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Research Article

The Polytopic-k-Step Fibonacci Sequences in Finite Groups

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We study the polytopic-k-step Fibonacci sequences, the polytopic-k-step Fibonacci sequences modulo m, and the polytopic-k-step Fibonacci sequences in finite groups. Also, we examine the periods of the polytopic-k-step Fibonacci sequences in semidihedral group SD_{2^m}.

1. Introduction

The well- known k-step Fibonacci sequence $\{F_n^k\}$ $(k \geq 2)$ is defined as

$$F_0^{(k)} = 0, \dots, F_{k-2}^{(k)} = 0, \qquad F_{k-1}^{(k)} = 1,$$
$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)} \quad \text{for } n \geq 0. \tag{1.1}$$

Let $\{a_j\}_{j=0}^{k-1}$ $(k \geq 2, a_{k-1} \neq 0)$ be a sequence of real numbers. A k-generalized Fibonacci sequence $\{V_n\}_{n=0}^{+\infty}$ is defined by the following linear recurrence relation of order k:

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{k-1} V_{n-k-1}, \quad \text{for } n \geq k-1, \tag{1.2}$$

where $V_0, \dots, V_{k-1}$ are specified by the initial conditions.

The k-step Fibonacci sequence, the k-generalized Fibonacci sequence, and their properties have been studied by several authors; see, for example, [1–5].

The k-step Fibonacci sequence is a special case of a sequence which is defined as a linear combination by Kalman as follows

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}, \tag{1.3}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [6], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \left[aij\right]_{k\times k} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.4}$$

Then, by an inductive argument he obtained

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}. \tag{1.5}$$

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \ldots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \ldots$ is simply periodic with period 6.

Definition 1.1. For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_i = a_{i+1}$, $0 \le i \le n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \ge 0$, is called the Fibonacci orbit of G with respect to the generating set A, denoted by $F_A(G)$. If $F_A(G)$ is periodic, then the length of the period of the sequence is called the Fibonacci length of G with respect to generating set A, written as $\mathrm{LEN}_A(G)$ [7].

Definition 1.2. For every integer k, where $2 \le k \le \mathrm{LEN}_A(G)$, the sequence $\{y_i\}_1^{\infty}$ of the elements of G defined by

$$\begin{aligned} y_i &= x_i, \quad i = 1, \ldots, k, \\ y_i &= \left(y_{i-k}\right)^{\alpha_1} \left(y_{i-k+1}\right)^{\alpha_2} \cdots \left(y_{i-1}\right)^{\alpha_k}, \quad i \ge k+1 \end{aligned} \tag{1.6}$$

is called a k-step generalized Fibonacci sequence of G, for some positive integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ [8].

Definition 1.3. A k-nacci sequence in a finite group is a sequence of group elements x_0, $x_1, x_2, x_3, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases} \tag{1.7}$$

We also require that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G seeded by $x_0, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, \ldots, x_{j-1})$ and its period is denoted by $P_k(G; x_0, \ldots, x_{j-1})$ [9].

The Fibonacci sequence, the k-nacci sequence, and the generalized order-k Pell sequence in finite groups have been studied by some authors, and different periods of these sequences in different finite groups have been obtained; see, for example, [7, 9–16]. Formulas which classified according to certain rules for this periods are critical to be used in cryptography, see, for example, [17–19]. Because the exponents of each term in the generalized Fibonacci sequence are determined randomly, classification according to certain rule of periods is resulting from application of this sequence in groups is possible, only if the exponent of each term are determined integers obtained according to a certain rule. Therefore, In this paper, by expanding the k-step Fibonacci sequence which is special type of the generalized Fibonacci sequences with polytopic numbers which are a well-known family of integers, we conveyed the sequence named the polytopic-k-step Fibonacci sequence that exponent of $n+t$nd term is determined that $\binom{\alpha+k-t-1}{k-t}$ formula to finite groups and named the polytopic-k-step Fibonacci sequence in finite groups as polytopic-k-nacci sequence. Because of varying both α and according to the number of step and the exponent of each term of this is determined according to a certain rule, the polytopic-k-step Fibonacci sequence is more useful and more general than the k-nacci sequences and the generalized order-k Pell sequence which varying only by the number of step. So that considered by different α value, different step values and different initial (seed) sets, different lineer recurrence sequences which are a special type of generalized Fibonacci sequences occur, and thus by conveying the polytopic-k-step Fibonacci sequence to finite groups, more useful and more general formulas than formulas used to obtain periods of the k-nacci and the generalized order-k Pell sequence in finite groups are obtained to be used in cryptography.

In this paper, the usual notation p is used for a prime number.

2. The Polytopic-k-Step Fibonacci Sequences

The well-known k-topic numbers are defined as

$$P_k(n) = \frac{n(n+1)(n+2)\cdots(n+r-1)}{k!} = \binom{n+k-1}{k}. \tag{2.1}$$

When $k = 2$, the k-topic numbers, $P_k(n)$, are reduced to the triangular numbers. In [20], Gandhi and Reddy obtained triangular numbers in the generalized Pell sequence $\{P_n^{(\alpha)}\}$ and generalized associated Pell sequence $\{Q_n^{(\alpha)}\}$ which are defined for a fixed $\alpha > 0$, respectively, as

$$\begin{gathered} P_0^{(\alpha)} = 0, \qquad P_1^{(\alpha)} = 1, \qquad P_{n+2}^{(\alpha)} = (\alpha+1)P_{n+1}^{(\alpha)} + \frac{\alpha(\alpha+1)}{2}P_n^{(\alpha)} \quad \text{for } n \geq 0, \\ Q_0^{(\alpha)} = Q_1^{(\alpha)} = 1, \qquad Q_{n+2}^{(\alpha)} = (\alpha+1)Q_{n+1}^{(\alpha)} + \frac{\alpha(\alpha+1)}{2}Q_n^{(\alpha)} \quad \text{for } n \geq 0. \end{gathered} \tag{2.2}$$

Now we define for a fixed integer $\alpha > 0$, a new sequence called the polytopic-k-step Fibonacci sequence $\{F_n^{(k,\alpha)}\}$, by

$$\begin{gathered} F_0^{(k,\alpha)} = 0, \dots, F_{k-2}^{(k,\alpha)} = 0, \qquad F_{k-1}^{(k,\alpha)} = 1, \\ F_{n+k}^{(k,\alpha)} = \alpha F_{n+k-1}^{(k,\alpha)} + \binom{\alpha+1}{2}F_{n+k-2}^{(k,\alpha)} + \cdots + \binom{\alpha+k-2}{k-1}F_{n+1}^{(k,\alpha)} + \binom{\alpha+k-1}{k}F_n^{(k,\alpha)} \quad \text{for } n \geq 0. \end{gathered} \tag{2.3}$$

Obviously, if we take $\alpha = 1$ in (2.3), then this sequence reduces to the well-known k-step Fibonacci sequence. When $\alpha \geq 2$ and $k = 2$ in (2.3), we call $\{F_n^{(2,\alpha)}\}$ the polytopic Fibonacci sequence.

By (2.3), we can write

$$\begin{bmatrix} F_{n+k}^{(k,\alpha)} \\ F_{n+k-1}^{(k,\alpha)} \\ F_{n+k-2}^{(k,\alpha)} \\ \vdots \\ F_{n+1}^{(k,\alpha)} \end{bmatrix} = \begin{bmatrix} \alpha & \binom{\alpha+1}{2} & \cdots & \binom{\alpha+k-2}{k-1} & \binom{\alpha+k-1}{k} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+k-1}^{(k,\alpha)} \\ F_{n+k-2}^{(k,\alpha)} \\ F_{n+k-3}^{(k,\alpha)} \\ \vdots \\ F_{n}^{(k,\alpha)} \end{bmatrix} \tag{2.4}$$

for the polytopic-k-step Fibonacci sequence. Let

$$M = [m_{ij}]_{k\times k} = \begin{bmatrix} \alpha & \binom{\alpha+1}{2} & \cdots & \binom{\alpha+k-2}{k-1} & \binom{\alpha+k-1}{k} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{2.5}$$

The matrix M is called the polytopic-k-step Fibonacci matrix.

We obtain that the polytopic Fibonacci sequences $\{F_n^{(2,\alpha)}\}$ are generated by a matrix Q^α for a fixed integer $\alpha \geq 2$:

$$Q^\alpha = \begin{bmatrix} \alpha & \dfrac{\alpha(\alpha+1)}{2} \\ 1 & 0 \end{bmatrix}, \qquad (Q^\alpha)^n = \begin{bmatrix} F_{n+1}^{(2,\alpha)} & \dfrac{\alpha(\alpha+1)}{2} F_n^{(2,\alpha)} \\ F_n^{(2,\alpha)} & \dfrac{\alpha(\alpha+1)}{2} F_{n-1}^{(2,\alpha)} \end{bmatrix} \tag{2.6}$$

which can be proved by mathematical induction.

3. The Polytopic-k-Step Fibonacci Sequences Modulo m

In this section we examine the polytopic-k-step Fibonacci sequences modulo m for $\alpha \geq 2$ and $k \geq 2$.

Reducing the polytopic-k-step Fibonacci sequence by a modulus m, we can get a repeating sequence denoted by

$$\left\{F^{(k,\alpha)}(m)\right\} = \left\{F_0^{(k,\alpha)}(m), F_1^{(k,\alpha)}(m), F_2^{(k,\alpha)}(m), \ldots, F_i^{(k,\alpha)}(m), \ldots\right\}, \tag{3.1}$$

where $F_i^{(k,\alpha)}(m) = F_i^{(k,\alpha)} (\bmod m)$. It has the same recurrence relation as in (2.3).

Theorem 3.1. *$\{F^{(k,\alpha)}(m)\}$ is a periodic sequence for $k \geq 2$ and $\alpha \geq 2$.*

Proof. Let $U_k = \{(x_1, x_2, \ldots, x_k) \mid 0 \leq x_i \leq m-1\}$. Then we have that $|U_k| = m^k$ is finite, that is, for any $a \geq 0$, there exist $b \geq a$ such that $F_{a+1}^{(k,\alpha)}(m) \equiv F_{b+1}^{(k,\alpha)}(m), \ldots, F_{a+k}^{(k,\alpha)}(m) \equiv F_{b+k}^{(k,\alpha)}(m)$. From the definition of the polytopic-k-step Fibonacci sequence $\{F_n^{(k,\alpha)}\}$ we have $F_{n+k}^{(k,\alpha)} = \alpha F_{n+k-1}^{(k,\alpha)} + \binom{\alpha+1}{2} F_{n+k-2}^{(k,\alpha)} + \cdots + \binom{\alpha+k-2}{k-1} F_{n+1}^{(k,\alpha)} + \binom{\alpha+k-1}{k} F_n^{(k,\alpha)}$, that is, $\binom{\alpha+k-1}{k} F_n^{(k,\alpha)} = F_{n+k}^{(k,\alpha)} - \alpha F_{n+k-1}^{(k,\alpha)} - \binom{\alpha+1}{2} F_{n+k-2}^{(k,\alpha)} - \cdots - \binom{\alpha+k-2}{k-1} F_{n+1}^{(k,\alpha)}$. Then we can easily get that $F_a^{(k,\alpha)}(m) \equiv F_b^{(k,\alpha)}(m)$, $F_{a-1}^{(k,\alpha)}(m) \equiv F_{b-1}^{(k,\alpha)}(m), \ldots, F_2^{(k,\alpha)}(m) \equiv F_{b-a+2}^{(k,\alpha)}(m)$ and $F_1^{(k,\alpha)}(m) \equiv F_{b-a+1}^{(k,\alpha)}(m)$, which implies that $\{F_n^{(k,\alpha)}\}$ is a periodic sequence.

Let $h_k^{(\alpha)}(m)$ denote the smallest period of $\{F^{(k,\alpha)}(m)\}$, called the period of the polytopic-k-step Fibonacci sequence modulo m. When $k = 2$, $h_2^{(\alpha)}(m)$ is the period of the polytopic Fibonacci sequence modulo m. □

Example 3.2. We have $\{F^{(3,4)}(3)\} = \{0, 0, 1, 1, 2, 2, 0, 0, 1, \ldots\}$ and then repeat. So we get $h_3^{(4)}(3) = 6$.

By elementary number theory it is easy to prove that if $m = \prod_{i=1}^{t} p_i^{e_i}$, $(t \geq 1)$, where p_i's are distinct primes, then $h_k^{(\alpha)}(m) = \text{lcm}[h_k^{(\alpha)}(p_i^{e_i})]$.

For a given matrix $A = [a_{ij}]$ with a_{ij}'s being integers, $A (\bmod m)$ means that every entry of A is reduced modulo m, that is, $A (\bmod m) = (a_{ij} (\bmod m))$. Let $\langle M \rangle_{p^a} = \{M^i (\bmod p^a) \mid i \geq 0\}$ be a cyclic group, and let $|\langle M \rangle_{p^a}|$ denote the order of $\langle M \rangle_{p^a}$ with $p \nmid \binom{\alpha+k-1}{k}$

(where by $p \nmid \binom{\alpha+k-1}{k}$ we mean that $\binom{\alpha+k-1}{k}$ is not divided by p) and T the transpose of a matrix. It is clear that

$$\left(M^i[1,0,0,\dots,0]^T\right)^T \pmod m = \left[F^{(k,\alpha)}_{i+k-1}(m), F^{(k,\alpha)}_{i+k-2}(m), \dots, F^{(k,\alpha)}_{i}(m)\right]. \tag{3.2}$$

We then obtain that $h^{(\alpha)}_k(m)$ is least positive integer $h^{(\alpha)}$ such that

$$\left(M^{h^{(\alpha)}}[1,0,0,\dots,0]^T\right)^T \pmod m = [1,0,0,\dots,0]. \tag{3.3}$$

Theorem 3.3. *Let $\alpha \ge 2$. If $p \nmid \binom{\alpha+k-1}{k}$, then $h^{(\alpha)}_k(p^a) = |\langle M\rangle_{p^a}|$.*

Proof. It is clear that $|\langle M\rangle_{p^a}|$ is divisible by $h^{(\alpha)}_k(p^a)$. Then we need only to prove that $h^{(\alpha)}_k(p^a)$ is divisible by $|\langle M\rangle_{p^a}|$. Let $h^{(\alpha)}_k(p^a) = n$. Then we have

$$M^n = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1k} \\ m_{21} & m_{22} & \cdots & m_{2k} \\ \vdots & \vdots & & \vdots \\ m_{k1} & m_{k2} & \cdots & m_{kk} \end{bmatrix}. \tag{3.4}$$

The elements of the matrix M^n are in the following forms:

$$\begin{gathered} m_{11} = F^{(k,\alpha)}_{n+k-1}, \qquad m_{21} = F^{(k,\alpha)}_{n+k-2}, \dots, m_{k1} = F^{(k,\alpha)}_{n}, \\ m_{ii} = \beta_1 F^{(k,\alpha)}_{n+k-2} + \beta_2 F^{(k,\alpha)}_{n+k-3} + \cdots + \beta_{k-1} F^{(k,\alpha)}_{n} + 1, \quad \text{for } 2 \le i \le k,\ \beta_1, \beta_2, \dots, \beta_{k-1} \ge 0, \\ m_{ij} = \eta_1 F^{(k,\alpha)}_{n+k-2} + \eta_2 F^{(k,\alpha)}_{n+k-3} + \cdots + \eta_{k-1} F^{(k,\alpha)}_{n} \quad \text{for } i \ne j,\ 1 \le i \le k,\ 2 \le j \le k, \eta_1, \eta_2, \dots, \eta_{k-1} \ge 0. \end{gathered} \tag{3.5}$$

We thus obtain that

$$\begin{gathered} m_{ii} \equiv 1 \left(\operatorname{mod} p^a\right), \quad \text{for } 1 \le i \le k, \\ m_{ij} \equiv 0 \left(\operatorname{mod} p^a\right), \quad \text{for } 1 \le i,\ j \le k \text{ such that } i \ne j. \end{gathered} \tag{3.6}$$

So we get that $M^n \equiv I \pmod{p^a}$, which yields that n is divisible by $|\langle M\rangle_{p^a}|$. We are done. $\square$

Theorem 3.4. *Let $\alpha \ge 2$, and let t be the largest positive integer such that $h^{(\alpha)}_k(p) = h^{(\alpha)}_k(p^t)$. Then $h^{(\alpha)}_k(p^a) = p^{a-t} h^{(\alpha)}_k(p)$ for every $a \ge t$. In particular, if $h^{(\alpha)}_k(p) \ne h^{(\alpha)}_k(p^2)$, then $h^{(\alpha)}_k(p^a) = p^{a-1} h^{(\alpha)}_k(p)$ holds for every $a > 1$.*

Proof. Let q be a positive integer. Since $M^{h_k^{(\alpha)}(p^{q+1})} \equiv I \pmod{p^{q+1}}$, that is, $M^{h_k^{(\alpha)}(p^{q+1})} \equiv I$ (mod p^q), we get that $h_k^{(\alpha)}(p^{q+1})$ is divided by $h_k^{(\alpha)}(p^q)$. On the other hand, writing $M^{h_k^{(\alpha)}(p^q)} = I + (a_{ij}^{(q)} p^q)$, we have

$$M^{h_k^{(\alpha)}(p^q)p} = \left(I + \left(a_{ij}^{(q)} p^q\right)\right)^p = \sum_{i=0}^{p} \binom{p}{i} \left(a_{ij}^{(q)} p^q\right)^i \equiv I \left(\bmod p^{q+1}\right), \tag{3.7}$$

which yields that $h_k^{(\alpha)}(p^q)p$ is divided by $h_k^{(\alpha)}(p^{q+1})$. Therefore, $h_k^{(\alpha)}(p^{q+1}) = h_k^{(\alpha)}(p^q)$ or $h_k^{(\alpha)}(p^{q+1}) = h_k^{(\alpha)}(p^q)p$, and the latter holds if, and only if, there is an $a_{ij}^{(q)}$ which is not divisible by p. Since $h_k^{(\alpha)}(p^t) \neq h_k^{(\alpha)}(p^{t+1})$, there is an $a_{ij}^{(t+1)}$ which is not divisible by p, thus, $h_k^{(\alpha)}(p^{t+1}) \neq h_k^{(\alpha)}(p^{t+2})$. The proof is finished by induction on t. □

Conjecture 3.5. *Let $\alpha \geq 2$. If $p \geq k$, then there exists a σ with $0 \leq \sigma \leq k$ such that $(p^{k+1} - p^{\sigma})$ is divided by $h_k^{(\alpha)}(p)$.*

Table 1 list some primes for which the conjecture is true when $k = 5$ and $\alpha = 5$.

4. The Polytopic-k-Nacci Sequences in Finite Groups

Definition 4.1. For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, we define the polytopic Fibonacci orbit $F_A^{\alpha}(G)$ with respect to the generating set A to be the sequence $\{x_i\}$ of the elements of G such that

$$\begin{gathered} x_i = a_{i+1}, \quad \text{for } 0 \leq i \leq n-1, \\ x_{i+n} = (x_i)^{\binom{\alpha+n-1}{n}} (x_{i+1})^{\binom{\alpha+n-2}{n-1}} \cdots (x_{i+n-2})^{\binom{\alpha+1}{2}} (x_{i+n-1})^{\alpha}, \quad \text{for } i \geq 0, \end{gathered} \tag{4.1}$$

Example 4.2. Let $G = \langle A \rangle$, where $A = \{a_1, a_2, a_3\}$. $F_A^{\alpha}(G)$ is

$$\begin{gathered} x_0 = a_1, \qquad x_1 = a_2, \qquad x_2 = a_3, \\ x_{i+3} = (x_i)^{\alpha(\alpha+1)(\alpha+2)/6} (x_{i+1})^{\alpha(\alpha+1)/2} (x_{i+2})^{\alpha}, \quad \text{for } i \geq 0. \end{gathered} \tag{4.2}$$

Definition 4.3. A polytopic-k-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, \ldots x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0^{\binom{\alpha+n-1}{n}} x_1^{\binom{\alpha+n-2}{n-1}} \cdots (x_{n-1})^{\alpha} & \text{for } j \leq n < k, \\ x_{n-k}^{\binom{\alpha+k-1}{k}} x_{n-k+1}^{\binom{\alpha+k-2}{k-1}} \cdots (x_{n-1})^{\alpha} & \text{for } n \geq k. \end{cases} \tag{4.3}$$

It is required that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus, forcing the polytopic-k-nacci sequence to reflect the structure of the group. We denote the polytopic-k-nacci sequence of a group G generated by $x_0, \ldots, x_{j-1}$ by $F_k^{\alpha}(G; x_0, \ldots, x_{j-1})$.

Table 1: The length of $h_5^{(5)}(p)$.

p	$h_5^{(5)}(p)$	Result
5	5	$h_5^{(5)}(p) \mid p^6 - p^5$
11	80525	$h_5^{(5)}(p) \mid p^6 - p$
13	15372	$h_5^{(5)}(p) \mid p^6 - 1$
23	145992	$h_5^{(5)}(p) \mid p^6 - 1$
29	24388	$h_5^{(5)}(p) \mid p^6 - p^3$
31	461760	$h_5^{(5)}(p) \mid p^6 - p^2$
43	1749132	$h_5^{(5)}(p) \mid p^6 - 1$
47	1661152	$h_5^{(5)}(p) \mid p^6 - 1$
53	2808	$h_5^{(5)}(p) \mid p^6 - p^4$
59	205378	$h_5^{(5)}(p) \mid p^6 - 1$
67	4030224	$h_5^{(5)}(p) \mid p^6 - p^2$
73	1419912	$h_5^{(5)}(p) \mid p^6 - p^2$
97	44264640	$h_5^{(5)}(p) \mid p^6 - p^2$
101	13136325	$h_5^{(5)}(p) \mid p^6 - 1$
223	52856154	$h_5^{(5)}(p) \mid p^6 - 1$
397	78804	$h_5^{(5)}(p) \mid p^6 - p^4$
419	12914277518098	$h_5^{(5)}(p) \mid p^6 - p$
523	47685222	$h_5^{(5)}(p) \mid p^6 - p^3$
607	16969333200	$h_5^{(5)}(p) \mid p^6 - p^2$
719	89206789920	$h_5^{(5)}(p) \mid p^6 - 1$
821	454331269680	$h_5^{(5)}(p) \mid p^6 - p^2$
853	529414856880	$h_5^{(5)}(p) \mid p^6 - p^2$
1009	518758082640	$h_5^{(5)}(p) \mid p^6 - 1$
1523	2319528	$h_5^{(5)}(p) \mid p^6 - p^4$
1613	2601768	$h_5^{(5)}(p) \mid p^6 - p^4$
2011	4044120	$h_5^{(5)}(p) \mid p^6 - p^4$
3011	27298090330	$h_5^{(5)}(p) \mid p^6 - p^3$
4021	262790931413426025	$h_5^{(5)}(p) \mid p^6 - p$
5059	43159140126	$h_5^{(5)}(p) \mid p^6 - p^3$
6037	132826492154616	$h_5^{(5)}(p) \mid p^6 - p^2$

Example 4.4. Let $G = \langle A \rangle$, where $A = \{a_1, a_2, a_3\}$. $F_4^{\alpha}(G; a_1, a_2, a_3)$ is

$$x_0 = a_1, \qquad x_1 = a_2, \qquad x_2 = a_3, \qquad x_3 = (x_0)^{\alpha(\alpha+1)(\alpha+2)/6}(x_1)^{\alpha(\alpha+1)/2}(x_2)^{\alpha},$$
$$x_{i+4} = (x_i)^{\alpha(\alpha+1)(\alpha+2)(\alpha+3)/24}(x_{i+1})^{\alpha(\alpha+1)(\alpha+2)/6}(x_{i+2})^{\alpha(\alpha+1)/2}(x_{i+3})^{\alpha} \quad \text{for } i \geq 0. \tag{4.4}$$

It is important to note that the polytopic Fibonacci orbit of a k-generated group is a polytopic-k-nacci sequence.

The classic polytopic Fibonacci sequence in the integers modulo m can be written as $F_2^{\alpha}(\mathbb{Z}_m; 0, 1)$. We call a polytopic-2-nacci sequence of a group of elements a polytopic Fibonacci sequence of a finite group.

Theorem 4.5. *A polytopic-k-nacci sequence in a finite group is periodic.*

Proof. The proof is similar to the proof of Theorem 1 in [6] and is omitted.

We denote the period of a polytopic-k-nacci sequence $F_k^{\alpha}(G; x_0, \ldots, x_{j-1})$ by $P_k^{\alpha}(G; x_0, \ldots, x_{j-1})$. When $\alpha = 1$, $F_k^{\alpha}(G; x_0, \ldots, x_{j-1})$ and $P_k^{\alpha}(G; x_0, \ldots, x_{j-1})$ are reduced to $F_k(G; x_0, \ldots, x_{j-1})$ and $P_k(G; x_0, \ldots, x_{j-1})$, respectively.

From the definition, it is clear that the period of a polytopic-k-nacci sequence in a finite group depends on the chosen generating set and the order in which the assignments of $x_0, x_1, \ldots x_{n-1}$ are made. □

Definition 4.6. Let G be a finite group. If there exists a polytopic-k-nacci sequence of the group G such that every element of the group G appears in the sequence, then the group G is called polytopic-k-nacci sequenceable.

It is important to note that the direct product of polytopic-k-nacci sequenceable groups is not necessarily polytopic-k-nacci sequenceable. Consider that the group $C_2 \times C_4$ is defined by the presentation

$$\left\langle x, y \mid x^2 = y^4 = e, xy = yx \right\rangle. \tag{4.5}$$

The polytopic Fibonacci sequences of the group $C_2 \times C_4$ for $\alpha = 2$ are

$$\begin{aligned} F_2^2(C_2 \times C_4; x, y) &= x, y, xy^2, y^3, x, y, \ldots, \\ F_2^2(C_2 \times C_4; y, x) &= y, x, y^3, xy^2, y, x, \ldots. \end{aligned} \tag{4.6}$$

Since the elements $e, xy,$ and xy^3 do not in either sequences, the group $C_2 \times C_4$ is not polytopic-2-nacci sequenceable.

The group $\langle x \rangle$ has a polytopic Fibonacci sequence

$$F_2^2(\langle x \rangle; e, x) = e, x, e, x, \ldots \tag{4.7}$$

and hence is polytopic-2-nacci sequenceable. The group $\langle y \rangle$ has a polytopic Fibonacci sequence

$$F_2^2(\langle y \rangle; e, y) = e, y, y^2, y^3, e, y, \ldots \tag{4.8}$$

and hence is polytopic-2-nacci sequenceable.

We will now address the periods of the polytopic-k-nacci sequences in specific classes of groups. A group SD_{2^m} is semidihedral group of order 2^m if

$$SD_{2^m} = \left\langle a, b \mid a^{2^{m-1}} = b^2 = e, b^{-1}ab = a^{-1+2^{m-2}} \right\rangle \tag{4.9}$$

for every $m \geq 4$. Note that the orders a and b are 2^{m-1} and 2, respectively.

Theorem 4.7. *The periods of the polytopic-k-nacci sequences in the group SD_{2^m} for initial (seed) set, a, b, and $\alpha = 2$ are as follows:*

(i) $P_k^2(\mathrm{SD}_{2^m}; a,\ b) = h_k^{(2)}(2^{m-2})$, *for* $2 \le k \le 4$.

(ii) $P_k^2(\mathrm{SD}_{2^m}; a,\ b) = h_k^{(2)}(2^{m-1})$, *for* $k \ge 5$.

Proof. (i) If $k = 2$, we have the polytopic-2-nacci sequence for $\alpha = 2$:

$$\begin{gathered}x_0 = a, \quad x_1 = b, \quad x_2 = a^3, \quad x_3 = a^{(2^{m-2}-1)\cdot 2\cdot 3}b, \\ x_4 = a^{3^2}, \quad x_5 = a^{(2^{m-2}-1)\cdot 2\cdot 3+(2^{m-2}-1)\cdot 2\cdot 3^2}b, \dots, \\ x_{2^{m-2}} = a^{3^{2^{m-3}}}, \quad x_{2^{m-2}+1} = a^{(2^{m-2}-1)\cdot 2\cdot 3+(2^{m-2}-1)\cdot 2\cdot 3^2+\cdots+(2^{m-2}-1)\cdot 2\cdot 3^{2^{m-3}}}b, \dots.\end{gathered} \tag{4.10}$$

By mathematical induction, it is easy to prove that

$$\begin{gathered}3^{2^{m-3}} \equiv 1 \left(\bmod 2^{m-1}\right), \left(2^{m-2}-1\right)\cdot 2\cdot 3 + \left(2^{m-2}-1\right)\cdot 2\cdot 3^2 \\ + \cdots + \left(2^{m-2}-1\right)\cdot 2\cdot 3^{2^{m-3}} \equiv 0 \left(\bmod 2^{m-1}\right).\end{gathered} \tag{4.11}$$

So we get $x_{2^{m-2}} = a^{3^{2^{m-3}}} = a$, $x_{2^{m-2}+1} = a^{(2^{m-2}-1)\cdot 2\cdot 3+(2^{m-2}-1)\cdot 2\cdot 3^2+\cdots+(2^{m-2}-1)\cdot 2\cdot 3^{2^{m-3}}}b = b$. It is easy to see that $h_2^{(2)}(2^{m-2}) = 2^{m-3}\cdot h_2^{(2)}(2) = 2^{m-3}\cdot 2 = 2^{m-2}$. Since the elements succeeding $x_{h_2^{(2)}(2^{m-2})}$, $x_{h_2^{(2)}(2^{m-2})+1}$, depend on a and b for their values, the cycle begins again with the $h_2^{(2)}(2^{m-2})$nd, that is, $x_{h_2^{(2)}(2^{m-2})} = x_0$ and $x_{h_2^{(2)}(2^{m-2})+1} = x_1$. Thus, the period of $F_2^2(\mathrm{SD}_{2^m}; a,\ b)$ is $h_2^{(2)}(2^{m-2})$.

If $k = 3$, we have the polytopic-3-nacci sequence for $\alpha = 2$:

$$\begin{gathered}x_0 = a, \quad x_1 = b, \quad x_2 = a^3, \quad x_3 = a^{4+(2^{m-2}-1)\cdot 2\cdot 3}b, \quad x_4 = a^{3^2}, \\ x_5 = a^{3\cdot 4+4+(2^{m-2}-1)\cdot 2\cdot 3+(2^{m-2}-1)\cdot 2\cdot 3^2}b, \quad x_6 = a^{3^3}, \dots, \\ x_{2^{m-2}} = a^{3^{2^{m-3}}}, \quad x_{2^{m-2}+1} = a^{3^{2^{m-3}-1}\cdot 4+3^{2^{m-3}-2}\cdot 4+\cdots+4+(2^{m-2}-1)\cdot 2\cdot 3+(2^{m-2}-1)\cdot 2\cdot 3^2+\cdots+(2^{m-2}-1)\cdot 2\cdot 3^{2^{m-3}}}b, \\ x_{2^{m-2}+2} = a^{3^{2^{m-3}+1}} \cdots.\end{gathered} \tag{4.12}$$

By mathematical induction, it is easy to prove that $3^{2^{m-3}-1}\cdot 4 + 3^{2^{m-3}-2}\cdot 4 + \cdots + 4 \equiv 0 (\bmod 2^{m-1})$, $3^{2^{m-3}+1} \equiv 3 (\bmod 2^{m-1})$. So we get $x_{2^{m-2}} = a^{3^{2^{m-3}}} = a, x_{2^{m-2}+1} = a^{3^{2^{m-3}-1}\cdot 4+3^{2^{m-3}-2}\cdot 4+\cdots+4+(2^{m-2}-1)\cdot 2\cdot 3+(2^{m-2}-1)\cdot 2\cdot 3^2+\cdots+(2^{m-2}-1)\cdot 2\cdot 3^{2^{m-3}}}b = b$, $x_{2^{m-2}+2} = a^{3^{2^{m-3}+1}} = a^3$. It is easy to see that $h_3^{(2)}(2^{m-2}) = 2^{m-3}\cdot h_3^{(2)}(2) = 2^{m-3}\cdot 2 = 2^{m-2}$. Since the elements succeeding $x_{h_3^{(2)}(2^{m-2})}, x_{h_3^{(2)}(2^{m-2})+1}, x_{h_3^{(2)}(2^{m-2})+2}$ depend on a, b, and a^3 for their values, the cycle begins again with $h_3^{(2)}(2^{m-2})$nd, that is $x_{h_3^{(2)}(2^{m-2})} = x_0, x_{h_3^{(2)}(2^{m-2})+1} = x_1$, and $x_{h_3^{(2)}(2^{m-2})+1} = x_2$. Thus, the period of $F_3^2(\mathrm{SD}_{2^m}; a,\ b)$ is $h_3^{(2)}(2^{m-2})$. The proof for $k = 4$ is similar and is omitted.

(ii) If $k \geq 5$, we have the polytopic-k-nacci sequence for $\alpha = 2$:

$$
\begin{gathered}
x_0 = a, \quad x_1 = b, \quad x_2 = a^3, \quad x_3 = a^{2^{m-1}-2}b, \quad x_4 = a^{14}, \\
x_5 = a^{u_1}, \quad x_6 = a^{u_2}, \dots, x_k = a^{u_{k-4}}, \\
x_{4h_k^{(2)}(2)} = a^{25}, \quad x_{4h_k^{(2)}(2)+1} = a^8 b, \quad x_{4h_k^{(2)}(2)+2} = a^{27}, \\
x_{4h_k^{(2)}(2)+3} = a^{2^{m-1}-2}b, \quad x_{4h_k^{(2)}(2)+4} = a^{30}, \\
x_{4h_k^{(2)}(2)+5} = a^{u_1+8\cdot\lambda_1}, \quad x_{4h_k^{(2)}(2)+6} = a^{u_2+8\cdot\lambda_2}, \dots, \quad x_{4h_k^{(2)}(2)+k} = a^{u_{k-4}+8\cdot\lambda_{k-4}}, \dots, \\
x_{i\cdot 4h_k^{(2)}(2)} = a^{1+24\cdot i}, \quad x_{i\cdot 4h_k^{(2)}(2)+1} = a^{8\cdot i}b, \quad x_{i.4h_k^{(2)}(2)+2} = a^{3+24\cdot i}, \\
x_{i\cdot 4h_k^{(2)}(2)+3} = a^{2^{m-1}-2}b, \quad x_{i\cdot 4h_k^{(2)}(2)+4} = a^{14+16\cdot i}, \quad x_{i\cdot 4h_k^{(2)}(2)+5} = a^{u_1+8\cdot i\cdot\lambda_1}, \\
x_{i\cdot 4h_k^{(2)}(2)+6} = a^{u_2+8\cdot i\cdot\lambda_2}, \dots, x_{i\cdot 4h_k^{(2)}(2)+k} = a^{u_{k-4}+8\cdot i\cdot\lambda_{k-4}}, \dots,
\end{gathered}
\tag{4.13}
$$

where $\lambda_1, \dots, \lambda_{k-4}$ are natural numbers and $u_1, \dots, u_{k-4}$ are even natural numbers. So we need the smallest $i \in \mathbb{N}$ such that $8 \cdot i = 2^{m-1}$. If we choose $i = 2^{m-4}$, we obtain $x_{h_k^{(2)}(2^{m-1})} = a = x_0, x_{h_k^{(2)}(2^{m-1})+1} = b = x_1, x_{h_k^{(2)}(2^{m-1})+2} = a^3 = x_2, x_{h_k^{(2)}(2^{m-1})+3} = a^{2^{m-1}-2}b = x_3, x_{h_k^{(2)}(2^{m-1})+4} = a^{14} = x_4$, $x_{h_k^{(2)}(2^{m-1})+5} = a^{u_1} = x_5, x_{h_k^{(2)}(2^{m-1})+6} = a^{u_2} = x_6, \dots, x_{h_k^{(2)}(2^{m-1})+k} = a^{u_{k-4}} = x_k$ since $2^{m-2} \cdot h_k^{(2)}(2) = h_k^{(2)}(2^{m-1})$. So we get $P_k^2(\mathrm{SD}_{2^m}; a, b) = h_k^{(2)}(2^{m-1})$ for $k \geq 5$. □

Theorem 4.8. *The periods of the the polytopic-k-nacci sequences in the group SD_{2^m} for initial (seed) sets b, a, and $\alpha = 2$ are as follows:*

(i) $P_k^2(\mathrm{SD}_{2^m}; b, a) = h_k^{(2)}(2^{m-2})$ *for* $2 \leq k \leq 3$,

(ii) $P_k^2(\mathrm{SD}_{2^m}; b, a) = h_k^{(2)}(2^{m-1})$ *for* $k \geq 4$.

Proof. The proof is similar to the proof of Theorem 4.5 and is omitted. □

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