\right) < \infty, \tag{2.22}$$

then (2.4) *holds true.*

Proof. In Theorem 2.4, we take

$$g_n(t) = \frac{|t|^{\beta}}{1 + |t|^{\beta}}, \quad 0 < \beta \leq 1, \; n \geq 1. \tag{2.23}$$

It is easy to check that $\{g_n(t), n \geq 1\}$ is a sequence of nonnegative, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for every $n \geq 1$. And

$$g_n(t) \geq \frac{1}{2}t^{\beta} \geq \frac{1}{2}t, \quad 0 < t \leq 1, \; 0 < \beta \leq 1. \tag{2.24}$$

Therefore, by Theorem 2.4, we can easily get (2.4). □

Acknowledgments

The authors are most grateful to the editor Carlo Piccardi and anonymous referee for careful reading of the paper and valuable suggestions which helped to improve an earlier version of this paper. This work is supported by the National Natural Science Foundation of China (10871001) and the Academic innovation team of Anhui University (KJTD001B).

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