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Research Article

On the Basic k-nacci Sequences in Finite Groups

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We define the basic k-nacci sequences and the basic periods of these sequences in finite groups, then we obtain the basic periods of the basic k-nacci sequences and the periods of the k-nacci sequences in symmetric group S_4, its subgroups, and binary polyhedral groups which related with these groups.

1. Introduction

The study of Fibonacci sequences in groups began with the earlier work of Wall [1], where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid-eighties, Wilcox extended the problem to Abelian groups [2]. The theory is expanded to some finite simple groups by Campbell et al. [3]. There, they defined the Fibonacci length of the Fibonacci orbit and the basic Fibonacci length of the basic Fibonacci orbit in a 2-generator group. The concept of Fibonacci length for more than two generators has also been considered; see, for example, [4, 5]. Also, the theory has been expanded to the nilpotent groups; see, for example, [6, 7]. Other works on Fibonacci length are discussed in, for example, [8–10]. Knox proved that the periods of k-nacci (k-step Fibonacci) sequences in dihedral groups were equal to $2k + 2$ [11]. Deveci, Karaduman, and Campbell examined the period of the k-nacci sequences in some finite binary polyhedral groups in [12]. Recently, k-nacci sequences have been investigated; see, for example, [13, 14].

This paper defines the basic k-nacci sequences and the periods of these sequences in finite groups and discusses the basic periods of the basic k-nacci sequences and the periods of the k-nacci sequences in the symmetric group S_4, alternating group A_4, D_2 four-group, and binary polyhedral groups $\langle 2,3,4\rangle$ and $\langle 2,3,3\rangle$ with related S_4 and A_4, respectively. We consider the groups S_4, A_4, $\langle 2,3,4\rangle$, and $\langle 2,3,3\rangle$ both as 2-generator and as 3-generator groups.

A *k-nacci sequence* in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases} \tag{1.1}$$

We also require that the initial elements of the sequence $x_0, x_1, x_2, \ldots, x_{j-1}$ generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \ldots, x_{j-1})$ [11].

A sequence of group elements is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the *period of the sequence*. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \ldots$ is periodic after the initial element a and has period 4. A sequence of group elements is *simply periodic* with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \ldots$ is simply periodic with period 6. In [11], Knox had denoted the period of a k-nacci sequence $F_k(G; x_0, x_1, \ldots, x_{j-1})$ by $P_k(G; x_0, x_1, \ldots, x_{j-1})$.

Definition 1.1. For a finitely generated group $G = \langle A\rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_i = a_{i+1}$, $0 \le i \le n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \ge 0$ is called the *Fibonacci orbit* of G with respect to the generating set A, denoted as $F_A(G)$ [4].

Definition 1.2. If $F_A(G)$ is simply periodic, then the period of the sequence is called the *Fibonacci length* of G with respect to generating set A, written, $LEN_A(G)$ [4].

Notice that the orbit of a k-generated group is a k-nacci sequence.

Let G be a finite j-generator group, and let X be the subset of $\underbrace{G \times G \times G \cdots \times G}_{j}$ such that $(x_0, x_1, \ldots, x_{j-1}) \in X$ if and only if G is generated by x_0, x_1, $\ldots$, x_{j-1}. We call $(x_0, x_1, \ldots, x_{j-1})$ a *generating j-tuple* for G.

2. Basic Period of Basic k-nacci Sequence

To examine the concept more fully, we study the action of automorphism group Aut$G$ of G on X and on the k-nacci sequences $F_k(G : x_0, x_1, \ldots x_{j-1})$, $(x_0, x_1, \ldots, x_{j-1}) \in X$. Now, Aut$G$ consists of all isomorphism $\theta : G \to G$ and if $\theta \in$ Aut$G$ and $(x_0, x_1, \ldots, x_{j-1}) \in X$, then $(x_0\theta, x_1\theta, \ldots, x_{j-1}\theta) \in X$.

For a subset $A \subseteq G$ and $\theta \in$ Aut$G$, the image of A under θ is

$$A\theta = \{a\theta : a \in A\}. \tag{2.1}$$

Definition 2.1. For a generating pair $(x, y) \in X$, the basic *Fibonacci orbit* $\overline{F}_{x,y}$ of the *basic length* m is defined by the sequence $\{b_i\}$ of elements of G such that

$$b_0 = x, \qquad b_1 = y, \qquad b_{i+2} = b_i b_{i+1}, \quad i \ge 0, \tag{2.2}$$

where $m \geq 1$ is the least integer with

$$b_0 = b_m\theta, \qquad b_1 = b_{m+1}\theta, \tag{2.3}$$

for some $\theta \in \mathrm{Aut}G$. Since b_m, b_{m+1} generate G, it follows that θ is uniquely determined. For more information, see [3].

Lemma 2.2. *Let $(x_0, x_1, \ldots, x_{j-1}) \in X$ and let $\theta \in \mathrm{Aut}\, G$, then $(F_k(G : x_0, x_1, \ldots x_{j-1}))\theta = F_k(G : x_0\theta, x_1\theta, \ldots x_{j-1}\theta)$.*

Proof. Let $F_k(G : x_0, x_1, \ldots x_{j-1}) = \{b_i\}$. The result is obvious since $\{b_i\}\theta = \{b_i\theta\}$ and

$$b_{i+k}\theta = (b_i b_{i+1} \cdots b_{i+k-1})\theta = b_i\theta b_{i+1}\theta \cdots b_{i+k-1}\theta. \tag{2.4}$$

Each generating *j-tuple* $(x_0, x_1, \ldots, x_{j-1}) \in X$ maps to $|\mathrm{Aut}G|$ distinct elements of X under the action of elements of AutG. Hence, there are

$$d_j(G) = |X| / |\mathrm{Aut}G|, \tag{2.5}$$

(where $|X|$ means the number of elements of X) nonisomorphic generating *j-tuples* for G. The notation $d_j(G)$ was introduced in [15].

Suppose that ω elements of AutG map $F_k(G : x_0, x_1, \ldots x_{j-1})$ into itself, then there are $|\mathrm{AutG}|/\omega$ distinct k-nacci sequences $F_k(G : x_0\theta, x_1\theta, \ldots x_{j-1}\theta)$ for $\theta \in \mathrm{Aut}G$. □

Definition 2.3. For a *j-tuple* $(x_0, x_1, \ldots, x_{j-1}) \in X$, the basic k-nacci sequence $\overline{F}_k(G : x_0, x_1, \ldots x_{j-1})$ of the basic period m is a sequence of group elements $b_0, b_1, b_2, \ldots, b_n, \ldots$ for which, given an initial (seed) set $b_0 = x_0,\ \ b_1 = x_1,\ \ b_2 = x_2, \ldots, b_{j-1} = x_{j-1}$, each element is defined by

$$b_n = \begin{cases} b_0 b_1 \cdots b_{n-1} & \text{for } j \leq n < k, \\ b_{n-k} b_{n-k+1} \cdots b_{n-1} & \text{for } n \geq k, \end{cases} \tag{2.6}$$

where $m \geq 1$ is the least integer with

$$b_0 = b_m\theta,\ \ b_1 = b_{m+1}\theta,\ \ b_2 = b_{m+2}\theta,\ \ldots,\ b_{k-1} = b_{m+k-1}\theta, \tag{2.7}$$

for some $\theta \in \mathrm{Aut}G$. Since G is a finite j-generator group and $b_m, b_{m+1}, \ldots, b_{m+j-1}$ generate G, it follows that θ is uniquely determined. The basic k-nacci sequence $\overline{F}_k(G : x_0, x_1, \ldots x_{j-1})$ is finite containing m element.

In this paper, we denote the basic period of the basic k-nacci sequence $\overline{F}_k(G : x_0, x_1, \ldots x_{j-1})$ by $BP_k(G; x_0, x_1, \ldots, x_{j-1})$.

From the definitions, it is clear that the periods of the k-nacci sequences and the basic k-nacci sequences in a finite group depend on the chosen generating set and the order of the generating elements.

Theorem 2.4. *Let G be a finite group and $(x_0, x_1, \ldots, x_{j-1}) \in X$. If $P_k(G; x_0, x_1, \ldots, x_{j-1}) = n$ and $BP_k(G; x_0, x_1, \ldots, x_{j-1}) = m$, then m divides n, and there are n/m elements of AutG which map $F_k(G : x_0, x_1, \ldots x_{j-1})$ into itself.*

Proof. We have $n = m\lambda$ where λ is the order of automorphism $\theta \in \mathrm{Aut}G$ since

$$F_k(G : x_0, x_1, \ldots x_{j-1}) = \overline{F}_k(G : x_0, x_1, \ldots x_{j-1}) \cup \overline{F}_k(G : x_0\theta, x_1\theta, \ldots x_{j-1}\theta) \cup \overline{F}_k\left(G : x_0\theta^2, x_1\theta^2, \ldots x_{j-1}\theta^2\right) \cup \cdots \tag{2.8}$$

and $BP_k(G; x_0, x_1, \ldots, x_{j-1}) = BP_k(G; x_0\theta, x_1\theta, \ldots, x_{j-1}\theta)$. Clearly, 1, θ, θ^2, $\ldots$, $\theta^{\lambda-1}$ map $F_k(G : x_0, x_1, \ldots x_{j-1})$ into itself. □

3. Applications

Definition 3.1. The *polyhedral group* (l, m, n) for $l, m, n > 1$ is defined by the presentation

$$\left\langle x, y, z : x^l = y^m = z^n = xyz = e \right\rangle, \tag{3.1}$$

or

$$\left\langle x, y : x^l = y^m = (xy)^n = e \right\rangle. \tag{3.2}$$

The *polyhedral group* (l, m, n) is finite if and only if the number

$$\mu = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn \tag{3.3}$$

is positive, that is, in the cases $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$. Its order is $2lmn/\mu$. A_4, S_4, and A_5 are the groups $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$, respectively. Also, the groups A_4, S_4, and A_5 being isomorphic to the groups of rotations of the regular tetrahedron, octahedron, and icosahedron. Using Tietze transformations, we may show that $(l, m, n) \cong (m, n, l) \cong (n, l, m)$. For more information on these groups, see, [16, 17, pp. 67-68].

Definition 3.2. The *binary polyhedral group* $\langle l, m, n\rangle$, for $l, m, n > 1$, is defined by the presentation

$$\left\langle x, y, z : x^l = y^m = z^n = xyz \right\rangle, \tag{3.4}$$

or

$$\left\langle x, y : x^l = y^m = (xy)^n \right\rangle. \tag{3.5}$$

The *binary polyhedral group* $\langle l, m, n\rangle$ is finite if and only if the number $k = lmn(1/l + 1/m + 1/n - 1) = mn + nl + lm - lmn$ is positive. Its order is $4lmn/k$.

For more information on these groups, see [17, pp. 68–71].

Definition 3.3. Let $f_n^{(k)}$ denote the nth member of the k-step Fibonacci sequence defined as

$$f_n^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \quad \text{for } n > k, \tag{3.6}$$

with boundary conditions $f_i^{(k)} = 0$ for $1 \leq i < k$ and $f_k^{(k)} = 1$. Reducing this sequence by a modulo m, we can get a repeating sequence, which we denote by

$$f(k,m) = \left(f_1^{(k,m)},\ f_2^{((k,m)},\ \ldots,\ f_n^{(k,m)} \ldots\right), \tag{3.7}$$

where $f_i^{(k,m)} = f_i^{(k)} (\operatorname{mod} m)$. We then have that $(f_1^{(k,m)},\ f_2^{(k,m)}, \ldots,\ f_k^{(k,m)}) = (0, 0, \ldots 0, 1)$, and it has the same recurrence relation as in (3.6) [18].

Theorem 3.4 ($f(k,m)$ is a periodic sequence [18]). *Let $h_k(m)$ denote the smallest period of $f(k,m)$, called the period of $f(k,m)$ or the wall number of the k-step Fibonacci sequence modulo m.*

Theorem 3.5. *The periods of the k-nacci sequences and the basic periods of the basic k-nacci sequences in the group S_4 are as follows.*

if the group is defined by the presentation $S_4 = \langle x, y, z : x^2 = y^3 = z^4 = xyz = e \rangle$, then

(i) *if $k = 2$, $P_2(S_4; y, z, x) = 18$ and $BP_2(S_4; y, z, x) = 9$,*

(ii) *if $k > 2$, $P_k(S_4; x, y, z) = 6k + 6$ and $BP_k(S_4; x, y, z) = 3k + 3$.*

If S_4 has the presentation $S_4 = \langle x, y : x^2 = y^3 = (xy)^4 = e \rangle$, then

(i′) *if $k = 2$, $P_2(S_4; x, y) = 18$ and $BP_2(S_4; x, y) = 9$,*

(ii′) *if $k > 2$, $P_k(S_4; x, y) = 6k + 6$ and $BP_k(S_4; x, y) = 3k + 3$.*

Proof. Firstly, let us consider the 3-generator case. We first note that $|x| = 2$, $|y| = 3$, and $|z| = 4$ (where $|x|$ means the order of x).

(i) If $k = 2$, we have the sequence for the generating triple (y, z, x),

$$\begin{gathered} y,\ z,\ x,\ y^2,\ xy^2,\ y^2xy^2,\ z^2y,\ z^2yz^3y,\ yxy, xyx, \\ xy^2, x, xy^2x,\ y^2x,\ yxy,\ yxz,\ zy,\ y^2xy^2,\ y,\ z,\ x, \ldots, \end{gathered} \tag{3.8}$$

which has period 18 and the basic period 9 since $x\theta = x$, $y\theta = xyx$, and $z\theta = xy^2$, where θ is the inner automorphism induced by conjugation by x.

(ii) If $k = 3$, we have the sequence for the generating triple (x, y, z),

$$\begin{gathered} x,\ y,\ z,\ e,\ x,\ y^2,\ xy^2,\ xzy^2,\ x,\ y,\ yxy^2,\ xzy^2,\ x, \\ y^2,\ yx,\ e,\ x,\ y,\ xy,\ z^2,\ x,\ y^2\ \ zy,\ z^2,\ x,\ y,\ z \ldots, \end{gathered} \tag{3.9}$$

which has period 24 and the basic period 12 since $x\theta = x$, $y\theta = y^2$, and $z\theta = yx$ where θ is an outer automorphism of order 2.

If $k \geq 4$, the first k elements of sequence for the generating triple (x, y, z) are

$$x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = xyz, \quad x_4 = (xyz)^2 \ldots, \; x_{k-1} = (xyz)^{2^{k-4}}. \tag{3.10}$$

Thus, using the above information, sequence reduces to

$$x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = e, \ldots, e, \; x_{k-1} = e, \tag{3.11}$$

where $x_j = e$ for $3 \leq j \leq k-1$. Thus,

$$\begin{gathered} x_k = e, \quad x_{k+1} = x, \quad x_{k+2} = y^2, \quad x_{k+3} = xy^2, \quad x_{k+4} = xzy^2, \\ x_{k+5} = e, \; \ldots, \; e, x_{2k+1} = e, \; x_{2k+2} = x, x_{2k+3} = y, \\ x_{2k+4} = yxy^2, \; x_{2k+5} = xzy^2, \; x_{2k+6} = e, \ldots, \; e, \; x_{3k+2} = e, \\ x_{3k+3} = x, x_{3k+4} = y^2, x_{3k+5} = yx, \; x_{3k+6} = e, \; \ldots, \; e, x_{4k+3} = e, \\ x_{4k+4} = x, x_{4k+5} = y, x_{4k+6} = xy, \; x_{4k+7} = z^2, \\ x_{4k+8} = e, \; \ldots, \; e, \; x_{5k+4} = e, \; x_{5k+5} = x, \; x_{5k+6} = y^2, \\ x_{5k+7} = zy, \; x_{5k+8} = z^2, \; x_{5k+9} = \; e, \ldots, \; e, \; x_{6k+5} = e, \end{gathered} \tag{3.12}$$

where $x_j = e$ for $k+5 \leq j \leq 2k+1$, $2k+6 \leq j \leq 3k+2$, $3k+6 \leq j \leq 4k+3$, $4k+8 \leq j \leq 5k+4$, and $5k+9 \leq j \leq 6k+5$.

We also have

$$x_{6k+6} = \prod_{i=5k+6}^{6k+5} x_i = x, \qquad x_{6k+7} = \prod_{i=5k+7}^{6k+6} x_i = y, \qquad x_{6k+8} = \prod_{i=5k+8}^{6k+7} x_i = z. \tag{3.13}$$

Since the elements succeeding x_{6k+6}, x_{6k+7}, and x_{6k+8} depend on x, y, and z for their values, the cycle begins again with the $6k+6^{\text{th}}$ element, that is, $x_0 = x_{6k+6}, \;\; x_1 = x_{6k+7}, \;\; x_2 = x_{6k+8}, \; \ldots$. Thus, $P_k(S_4; x, y, z) = 6k+6$.

It is easy to see from the above sequence that

$$x_{3k+3} = x, \quad x_{3k+4} = y^2, \quad x_{3k+5} = yx, \quad x_{3k+6} = e, \; \ldots, e, \quad x_{4k+2} = e. \tag{3.14}$$

$BP_k(S_4; x, y, z) = 3k+3$ since $x\theta = x$, $y\theta = y^2$, and $z\theta = yx$ where θ is an outer automorphism of order 2.

Secondly, let us consider the 2-generator case. We first note that $|x| = 2$, $|y| = 3$, and $|xy| = 4$.

(i′) If $k = 2$, $P_2(S_4; x, y) = 18$ and $BP_2(S_4; x, y) = 9$ since $x\theta = x$ and $y\theta = xyx$ where θ is the inner automorphism induced by conjugation by x.

(ii′) If $k > 2$, $P_k(S_4; x, y) = 6k + 6$ and $BP_k(S_4; x, y) = 3k + 3$ since $x\theta = x$ and $y\theta = y^2$ where θ is an outer automorphism of order 2.

The proofs are similar to above and are omitted. □

Theorem 3.6. *The periods of the k-nacci sequences and the basic periods of the basic k-nacci sequences in the binary polyhedral group $\langle 2,3,4\rangle$ are as follows.*

If the group is defined by the presentation $\langle 2,3,4\rangle = \langle x, y, z : x^2 = y^3 = z^4 = xyz\rangle$, then

(i) *if $k = 2$, $P_2(\langle 2,3,4\rangle; y, z, x) = 18$ and $BP_2(\langle 2,3,4\rangle; y, z, x) = 9$,*

(ii) *if $k > 2$, $P_k(\langle 2,3,4\rangle; x, y, z) = 6k + 6$ and $BP_k(\langle 2,3,4\rangle; x, y, z) = 6k + 6$.*

If the group is defined by the presentation $\langle 2,3,4\rangle = \langle x, y : x^2 = y^3 = (xy)^4\rangle$, then

(i′) *if $k = 2$, $P_2(\langle 2,3,4\rangle; x, y) = 18$ and $BP_2(\langle 2,3,4\rangle; x, y) = 9$,*

(ii′) *if $k > 2$, $P_k(\langle 2,3,4\rangle; x, y) = 6k + 6$ and $BP_k(\langle 2,3,4\rangle; x, y) = 6k + 6$.*

Proof. Firstly, let us consider the 2-generator case. We first note that $|x| = 4$, $|y| = 6$, and $|xy| = 8$.

(i′) If $k = 2$, we have the sequence for the generating pair (x, y),

$$\begin{gathered} x,\ y,\ xy,\ yxy,\ xy^2xy,\ xyxy^2x,\ y^2xy^2,\ xy^5x,\ xy,\ x^3, \\ xyx^3,\ yx^3,\ y^2xy^2,\ y^2xyx,\ yxy^2,\ yxy, y^2,\ y^4x,\ x,\ y,\ \ldots, \end{gathered} \tag{3.15}$$

which has period 18 and the basic period 9 since $x\theta = x^3$ and $y\theta = x^3yx$ where θ is a outer automorphism of order 2.

(ii′) If $k = 3$, we have the sequence for the generating pair (x, y),

$$\begin{gathered} x,\ y,\ xy,\ (xy)^2,\ x,\ y^2,\ y^5xy,\ (xy)^2,\ x,\ y,\ (xy)^3,\ (xy)^4, x^3, \\ y^2, xy^2, (yx)^2, x^3, y,\ yxy^2, (yx)^2,\ x^3,\ y^2,\ y^4x,\ e,\ x,\ y,\ xy,\ \ldots, \end{gathered} \tag{3.16}$$

which has period 24 and the basic period 24 since $x\theta = x$ and $y\theta = y$ where θ is an inner automorphism induced by conjugation by x^2.

If $k = 4$, we have the sequence for the generating pair (x, y),

$$\begin{gathered} x,\ y,\ xy,\ (xy)^2,\ (xy)^4,\ x^3,\ y^2,\ y^5xy,\ (xy)^2,\ e,\ x, \\ y,\ (xy)^3,\ (xy)^4,\ e,\ x^3, y^2,\ xy^2,\ (yx)^2,\ x^2,\ x,\ y, \\ yxy^2,\ (yx)^2,\ e,\ x^3,\ y^2,\ y^4x,\ e,\ e,\ x,\ y,\ xy,\ (xy)^2,\ \ldots, \end{gathered} \tag{3.17}$$

which has period 30 and the basic period 30 since $x\theta = x$ and $y\theta = y$ where θ is an inner automorphism induced by conjugation by x^2.

If $k \geq 5$, the first k elements of sequence for the generating pair (x, y) are

$$x_0 = x,\ x_1 = y,\ x_2 = xy,\ x_3 = (xy)^2, x_4 = (xy)^4,\ x_5 = (xy)^8\ \ldots,\ x_{k-1} = (xy)^{2^{k-3}}. \tag{3.18}$$

Thus, using the above information, sequence reduces to

$$x_0 = x,\ x_1 = y,\ x_2 = xy,\ x_3 = (xy)^2, x_4 = (xy)^4,\ x_5 = e,\ \ldots,\ e,\ x_{k-1} = e, \tag{3.19}$$

where $x_j = e$ for $5 \leq j \leq k - 1$. Thus,

$$\begin{gathered} x_k = e,\ x_{k+1} = x^3,\ x_{k+2} = y^2,\ x_{k+3} = y^5xy, \\ x_{k+4} = (xy)^2,\ x_{k+5} = e,\ \ldots,\ e,\ x_{2k+1} = e,\ x_{2k+2} = x, \end{gathered} \tag{3.20}$$

$$\begin{gathered} x_{2k+3} = y,\ x_{2k+4} = (xy)^3,\ x_{2k+5} = (xy)^4,\ x_{2k+6} = e,\ \ldots,\ e, \\ x_{3k+2} = e,\ x_{3k+3} = x^3,\ x_{3k+4} = y^2,\ x_{3k+5} = xy^2, \\ x_{3k+6} = (yx)^2,\ x_{3k+7} = x^2, x_{3k+8} = e \ldots,\ e,\ x_{4k+3} = e, \\ x_{4k+4} = x,\ x_{4k+5} = y,\ x_{4k+6} = yxy^2 x_{4k+7} = (yx)^2, \\ x_{4k+8} = e,\ \ldots,\ e,\ x_{5k+4} = e, x_{5k+5} = x^3,\ x_{5k+6} = y^2, \\ x_{5k+7} = y^4x,\ x_{5k+8} = e, \ldots,\ e,\ x_{6k+5} = e, \end{gathered} \tag{3.21}$$

where $x_j = e$ for $k + 5 \leq j \leq 2k + 1$, $2k + 6 \leq j \leq 3k + 2$, $3k + 8 \leq j \leq 4k + 3$, $4k + 8 \leq j \leq 5k + 4$, and $5k + 8 \leq j \leq 6k + 5$.

We also have

$$x_{6k+6} = \prod_{i=5k+6}^{6k+5} x_i = x, \qquad x_{6k+7} = \prod_{i=5k+7}^{6k+6} x_i = y. \tag{3.22}$$

Since the elements succeeding x_{6k+6}, x_{6k+7} depend on x and y for their values, the cycle begins again with the $6k + 6$th element, that is, $x_0 = x_{6k+6}$, $x_1 = x_{6k+7}$, …. Thus, $P_k(\langle 2,3,4\rangle; x, y) = 6k + 6$ and $BP_k(\langle 2,3,4\rangle; x, y) = 6k + 6$ since $x\theta = x$ and $y\theta = y$ where θ is an inner automorphism induced by conjugation by x^2.

Secondly, let us consider the 3-generator case. We first note that $|x| = 4$, $|y| = 6$, and $|z| = 8$.

(i) If $k = 2$, $P_2(\langle 2,3,4\rangle; y, z, x) = 18$ and $BP_2(\langle 2,3,4\rangle; y, z, x) = 9$ since $x\theta = x^3$, $y\theta = x^3yx$, and $z\theta = xy^2$ where θ is an outer automorphism of order 2.

(ii) If $k > 2$, $P_k(\langle 2,3,4\rangle; x, y, z) = 6k + 6$ and $BP_k(\langle 2,3,4\rangle; x, y, z) = 6k + 6$ since $x\theta = x$ and $y\theta = y$ where θ is an inner automorphism induced by conjugation by x^2.

The proofs are similar to the proofs of Theorems 3.5.(i) and 3.5.(ii) and are omitted. □

Theorem 3.7. *The periods of the k-nacci sequences and the basic periods of the basic k-nacci sequences in the group A_4 are as follows.*

If the group is defined by the presentation $A_4 = \langle x, y, z : x^2 = y^3 = z^3 = xyz = e\rangle$, *then*

(i) *if* $k = 2$, $P_2(A_4; y, z, x) = 16$ *and* $BP_2(A_4; y, z, x) = 4$,

(ii) *if* $k > 2$,

$$P_k(A_4; x, y, z) = \begin{cases} 3BP_k(A_4; x, y, z), & k \equiv 0 \mod 4, \\ 2BP_k(A_4; x, y, z), & k \equiv 2 \mod 4, \\ 2BP_k(A_4; x, y, z), & \text{otherwise}, \end{cases}$$
$$BP_k(A_4; x, y, z) = \begin{cases} u_1 h_k(3), & k \equiv 0 \mod 4, \\ u_2 h_k(3), & k \equiv 2 \mod 4, \\ u_3 h_k(3), & \text{otherwise}, \end{cases} \quad (3.23)$$

where $u_1, u_2, u_3 \in N$, *and* $h_k(3)$ *denote the wall number of the k-step Fibonacci sequence modulo 3.*

If the group is defined by the presentation $A_4 = \langle x, y : x^2 = y^3 = (xy)^3 = e\rangle$, *then*

(i′) *if* $k = 2$, $P_2(A_4; x, y) = 16$ *and* $BP_2(A_4; x, y) = 4$,

(ii′) *if* $k > 2$,

$$P_k(A_4; x, y) = \begin{cases} 3BP_k(A_4; x, y), & k \equiv 0 \mod 4, \\ 2BP_k(A_4; x, y), & k \equiv 2 \mod 4, \\ 2BP_k(A_4; x, y), & \text{otherwise}, \end{cases}$$
$$BP_k(A_4; x, y) = \begin{cases} u_1 h_k(3), & k \equiv 0 \mod 4, \\ u_2 h_k(3), & k \equiv 2 \mod 4, \\ u_3 h_k(3), & \text{otherwise}, \end{cases} \quad (3.24)$$

where $u_1, u_2, u_3 \in N$.

Proof. Firstly, let us consider the 2-generator case. We process as similar to the proof of Theorem 3.6 We first note that $|x| = 2$, $|y| = 3$, and $|xy| = 3$.

(i′) If $k = 2$, we have the sequence for the generating pair (x, y),

$$x,\ y,\ xy,\ yxy,\ yxy^2,\ (xy)^2,\ xy^2,\ y,\ x,$$
$$yx,\ xyx,\ y^2x,\ yxy^2,\ yxy,\ y^2,\ yx,\ x,\ y,\ \dots, \quad (3.25)$$

which has period 16 and the basic period 4 since $x\theta = yxy^2$ and $y\theta = yxy$ where θ is an outer automorphism of order 4.

(ii′) If $k > 2$,

let k be even, then the first k elements of sequence for the generating pair (x, y) are

$$x_0 = x,\ x_1 = y,\ x_2 = xy,\ x_3 = (xy)^2, x_4 = xy,\ x_5 = (xy)^2\ \ldots,\ x_{k-2} = xy,\ x_{k-1} = (xy)^2. \tag{3.26}$$

If $k \equiv 0 \bmod 4$,

$$\begin{gathered} x_{u_1 h_k(3)-(k-2)} = e,\quad x_{u_1 h_k(3)-(k-1)} = e, \ldots,\ e, \\ x_{u_1 h_k(3)-1} = e,\quad x_{u_1 h_k(3)} = y^2 xy,\quad x_{u_1 h_k(3)+1} = yx,\ \ldots. \end{gathered} \tag{3.27}$$

$P_k(A_4; x, y) = 3BP_k(A_4; x, y)$ and $BP_k(A_4; x, y) = u_1 h_k(3)$ since $x\theta = yxy^2$ and $y\theta = xyx$ where θ is an outer automorphism of order 3.

If $k \equiv 2 \bmod 4$,

$$\begin{gathered} x_{u_2 h_k(3)-(k-2)} = e,\quad x_{u_2 h_k(3)-(k-1)} = e, \ldots,\ e, \\ x_{u_2 h_k(3)-1} = e,\quad x_{u_2 h_k(3)} = x,\quad x_{u_2 h_k(3)+1} = xy,\ \ldots. \end{gathered} \tag{3.28}$$

$P_k(A_4; x, y) = 2BP_k(A_4; x, y)$ and $BP_k(A_4; x, y) = u_2 h_k(3)$ since $x\theta = x$ and $y\theta = xy$ where θ is a outer automorphism of order 2.

Let k be odd, then the first k elements of sequence are for the generating pair (x, y),

$$x_0 = x,\ x_1 = y,\ x_2 = xy,\ x_3 = (xy)^2, x_4 = xy,\ x_5 = (xy)^2 \ldots,\ x_{k-2} = (xy)^2,\ x_{k-1} = xy. \tag{3.29}$$

Also,

$$\begin{gathered} x_{u_3 h_k(3)-(k-2)} = e,\quad x_{u_3 h_k(3)-(k-1)} = e, \ldots,\ e, \\ x_{u_3 h_k(3)-1} = e,\quad x_{u_3 h_k(3)} = x,\quad x_{u_3 h_k(3)+1} = yx,\ \ldots. \end{gathered} \tag{3.30}$$

$P_k(A_4; x, y) = 2BP_k(A_4; x, y)$ and $BP_k(A_4; x, y) = u_3 h_k(3)$ since $x\theta = x$ and $y\theta = yx$ where θ is an outer automorphism of order 2.

Secondly, let us consider the 3-generator case. We first note that $|x| = 2$, $|y| = 3$, and $|z| = 3$.

(i) If $k = 2$, $P_2(A_4; y, z, x) = 16$ and $BP_2(A_4; y, z, x) = 4$ since $x\theta = y^2 xy$, $y\theta = yxy$, and $z\theta = yx$ where θ is an outer automorphism of order 4.

(ii) If $k > 2$,
let $k \equiv 0 \bmod 4$, then $P_k(A_4; x, y, z) = 3BP_k(A_4; x, y, z)$ and $BP_k(A_4; x, y, z) = u_1 h_k(3)$ since $x\theta = y^2 xy$, $y\theta = xyx$, and $z\theta = zx$ where θ is an outer

automorphism of order 3; let $k \equiv 2 \mod 4$, then $P_k(A_4; x, y, z) = 2BP_k(A_4; x, y, z)$ and $BP_k(A_4; x, y, z) = u_2 h_k(3)$ since $x\theta = x$, $y\theta = yx$, and $z\theta = yz^2$ where θ is an outer automorphism of order 2; let k be odd; then $P_k(A_4; x, y, z) = 2BP_k(A_4; x, y, z)$ and $BP_k(A_4; x, y, z) = u_3 h_k(3)$ since $x\theta = x$, $y\theta = xy$, and $z\theta = zx$ where θ is an outer automorphism of order 2.

The proofs are similar to the proofs of Theorems 3.5.(i) and 3.5.(i.i) and are omitted. □

Theorem 3.8. *The periods of the k-nacci sequences and the basic periods of the basic k-nacci sequences in the binary polyhedral group $\langle 2,3,3\rangle$ are as follows.*

If the group is defined by the presentation $\langle 2,3,3\rangle = \langle x, y, z : x^2 = y^3 = z^3 = xyz\rangle$, then

(i) *if $k = 2$, $P_2(\langle 2,3,3\rangle; y, z, x) = 48$ and $BP_2(\langle 2,3,3\rangle; y, z, x) = 12$,*

(ii) *if $k > 2$,*

$$P_k(\langle 2,3,3\rangle; x, y, z) = \begin{cases} 3BP_k(\langle 2,3,3\rangle; x, y, z), & k \equiv 0 \mod 4, \\ BP_k(\langle 2,3,3\rangle; x, y, z), & k \not\equiv 0 \mod 4, \end{cases} \tag{3.31}$$

$$BP_k(\langle 2,3,3\rangle; x, y, z) = \begin{cases} v_1 h_k(6), & k \equiv 0 \mod 4, \\ v_2 h_k(6), & k \not\equiv 0 \mod 4, \end{cases} \tag{3.32}$$

where $v_1, v_2 \in N$, and $h_k(6)$ denote the wall number of the k-step Fibonacci sequence modulo 6.

If the group is defined by the presentation $\langle 2,3,3\rangle = \langle x, y : x^2 = y^3 = (xy)^3\rangle$, then

(i′) *if $k = 2$, $P_2(\langle 2,3,3\rangle; x, y) = 48$ and $BP_2(\langle 2,3,3\rangle; x, y) = 12$,*

(ii′) *if $k > 2$,*

$$P_k(\langle 2,3,3\rangle; x, y) = \begin{cases} 3BP_k(\langle 2,3,3\rangle; x, y), & k \equiv 0 \mod 4, \\ BP_k(\langle 2,3,3\rangle; x, y), & k \not\equiv 0 \mod 4, \end{cases} \tag{3.33}$$

$$BP_k(\langle 2,3,3\rangle; x, y) = \begin{cases} v_1 h_k(6), & k \equiv 0 \mod 4, \\ v_2 h_k(6), & k \not\equiv 0 \mod 4, \end{cases} \tag{3.34}$$

where $v_1, v_2 \in N$.

Proof. Firstly, let us consider the 3-generator case. We first note that $|x| = 4$, $|y| = 6$, and $|z| = 6$.

(i) If $k = 2$, $P_2(\langle 2,3,3\rangle; y, z, x) = 48$ and $BP_2(\langle 2,3,3\rangle; y, z, x) = 12$ since $x\theta = y^2xy$, $y\theta = xz^4x$, and $z\theta = y^2xy^2$ where θ is an outer automorphism of order 4.

(ii) If $k > 2$,
let $k \equiv 0 \bmod 4$, then $P_k(\langle 2,3,3\rangle; x,y,z) = 3BP_k(\langle 2,3,3\rangle; x,y,z)$ and $BP_k(\langle 2,3,3\rangle; x,y,z) = v_1 h_k(6)$ since $x\theta = yxy^5$, $y\theta = z^3xy$, and $z\theta = xy^2x$ where θ is an inner automorphism induced by conjugation by z^3yx;

let $k \not\equiv 0 \bmod 4$, then $P_k(\langle 2,3,3\rangle; x,y,z) = BP_k(\langle 2,3,3\rangle; x,y,z)$ and $BP_k(\langle 2,3,3\rangle; x,y,z) = v_2 h_k(6)$ since $x\theta = x$, $y\theta = y$, and $z\theta = z$ where θ is an inner automorphism induced by conjugation by $\mathbf{x^2}$.

The proofs are similar to the proofs of Theorems 3.5.(i) and 3.5.(ii) and are omitted.

Secondly, let us consider the 2-generator case. We first note that $|x| = 4$, $|y| = 6$, and $|xy| = 6$.

(i′) If $k = 2, P_2(\langle 2,3,3\rangle; x,y) = 48$ and $BP_2(\langle 2,3,3\rangle; x,y) = 12$ since $x\theta = yxy^2$ and $y\theta = y^2x$ where θ is an outer automorphism of order 4.

(ii′) If $k > 2$,
let $k \equiv 0 \bmod 4$, then $P_k(\langle 2,3,3\rangle; x,y) = 3BP_k(\langle 2,3,3\rangle; x,y)$ and $BP_k(\langle 2,3,3\rangle; x,y) = v_1 h_k(6)$ since $x\theta = y^5xy$, $y\theta = yx$, and $z\theta = xy^2x$ where θ is an inner automorphism induced by conjugation by y^5x,

let $k \not\equiv 0 \bmod 4$, then $P_k(\langle 2,3,3\rangle; x,y) = BP_k(\langle 2,3,3\rangle; x,y)$ and $BP_k(\langle 2,3,3\rangle; x,y) = v_2 h_k(6)$ since $x\theta = x$ and $y\theta = y$ where θ is an inner automorphism induced by conjugation by x^2.

The proofs are similar to the proofs of Theorem 3.6.(i′) and Theorem 3.6.(ii′) and are omitted. □

Theorem 3.9. *The periods of the k-nacci sequences are* $k + 1$, *and the basic period of the basic k-nacci sequences is* $k + 1$ *in* D_2 *four-group.*

Proof. We have the presentation $D_2 = \langle x, y : x^2 = y^2 = e,\ \ xy = yx\rangle$. $P_k(D_2; x, y) = k + 1$; see [14] for a proof and $BP_k(D_2; x, y) = k + 1$ since $x\theta = x$ and $y\theta = y$ where θ is an inner automorphism induced by conjugation by x. □

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