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Research Article

Existence and Uniqueness Theorem for Stochastic Differential Equations with Self-Exciting Switching

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We introduce a new kind of equation, stochastic differential equations with self-exciting switching. Firstly, we give some preliminaries for this kind of equation, and then, we get the main results of our paper; that is, we gave the sufficient condition which can guarantee the existence and uniqueness of the solution.

1. Introduction

In this paper, we propose a new kind of stochastic differential equation (SDE) with self-exciting switching which has the form

$$dX(t) = b(J(t), X(t))dt + \sigma(J(t), X(t))dB(t), \tag{1.1}$$

where $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are appropriate functions, $B(\cdot)$ is a multidimensional standard Brownian motion, and $J(\cdot)$ is a switching process taking values in the space $\mathcal{M} = \{0, 1, 2, \ldots, h, h \leq \infty\}$ and the value of $J(\cdot)$ depends on $X(\cdot)$. The dependence of $J(\cdot)$ on $X(\cdot)$ is given in terms of the value of $X(\cdot)$ to be specified later. Throughout the whole paper, we call such equations SDEs with self-exciting. One of the distinct features is that in these systems, discrete events are highly correlate with continuous dynamics. The SDEs with self-exciting is different from the SDEs with Markovian switching, which is discussed in [1] because for Markovian switching-diffusion processes, the switching process is a continuous-time Markov chain, which is independent of the Brownian motion. It is also different from SDEs with continuous-state-dependent switching (see, e.g., [2]) in which the dependence of the switching process on $X(\cdot)$ is in terms of transition probabilities. So, the equation discussed in our paper is not only different from but not included in these two kind of equations. The equation in our paper can be regarded as the general SDEs switching from one to the other according to the movement of the control, and the switching is determined by the state of the solution of the equation. Moreover, the state space of the switching process in our paper is $\mathcal{M} = \{0, 1, 2, \dots, h\}$, $h \leq \infty$, while it is a finite state space in [1, 2].

Owing to the increasing demands on regime-switching diffusions from emerging applications in financial engineering and wireless communications, much attention has been paid to switching diffusion processes. The introduction of hybrid models makes it possible to describe stochastic volatility in a relatively simple manner. One of the early efforts of using the continuous-state-dependent hybrid models for financial applications can be traced back to [3], in which both the appreciation rate and the volatility of a stock depend on a continuous-time Markov chain.

Our model can be used to describe the mechanism of a market. For example, let $X(\cdot)$ be a market, and the central bank would adjust the bank interest rates according to the changes of the market aperiodically. The adjustment often occurs on a random moment, and often, the adjusted interest rates will sustain for a period of time. Another example is from stock market, in the simplest case, a stock market may be considered to have finite or infinite "modes" or "regimes" (determined by $J(\cdot)$), up and down, resulting from the value of $X(\cdot)$, which can be interpreted as the state of the underlying economy, the general mood of investors in the market, and so on. The rationale is that in the different modes or regimes, the volatility and return rates are very different, and they may change their value according to the value of the stock. Every changing happens on a random time, and the value will keep for a period. This can be described by the changing of the value of $J(\cdot)$ on the surface S_k.

As far as we know, there are no papers which studied this kinds of equations. So motivated by the arising applications, we firstly study the existence and uniqueness of the solution of such equation, and this is a fundamental work for the further studying of the other properties of SDEs with self-exciting. We divided the whole space $\mathbb{R}^n$ into h parts, and the changing of the value of $J(\cdot)$ only occurs on a random time when the stochastic process $X(t)$ touch the curved surface S_k, and it does not change its value until the sample path of the solution process touch S_{k-1} or S_{k+1} for the first time. That is to say that the switching of the equation only occurs on the surface, and it keeps the state until $X(t)$ contact the adjacent surface.

The rest of the paper is arranged as follows. Section 2 begins with some preliminaries of the SDEs with self-exciting switching together with the switching mechanism of the equation. In Section 3, we give the main results of our paper, that is, the existence and uniqueness of the solution to (2.1). In order to illustrate the difference of our model with other switching process, we give some conclusions and discussions in Section 4 at the end of our paper.

2. SDE with Self-Exciting Switching

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets); that is, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is complete. Let $B(t) = (B_t^1, \dots, B_t^m)^T$ be a given m-dimensional standard Brownian motion defined on this probability space. If $x \in \mathbb{R}^n$, its norm is denoted by $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. For convenience, we let

(a) $1L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$: the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variables ξ with $E|\xi|^p < \infty$,

(b) $\mathcal{L}^p([a, b]; \mathbb{R}^n)$: the the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b |f(t)|^p dt < \infty$ a.s.,

(c) $\mathcal{M}^2([a,b];\mathbb{R}^n)$: the family of processes $\{f(t)\}_{a\le t\le b}$ in $\mathcal{L}^p([a,b];\mathbb{R}^n)$ such that $E\int_a^b |f(t)|^p dt < \infty$.

Define $U_k, k \in \mathcal{M} = \{0,1,2,\ldots,h,\ h \le \infty\}$ such that $U_0 \cup U_1 \cup \cdots \cup U_h = \mathbb{R}^n$ and $U_i \cap U_j = \emptyset$ for $i \ne j$. Consider the following stochastic differential equation:

$$dX(t) = b(X(t), J(t))dt + \sigma(X(t), J(t))dB(t), \tag{2.1}$$

with the initial data $X(0) = x_0 \in L^2_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)$ and $J(0) = J_0$, where J_0 is a $\mathcal{M}$-value $\mathcal{F}_0$-measurable random variable. Here, we let

$$\begin{aligned} U_k &= \{x \in \mathbb{R}^n \mid k \le |x| < k+1\}, \\ S_k &= \{x \in \mathbb{R}^n \mid |x| = k\}\ , \end{aligned} \tag{2.2}$$

for $k = 0,1,2,3,\ldots,h,\ h \le \infty$. Then, $U_0 \cup U_1 \cup \cdots \cup U_h = \mathbb{R}^n$ and $U_i \cap U_j = \emptyset$ for $i \ne j$. Hence (2.1) has h regimes, and we assume that $J(t)$ is described by

(i) if $X(0) \in S_k$ then, $J(0) = k$ and $J(t) = k$ until $X(t) \in S_{k-1}$ or $X(t) \in S_{k+1}$,

(ii) if $k < X(0) < k+1$, then

  (a) $J(0) = k$ and $J(t) = k$ until $X(t) \in S_{k-1}$ or $X(t) \in S_{k+1}$, or

  (b) $J(0) = k+1$ and $J(t) = k+1$ until $X(t) \in S_k$ or $X(t) \in S_{k+2}$,

(iii) for any $k \in \mathcal{M}$, $J(t) = k$ if $X(t) \in S_k$.

By the same procedure, we will give the value of $J(t)$ in view of the state of $X(t)$. From the description of $J(t)$ above, we observe that the coexistence of continuous dynamics $X(t)$ and discrete events $J(t)$ in (2.1). The value of $J(t)$ depends on the state of the solution process, and then, $J(t)$ controls the equation. The switching of (2.1) occurs on the curved surface S_k, that is, $J(t)$ changes its value only when $X(t)$ touch on S_k. We call (2.1) is the stochastic differential equations (SDEs) with self-exciting switching. Note that the state space of $\mathcal{M}$ may have infinite states, and this is different from the equations with finite switchings.

3. Existence and Uniqueness for SDE with Self-Exciting Switching

In this section, we will establish the existence-and-uniqueness theorem for SDEs with self-exciting switching which has the form of (2.1). We first give the definition of the solution to (2.1).

Definition 3.1. An $\mathbb{R}^n \times \mathcal{M}$-valued stochastic process $\{(X(t), J(t))\}_{0\le t\le T}$ is called the *solution* to (2.1) if it satisfies

(a) $X(t)$ is t-continuous and $\mathcal{F}_t$-adapted,

(b) $\{b(X(t), J(t))\}_{0\le t\le T} \in \mathcal{L}^1([0,T];\mathbb{R}^n)$ and $\{\sigma(X(t), J(t))\} \in \mathcal{L}^2([0,T];\mathbb{R}^{n\times m})$,

(c) for each $t \in [0,T]$, equation

$$X(t) = x_0 + \int_0^t b(X(s), J(s))\mathrm{d}s + \int_0^t \sigma(X(s), J(s))\mathrm{d}B(s) \tag{3.1}$$

hold with probability 1.

The assertion of uniqueness means that if $X(t,\omega)$ and $\widetilde{X}(t,\omega)$ are two t-continuous process, the pair of the process $(X(t,\omega), J(t,\omega))$ and $(\widetilde{X}(t,\omega), \widetilde{J}(t,\omega))$ satisfy the definition of the solution to (2.1) above, then

$$P\Big(\omega : X(t,\omega) = \widetilde{X}(t,\omega), J(t,\omega) = \widetilde{J}(t,\omega)\ \forall t \in [0,T]\Big) = 1, \tag{3.2}$$

that is, $(X(t,\omega), J(t,\omega))$ is indistinguishable from any other solution $(\widetilde{X}(t,\omega), \widetilde{J}(t,\omega))$.

Let $b(x,i)$ and $\sigma(x,i)$ be an n-vector and $n \times m$-matrix valued functions, respectively, defined for $(x,i) \in \mathbb{R}^n \times \mathcal{M}$. In order to give the existence-and-uniqueness theorem, we define the stopping times sequences $\{\tau_k\}_{k\geq 0}$ on the basis of the movement of $J(t)$ in the following ways:

$$\begin{aligned}
\tau_0 &= 0, \\
\tau_1 &= \inf\big\{t > \tau_0 \mid X(t) \in S_{J(\tau_0)-1} \cup S_{J(\tau_0)+1}\big\}, \\
&\cdots \\
\tau_{k+1} &= \inf\big\{t > \tau_k \mid X(t) \in S_{J(\tau_k)-1} \cup S_{J(\tau_k)+1}\big\}.
\end{aligned} \tag{3.3}$$

Remark 3.2. From the description of $J(t)$ in Section 2, we get the stopping times sequences $\{\tau_k\}_{k>0}$ are the switching points of $J(t)$ and satisfy

(i) every switching point $\{\tau_k\}_{k\geq 0}, k \in \mathcal{M}$ is not a cluster point, since the distance of every curved surface S_k is a positive number,

(ii) for almost every $\omega \in \Omega$, there is a finite $\overline{k} = \overline{k}(\omega)$ for $0 = \tau_0 < \tau_1 < \cdots < \tau_{\overline{k}} = T$ and $\tau_k = T$ if $k > \overline{k}$,

(iii) $J(\cdot)$ is a random constant on every interval $[\tau_k, \tau_{k+1})$, namely,

$$J(t) = J(\tau_k) \quad \text{on } \tau_k \leq t < \tau_{k+1} \text{ for } k \in \mathcal{M}. \tag{3.4}$$

Theorem 3.3. *Assume that there exist two positive constants C and D, for all $x, y \in \mathbb{R}^n$, $t \in [0,T]$ and $i \in \mathcal{M}$, such that (linear growth condition)*

$$|b(x,i)|^2 \vee |\sigma(x,i)|^2 \leq C\Big(1 + |x|^2\Big), \tag{3.5}$$

(Lipschitz condition)

$$\big|b(x,i) - \sigma(y,i)\big|^2 \vee \big|\sigma(x,i) - \sigma(y,i)\big|^2 \leq D\big|x - y\big|^2. \tag{3.6}$$

Then, there exists a unique solution $X(t)$ to (2.1)*, and, moreover,*

$$E\left(\sup_{0\leq t\leq T} |X(t)|^2\right) \leq \Big(1 + 3E|x_0|^2\Big) e^{3CT(T-4)}. \tag{3.7}$$

Proof. From the description of $J(\cdot)$, we can get almost every sample path of $J(\cdot)$ is a right-continuous step function with a finite number of simple jumps on $[0,T]$. For $t \in [\tau_0, \tau_1]$, (2.1) becomes

$$dX(t) = b(X(t), J(\tau_0))dt + \sigma(X(t), J(\tau_0))dB(t), \tag{3.8}$$

with the initial data $X(0) = x_0$ and $J(0) = J(\tau_0) = J_0$, which is determined by the value of x_0. By the classical existence and uniqueness theorem for SDE, (3.8) has a unique solution which belongs to $\mathcal{M}^2([\tau_0, \tau_1]; \mathbb{R}^n)$. In particular, $X(\tau_1) \in L^2_{\mathcal{F}_{\tau_1}}(\Omega; \mathbb{R}^n)$. We next consider (2.1) on $t \in [\tau_1, \tau_2]$ which becomes

$$dX(t) = b(X(t), J(\tau_1))dt + \sigma(X(t), J(\tau_1))dB(t), \tag{3.9}$$

with the initial data $X(\tau_1)$ and $J(\tau_1)$, the value of $J(\tau_1)$ is a random constant determined by $X(\tau_1)$ which touch the curved surface S_1 at τ_1. Again, by the classical existence and uniqueness theorem for SDE, (3.9) has a unique solution which belongs to $\mathcal{M}^2([\tau_1, \tau_2]; \mathbb{R}^n)$. We see that (2.1) has a unique solution $X(t)$ on $[0,T]$ by repeating this procedure. In the same way, as the mean square estimation of the SDEs was proved, we can obtain the assertion (3.7) (see, e.g., [4]). Moreover, we get the solution of (2.1) that cannot be explosive under the conditions of this Theorem, so $J(t) < \infty$ for any $t \in [0,T]$; that is, $J(t)$ takes value in a finite state space and (2.1) can be regard as the finite equations switching from one to the other according to the movement of $J(t)$. □

Remark 3.4. We can prove that the solution to (2.1) has *semigroup property*. In fact, we denote the solution to (2.1) by the pair of the process $(X(t; 0, x_0, J_0), J(t; 0, x_0, J_0))$. Note that

$$X(t) = X(s) + \int_s^t b(X(u), J(u))\mathrm{d}u + \int_s^t \sigma(X(u), J(u))\mathrm{d}B(u) \quad \forall s \le t \le T. \tag{3.10}$$

Then, $X(t)$ can be regarded as the solution of (2.1) with the initial data $X(s) = X(s; 0, x_0, J_0)$ and $J(s) = J(s; 0, x_0, J_0)$. So, we have

$$\begin{aligned}&(X(t; 0, x_0, J_0), J(t; 0, x_0, J_0))\\ &\quad = (X(t; s, X(s; 0, x_0, J_0), J(s; 0, x_0, J_0)), J(t; s, X(s; 0, x_0, J_0), J(s; 0, x_0, J_0)))\end{aligned} \tag{3.11}$$

for $0 \le s \le t \le T$. This demonstrates that the pair of the solution process $(X(t; 0, x_0, J_0), J(t; 0, x_0, J_0))$ has *flow* or *semigroup property*.

Next, we will give the conditions which can guarantee the existence of a unique maximal local solution. For the readers convenience, we give the definition though it is similar to the definition of maximal local solution of SDE.

Definition 3.5. Let σ_∞ be a stopping time such that $0 \le \sigma_\infty \le T$ a.s. The pair of the process $(X(t), J(t))$ is called a *local solution* to (2.1) if

(a) $X(t)$ is an $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted continuous stochastic process on $[0, \sigma_\infty)$;

(b) $J(\cdot)$ is a right-continuous step function with a finite number of simple jumps on $[0, \sigma_\infty)$,

(c) there exists a nondecreasing sequence $\{\sigma_k\}_{k\geq 1}$ of stopping times such that $0 \leq \sigma_k \uparrow \sigma_\infty$ a.s., and

$$X(t) = X(0) + \int_0^{t\wedge\sigma_k} b(X(s), J(s))ds + \int_0^{t\wedge\sigma_k} \sigma(X(s), J(s))dB(s) \tag{3.12}$$

holds for any $t \in [0, T)$ and $k \geq 1$ with probability 1.

If, furthermore,

$$\begin{gathered}\lim_{t\to\sigma_\infty} \sup|X(t)| = \infty \quad \text{whenever } \sigma_\infty < T, \text{a.s.}, \\ \lim_{t\to\sigma_\infty} J(t) = \infty \quad \text{whenever } \sigma_\infty < T, \text{a.s.},\end{gathered} \tag{3.13}$$

then it is called a *maximal local solution*, and σ_∞ is called the *explosion time*.

The uniqueness of a maximal local solution $\{(X(t), J(t)) : 0 \leq t < \sigma_\infty\}$ means that any other maximal solution $\{(\overline{X}(t), \overline{J}(t)) : 0 \leq t < \overline{\sigma}_\infty\}$ is indistinguishable from it, namely, $X(t) = \overline{X}(t)$, $J(t) = \overline{J}(t)$, and $\sigma_\infty = \overline{\sigma}_\infty$ for $0 \leq t < \sigma_\infty$ with probability 1.

Remark 3.6. The value of h may, ∞, and from the definition of maximal local solution, we find that (2.1) may be switching between infinite equations according to the description of $J(t)$ of our paper.

Next, we will give the conditions which can guarantee the existence and uniqueness of maximal local solution to (2.1).

Theorem 3.7. *Assume that for every integer $k \geq 1$, there exists a positive constant h_k such that, for all $t \in [0, T], i \in \mathcal{M}$ and those $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq k$,*

$$\left|b(x, i) - \sigma(y, i)\right|^2 \vee \left|\sigma(x, i) - \sigma(y, i)\right|^2 \leq h_k\left|x - y\right|^2, \tag{3.14}$$

that is, local Lipschitz condition holds. Then, there exists a unique maximal local solution to (2.1).

Proof. For each $N \geq 1$, define the truncation function as

$$b_N(x, i) = \begin{cases} b(x, i), & |x| \leq N, \\ b\left(\dfrac{Nx}{|x|}, i\right), & |x| > N, \end{cases} \tag{3.15}$$

and $\sigma_N(x, i)$ defined similarly. Then b_N and σ_N satisfy the Lipschitz condition and the linear growth condition. Then, Theorem 3.3 implies that there is a unique solution $(X_N(t), J_N(t)$ to

$$dX_N(t) = b_N(X_N(t), J_N(t))dt + \sigma_N(X_N(t), J_N(t))dB(t), \quad t \in [0, T], \tag{3.16}$$

with the initial data $X(0) = x_0$ and $J(0) = J_0$ which is determined by the value of x_0 and $X_N(t)$ in $\mathcal{M}^2([0, T]; \mathbb{R}^n)$, $J_N(t) \in \mathcal{M}$. Define the stopping time as

$$\sigma_N = T \wedge \inf\{t \in [0, T] : |X_N(t)| \leq N\}. \tag{3.17}$$

It is easy to show that

$$X_N(t) = X_{N+1}(t), \qquad J_N(t) = J_{N+1}(t) \quad \text{if } 0 \le t \le \sigma_N. \tag{3.18}$$

This implies that σ_N is increasing and has its limit $\sigma_\infty = \lim_{N\to\infty}\sigma_N$. Define $\{X(t) : 0 \le t < \sigma_\infty\}$ by

$$X(t) = X_N(t), \quad t \in (\sigma_{N-1}, \sigma_N), \quad N \ge 1, \tag{3.19}$$

and $\{J(t) : 0 \le t < \sigma_\infty\}$ by

$$J(t) = J_N(t), \quad t \in (\sigma_{N-1}, \sigma_N), N \ge 1, \tag{3.20}$$

where $\sigma_0 = 0$. Equation (3.18) implies that $X(t\wedge\sigma_N) = X_N(t\wedge\sigma_N)$ and $J(t\wedge\sigma_N) = J_N(t\wedge\sigma_N)$. Therefore, we observe from (3.16) that

$$\begin{aligned} X(t \wedge \sigma_N) &= x_0 + \int_0^{t\wedge\sigma_N} b_N(X(s), J(s))ds + \int_0^{t\wedge\sigma_N} \sigma_N(X(s), J(s))dB(s) \\ &= x_0 + \int_0^{t\wedge\sigma_N} b(X(s), J(s))ds + \int_0^{t\wedge\sigma_N} \sigma(X(s), J(s))dB(s), \end{aligned} \tag{3.21}$$

for any $t \in [0, T)$ and $N \ge 1$. It is also easy to see if $\sigma_\infty < T$, then

$$\begin{aligned} \lim_{t\to\sigma_\infty} \sup|X(t)| &\ge \lim_{N\to\infty} \sup|X(\sigma_N)| = \lim_{N\to\infty} \sup|X_K(\sigma_N)| = \infty, \\ \lim_{t\to\sigma_\infty} J(t) &\ge \lim_{N\to\infty} J(\sigma_N) = \lim_{N\to\infty} \sup J_K(\sigma_N) = \infty. \end{aligned} \tag{3.22}$$

Hence, $\{(X(t), J(t)) : 0 \le t < \sigma_\infty\}$ is a maximal local solution. To show the uniqueness, let $\{(\overline{X}(t), \overline{J}(t)) : 0 \le t < \overline{\sigma}_\infty\}$ be another maximal local solution. Define

$$\overline{\sigma}_N = \overline{\sigma}_\infty \wedge \inf\left\{t \in [0, \overline{\sigma}_\infty) : \left|\overline{X}(t)\right| \ge N\right\}. \tag{3.23}$$

It is easy to show that $\overline{\sigma}_N \to \overline{\sigma}_\infty$ a.s. as $N \to \infty$ and

$$\mathbb{P}\left\{X(t) = \overline{X}(t), J(t) = \overline{J}(t) \forall\, t \in [0, \sigma_N \wedge \overline{\sigma}_N)\right\} = 1 \quad \forall\, N \ge 1. \tag{3.24}$$

Letting $N \to \infty$ yields that

$$\mathbb{P}\left\{X(t) = \overline{X}(t), J(t) = \overline{J}(t) \forall\, t \in [0, \sigma_\infty \wedge \overline{\sigma}_\infty)\right\} = 1. \tag{3.25}$$

We claim that

$$\sigma_\infty = \overline{\sigma}_\infty \text{ a.s.} \tag{3.26}$$

If this is not true, then $\{\sigma_\infty < \overline{\sigma}_\infty\}$ or $\{\sigma_\infty > \overline{\sigma}_\infty\}$. Then, for almost all $\omega \in \{\sigma_\infty < \overline{\sigma}_\infty\}$, we have

$$\left|\overline{X}(\sigma_\infty, \omega)\right| = \lim_{N\to\infty}\left|\overline{X}(\sigma_N, \omega)\right| = \lim_{N\to\infty}|X(\sigma_N, \omega)| = \infty, \tag{3.27}$$

which contradicts the fact that $\overline{X}(t,\omega)$ is continuous on $t \in [0, \overline{\sigma}_\infty)$. So we must have $\sigma_\infty \geq \overline{\sigma}_\infty$ a.s. Similarly, we can show $\sigma_\infty \leq \overline{\sigma}_\infty$ a.s. Therefore, we must have $\sigma_\infty = \overline{\sigma}_\infty$ a.s.. □

Corollary 3.8. *For $i \in \mathcal{M}$ and $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq N$, suppose that the local Lipschitz condition*

$$\left|b(x,i) - \sigma(y,i)\right|^2 \vee \left|\sigma(x,i) - \sigma(y,i)\right|^2 \leq h_N\left|x - y\right|^2, \tag{3.28}$$

and linear growth hold. Then, we still have the conclusions of Theorem 3.3.

Proof. For each $N \geq 1$, define the truncation function $b_N(x,i)$ and $\sigma_N(x,i)$ as in the proof of Theorem 3.7. Then, b_N and σ_N satisfy the linear growth condition (3.5) and the Lipschitz condition (3.6). So, we can prove that

$$E\left(\sup_{0\leq t\leq T}|X_N(t)|^2\right) \leq \left(1 + 3E|x_0|^2\right)e^{3C(T)(T+4)}. \tag{3.29}$$

That is,

$$E\left(\sup_{0\leq t\leq \sigma_N}|X(t)|^2\right) \leq \left(1 + 3E|x_0|^2\right)e^{3C(T)(T+4)}. \tag{3.30}$$

Letting $N \to \infty$, we get

$$E\left(\sup_{0\leq t\leq \sigma_\infty}|X(t)|^2\right) \leq \left(1 + 3E|x_0|^2\right)e^{3C(T)(T+4)}. \tag{3.31}$$

This implies that $\sigma_\infty = T$ a.s. and $J(t) < \infty$ for $t \in [0, T]$. So, we can get the the conclusion of Theorem 3.3. □

Theorem 3.9. *Assume that the monotone condition holds; that is, there exists a positive constant C such that*

$$x^T b(x,i) + \frac{1}{2}|\sigma(x,i)|^2 \leq C\left(1 + |x|^2\right), \tag{3.32}$$

for all $(x,i) \in \mathbb{R}^n \times \mathcal{M}$. If also local Lipschitz condition (3.28) holds, then there exists a unique solution $X(t)$ to (2.1) in $\mathcal{M}^2([0,T];\mathbb{R}^n)$.

The local Lipschitz condition guarantees that the solution exists in $[0, \sigma_\infty)$ *while the monotone condition guarantees that the solution exists on the whole interval* $[0, T]$. *This Theorem can be proved in a similar way as Corollary 3.8, and we omit it here.*

To give more general result, we suppose that the transition probability of $J(t)$ *is dependent on the value of* $X(t)$*; that is, for* $i \neq j$,

$$\mathbb{P}\{J(t+\Delta) = j \mid J(t) = i, X(s), J(s), s \le t\} = \gamma_{ij}(X(t))\Delta + o(\Delta). \tag{3.33}$$

Then, the evolution of the switching process $J(\cdot)$ *can be represented by a stochastic integral with respect to a Poisson random measure (see, e.g., [1, 5]). Suppose that* $\Gamma(x) = (\gamma_{ij}(x))$ *satisfies the* q*-property (see, e.g., [2, 6]). For a suitable function* $h(\cdot,\cdot)$,

$$\Gamma(x)h(x,\cdot)(i) = \sum_{j\in\mathbb{S}} \gamma_{ij}(x)\big(h(x,j) - h(x,i)\big), \quad \text{for each } i \in \mathcal{M}. \tag{3.34}$$

Let $C^2(\mathbb{R}^n \times \mathcal{M}; \mathbb{R}_+)$ *denote the family of all nonnegative functions* $V(x,i)$ *on* $\mathbb{R}^n \times \mathcal{M}$ *which are continuously twice differentiable in* x. *If* $V(\cdot,i) \in C^2(\mathbb{R}^n \times \mathcal{M}; \mathbb{R}_+)$, *we introduce an important operator* $\mathcal{L}V(\cdot,i)$ *(see, e.g., [2]) associated with the process* $(X(t), J(t))$ *defined by the* (2.1) *as follows:*

$$\mathcal{L}V(\cdot,i) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x,i)\frac{\partial^2 V(x,i)}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x,i)\frac{\partial V(x,i)}{\partial x_j} + \Gamma(x)V(x,\cdot)(i), \tag{3.35}$$

where $a = \sigma\sigma^T$. *Then, the generalized Itô formula (see [7, Lemma 3.1]) reads as follows: for* $V(x,i) \in \mathbb{R}^n \times \mathbb{S}$,

$$EV(X(\tau_2), r(\tau_2)) = EV(X(\tau_1), r(\tau_1)) + E\int_{\tau_1}^{\tau_2} \mathcal{L}V(X(s), r(s))ds, \tag{3.36}$$

with any stopping times $0 \le \tau_1 \le \tau_2 < \infty$ *as long as the integrations involved exist and are finite.*

Theorem 3.10. *Assume that there is a function* $V(x,i) \in C^2(\mathbb{R}^n \times \mathcal{M}; \mathbb{R}_+)$ *and a constant* $\alpha > 0$ *such that*

$$\lim_{|x|\to\infty}\left(\inf_{i\in\mathcal{M}} V(x,i)\right) = \infty, \tag{3.37}$$

$$\mathcal{L}V(x,i) \le \alpha(1 + V(x,i)), \quad \forall (x,i) \in \mathbb{R}^n \times \mathcal{M}. \tag{3.38}$$

If also the local Lipschitz condition (3.28) *holds, then there exists a unique global solution* $(X(t), J(t))$ *to* (2.1).

Proof. It is easy to observe from Theorem 3.7 and the local Lipschitz condition (3.28) that the maximal solution $(X(t), J(t))$ exists on $[0, \sigma_\infty)$, where σ_∞ is the explosion time. In order to prove the existence of the unique global solution, we only need to show that $\sigma_\infty = \infty$ a.s. If this is not true, then for any $\varepsilon > 0$, there exist a positive constant T such that

$$\mathbb{P}\{\sigma_\infty \le T\} > \varepsilon. \tag{3.39}$$

For $V(x,i) \in C^2(\mathbb{R}^n \times \mathcal{M}; \mathbb{R}_+)$, we define

$$\rho_k = \inf\{V(x,i) : |x| \geq k, t \in [0,T], i \in \mathcal{M}\}. \tag{3.40}$$

Then, by the condition (3.37), we get that $\rho_k \to \infty$ when $k \to \infty$. On the other hand, we define the sequences of stopping time

$$\sigma_k = \inf\{t \geq 0 : |X(t)| \geq k\}, \tag{3.41}$$

for each $k \geq 1$. Then, there exists a sufficiently large integer N such that

$$\mathbb{P}\{\sigma_k \leq T\} > \frac{\varepsilon}{2}, \quad \forall k \geq N, \tag{3.42}$$

since $\sigma_k \to \sigma_\infty$ a.s. as $k \to \infty$. For any $t \in [0,T]$, by (3.34), and condition (3.38) we have that

$$\begin{aligned} EV(X(t \wedge \sigma_k), J(t \wedge \sigma_k)) &= V(x_0, J_0) + E\int_0^{t\wedge\sigma_k} \mathcal{L}V(X(s), J(s))ds \\ &\leq V(x_0, J_0) + \alpha T + \alpha\int_0^t EV(X(s \wedge \sigma_k), J(s \wedge \sigma_k))ds. \end{aligned} \tag{3.43}$$

So, the Gronwall inequality implies that

$$EV(X(T \wedge \sigma_k), J(T \wedge \sigma_k)) \leq [V(x_0, J_0) + \alpha T]e^{\alpha T}. \tag{3.44}$$

Therefore,

$$E\big(I_{\{\sigma_k \leq T\}} V(X(\sigma_k), J(\sigma_k))\big) \leq [V(x_0, J_0) + \alpha T]e^{\alpha^T}. \tag{3.45}$$

By the definition of ρ_k, we have

$$\rho_k \mathbb{P}\{\sigma_k \leq T\} \leq E\big(I_{\{\sigma_k \leq T\}} V(X(\sigma_k), J(\sigma_k))\big). \tag{3.46}$$

Fix $k \geq N$, we observe from (3.42) and (3.45) that

$$[V(x_0, J_0) + \alpha T]e^{\alpha T} \geq \rho_k \mathbb{P}\{\sigma_k \leq T\} \geq \frac{\varepsilon\rho_k}{2}. \tag{3.47}$$

Letting $k \to \infty$ yields a contradiction so, we must have $\sigma_\infty = \infty$ a.s. □

Remark 3.11. For the equation in our paper, $J(t)$ may take values in a infinite state space; that is, the equation may be switching between infinite equations, and this is drastically different from stochastic differential equations with Markovian switching in [1] and stochastic differential equations with continuous-state-dependent switching in [2], because there the switching process taking values in a finite state space.

4. Conclusions and Discussions

This paper mainly discussed a new kind of SDE, that is, the SDE with self-exciting switching. As far as we know, there are no papers which involve this kind of equations. The main feature of this equation is that the switching depends on the state of the solution, and the solution is also affected by the switching through the equation. The dependence of $J(\cdot)$ on $X(\cdot)$ is not exactly the same as the equation discussed in [2]. Moreover, the equation may switch between infinite equations. In order to illustrate the difference of the equation with the switching, we give the following example. Consider the equation

$$dX(t) = b(X(t), J(t))dt + \sigma(X(t), J(t))dB(t), \quad t \geq 0. \tag{4.1}$$

In view of [2], the switching process $J(t)$ has finite states with generator $Q(x)$; for example, we assume that it has two states and the x-dependent generator is expressed as

$$Q(x) = \begin{pmatrix} -5 - \sin x & 5 + \sin x \\ 2 + \sin^2 x & -2 - \sin^2 x \end{pmatrix}. \tag{4.2}$$

Then, (4.1) can be regard as the following two equations:

$$\begin{aligned} dX(t) &= b(X(t), 1)dt + \sigma(X(t), 1)dB(t)\ , \\ dX(t) &= b(X(t), 2)dt + \sigma(X(t), 2)dB(t), \end{aligned} \tag{4.3}$$

switching back and forth from one to the other according to the movement of the jump process $J(t)$ which has the generator $Q(x)$. That is to say that the dependence of $J(t)$ on x is given in terms of transition probabilities of $X(t)$. If $Q(x) = Q$ that generates a Markov chain independent of the Brownian motion, then (4.1) can be regard as the two equations switching from one to the other according to the movement of the Markov chain, and this is the equation with Markovian switching which is discussed in [1].

For example, we define U_1 and U_2 such that $U_1 \cup U_2 = \mathbb{R}^n$ and $U_1 \cap U_2 = \emptyset$. According to the description of our paper, $J(t)$ is defined by

$$J(t) = \begin{cases} 1 & \text{if } X(t) \in U_1, \\ 2 & \text{if } X(t) \in U_2. \end{cases} \tag{4.4}$$

So (4.1) can be regard as the two (4.3) switching back and forth from one to the other according to the movement of $J(t)$. Of course, if we define $U_1, U_2, \ldots$ such that $\cup_{i=1}^{\infty} \mathrm{U}_i = \mathbb{R}^n$ and $U_i \cap U_j = \emptyset$ for $i \neq j$, then $J(t)$ will be defined by

$$J(t) = \begin{cases} 1 & \text{if } X(t) \in U_1, \\ 2 & \text{if } X(t) \in U_2, \\ & \cdots, \end{cases} \tag{4.5}$$

that is, $J(t)$ take values in a infinite space $\mathcal{M} = \{1, 2, \ldots, \infty\}$. Noting that the value of $J(t)$ depends on the value of the solution of equation, the equation in our paper is different from the equation discussed in [1, 2].

In the future, we will concentrate all our efforts on other properties of this kind of equation discussed in our paper, such as L^p-estimates, kinds of stabilities and other asymptotic properties. Most importantly, SDE with self-exciting switching can be used to describe many practical models. So, the work in our paper is a foundation of the future work on this aspect.

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References

[1] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, UK, 2006.
[2] G. Yin and C. Zhu, "Properties of solutions of stochastic differential equations with continuous-state-dependent switching," *Journal of Differential Equations*, vol. 249, no. 10, pp. 2409–2439, 2010.
[3] G. Barone-Adesi and R. Whaley, "Efficient analytic approximation of American option values," *The Journal of Finance*, vol. 42, pp. 301–320, 1987.
[4] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Series in Mathematics & Applications, Horwood Publishing, Chichester, UK, 1997.
[5] A. V. Skorohod, *Asymptotic Methods in the Theory of Stochastic Differential Equations*, Amer. Math. Soc., Providence, RI, USA, 1989.
[6] G. G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, vol. 37 of *Applications of Mathematics*, Springer, New York, NY, USA, 1998.
[7] V. Kolmanovskii, N. Koroleva, T. Maizenberg, X. Mao, and A. Matasov, "Neutral stochastic differential delay equations with Markovian switching," *Stochastic Analysis and Applications*, vol. 21, no. 4, pp. 819–847, 2003.