Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 826164, 6 pages
doi:10.1155/2011/826164

Research Article

Normal Criteria of Function Families Related to a Result of Drasin

Zhu Bing,[1] Lin Jianming,[2] and Yuan Wenjun[3]

[1] *College of Computer Engineering Technology, Guangdong Institute of Science and Technology, Zhuhai 519090, China*
[2] *School of Economic and Management, Guangzhou University of Chinese Medicine, Guangzhou 512009, China*
[3] *School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China*

Correspondence should be addressed to Yuan Wenjun, wjyuan1957@126.com

Received 7 April 2011; Accepted 20 May 2011

Academic Editor: M. De la Sen

We study the normality of families of meromorphic functions related to a result of Drasin. We consider whether a family meromorphic functions $\mathcal{F}$ whose each function does not take zero is normal in D, if for every pair of functions f and g in $\mathcal{F}$, $f(z)$ and $g(z)$ share ∞ or $H(f)-1$ and $H(g)-1$ share 0, where $H(f) := f^{(k)}(z) + a_{k-1}f^{(k-1)}(z) + \cdots , a_0 f(z)$. Some examples show that the conditions in our results are best possible.

1. Introduction and Main Result

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in a domain $D \subsetneq \mathbb{C}$, and let a be a finite complex value. We say that f and g share a CM (or IM) in D provided that $f-a$ and $g-a$ have the same zeros counting (or ignoring) multiplicity in D. When $a=\infty$, the zeros of $f-a$ mean the poles of f (see [1]). It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory (see [2–4] or [1]).

Influenced from Bloch's principle [5], every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to a constant makes a family of meromorphic functions in a domain D normal. Although the principle is false in general (see [6]), many authors proved normality criterion for families of meromorphic functions corresponding to Liouville-Picard type theorem (see [7] or [4]).

It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick [8] first proved an interesting result that a family of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, more results about normality criteria concerning shared values can be found, for instance, (see [9–13]) and so on. In recent years, this subject has attracted the attention of many researchers worldwide.

The following result is due to Schiff [14].

Theorem 1.1. *Let $\mathcal{F}$ be a family holomorphic functions in D, and $a_j \in \mathbb{C}, j = 0,1,\ldots,k-1, k$ a positive integer. If $f(z) \neq 0$ and the zeros of $H(f) - 1$ are of multiplicity $\geq p > k+1$ for each $f(z) \in \mathcal{F}$, where $H(f) := f^{(k)}(z) + a_{k-1}f^{(k-1)}(z) + \cdots, a_0 f(z)$ and p is a positive integer, then $\mathcal{F}$ is normal in D.*

In 2001, Fang and Yuan [15] extended Theorem 1.1 as follows.

Theorem 1.2. *Let $\mathcal{F}$ be a family holomorphic functions in D, and $a_j \in \mathbb{C}, j = 0,1,\ldots,k-1, k$ a positive integer. If $f(z) \neq 0$ and the zeros of $H(f) - 1$ are of multiplicity ≥ 2 for each $f(z) \in \mathcal{F}$, where $H(f) := f^{(k)}(z) + a_{k-1}f^{(k-1)}(z) + \cdots, a_0 f(z)$, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether or not Theorems 1.1 and 1.2 hold for meromorphic case or holomorphic case by the idea of shared values. In this paper, we answer above question and prove the following results.

Theorem 1.3. *Let $\mathcal{F}$ be a family meromorphic functions in D, and $a_j \in \mathbb{C}, j = 0,1,\ldots,k-1, k$ a positive integer. If (i) $f(z) \neq 0$ and the zeros of $H(f) - 1$ are of multiplicity ≥ 2 for each $f(z) \in \mathcal{F}$, (ii) $f(z)$ and $g(z)$ share ∞ for every pair of functions $f(z)$ and $g(z)$ in $\mathcal{F}$, where $H(f) := f^{(k)}(z) + a_{k-1}f^{(k-1)}(z) + \cdots, a_0 f(z)$, then $\mathcal{F}$ is normal in D.*

Theorem 1.4. *Let $\mathcal{F}$ be a family holomorphic functions in D, and $a_j \in \mathbb{C}, j = 0,1,\ldots,k-1, k$ a positive integer. If (i) $f(z) \neq 0$ for each $f(z) \in \mathcal{F}$, (ii) $H(f) - 1$ and $H(g) - 1$ share the value 0 for every pair of functions $f(z)$ and $g(z)$ in $\mathcal{F}$, where $H(f) := f^{(k)}(z) + a_{k-1}f^{(k-1)}(z) + \cdots, a_0 f(z)$, then $\mathcal{F}$ is normal in D.*

Example 1.5. The family of holomorphic functions $\mathcal{F} = \{f_n(z) = nze^z - ne^z : n = 1,2,\ldots\}$ is not normal in $D = \{z : |z| < 1\}$. Obviously $f_n^{(k)} - f_n = ke^z \neq 0$. On the other hand, $f_n(0) = -n$, $f_n(1/\sqrt{n}) = -1/2 + (1/2\sqrt{n}) + o(1/n) \rightarrow -1/2$, as $n \rightarrow \infty$. This implies that the family $\mathcal{F}$ fails to be equicontinuous at 0, and thus $\mathcal{F}$ is not normal at 0.

Example 1.6. The family of holomorphic functions $\mathcal{F} = \{f_n(z) = n(z+1) - 1 : n = 1,2,\ldots\}$ is normal in $D = \{z : |z| < 1\}$. Obviously, $f_n(z) \neq 0$ and $f_n' - f_n = -nz + 1$. So for each pair m, n, $f_n' - f_n$ and $f_m' - f_m$ share the value 1 in D. Theorem 1.4 implies that the family $\mathcal{F}$ is normal in D.

Example 1.7. The family of meromorphic functions $\mathcal{F} = \{f_n(z) = z/n - 1 : n = 1,2,\ldots\}$ is normal in $D = \{z : |z| < 1\}$. The reason is the conditions of Theorems 1.3 and 1.4 hold, that is, $f_n(z) \neq 0$ and $f_n' - f_n = (1-z)/n + 1 \neq 1$ for each $f_n(z) \in \mathcal{F}$ in $D = \{z : |z| < 1\}$.

Example 1.8. The family of meromorphic functions $\mathcal{F} = \{f_n(z) = n/z - 1 : n = 1,2,\ldots\}$ is normal in $D = \{z : |z| < 1\}$. The reason is the conditions of Theorem 1.3 hold, that is, $f_n(z) \neq 0$ and has only one pole 0 for each $f(z) \in \mathcal{F}$; $f_n^{(k)} - f_n - 1 = n((-1)k! - z^k)/z^{k+1} \neq 0$ for each $f_n(z) \in \mathcal{F}$ in $D = \{z : |z| < 1\}$.

Remark 1.9. Example 1.5 shows that $H(g)-1$ is not replaced by $H(g)$ in Theorems 1.3 and 1.4. is not valid when $n = 1$. All of Examples 1.6, 1.7 and 1.8 show that Theorems 1.3 and 1.4 occur.

2. Preliminary Lemmas

In order to prove our result, we need the following lemmas. The first one extends a famous result by Zalcman [16] concerning normal families.

Lemma 2.1 (see [17]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc satisfying all zeros of functions in $\mathcal{F}$ having multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$. Let α be a real number satisfying $-p < \alpha < q$. Then, $\mathcal{F}$ is not normal at 0 if and only if there exist*

(a) *a number $0 < r < 1$,*

(b) *points z_n with $|z_n| < r$,*

(c) *functions $f_n \in F$,*

(d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := \rho^{\alpha} f_n(z_n + \rho_n \zeta)$ converges spherically uniformly on each compact subset of $\mathbb{C}$ to a nonconstant meromorphic function $g(\zeta)$, whose all zeros of functions in $\mathcal{F}$ have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$ and order is at most 2.

Remark 2.2. If $\mathcal{F}$ is a family of holomorphic functions on the unit disc in Lemma 2.1, then $g(\zeta)$ is a nonconstant entire function whose order is at most 1.

The order of g is defined by using the Nevanlinna's characteristic function $T(r,g)$

$$\rho(g) = \lim_{r\to\infty} \sup \frac{\log T(r,g)}{\log r}. \tag{2.1}$$

Here, $g^{\#}(\xi)$ denotes the spherical derivative

$$g^{\#}(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}. \tag{2.2}$$

Lemma 2.3 (see [3]). *Let $f(z)$ be a meromorphic function in $\mathbb{C}$, then*

$$T(r,f) \leq \left(2+\frac{1}{k}\right)N\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{k}\right)\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f), \tag{2.3}$$

$$T(r,f) \leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f). \tag{2.4}$$

Remark 2.4. Both (2.3) and (2.4) are called Hayman inequality and Milloux inequality, respectively.

2.1. Proof of the Results

Proof of Theorem 1.3. Suppose that $\mathcal{F}$ is not normal in D. Then, there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0. Without loss of generality, we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$ and functions $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi) \Longrightarrow g(\xi), \tag{2.5}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\mathbb{C}$. Moreover, the order of g is less than 2.

Since $f_j(z) \in \mathcal{F}$, by Hurwitz's theorem, we see $g \neq 0$.

From (2.5), we know

$$g_j^{(k)}(\xi) = f_j^{(k)}(z_j + \rho_j \xi) \Longrightarrow g^{(k)}(\xi), \tag{2.6}$$

$$g_j^{(k)}(\xi) - 1 + \sum_{i=1}^{k-1} \rho_j^{k-i} a_i g_j^{(i)}(z_j + \rho_j \xi) = H(f_j(z_j + \rho_j \xi)) - 1 \Longrightarrow g^{(k)}(\xi) - 1, \tag{2.7}$$

also locally uniformly with respect to the spherical metric.

If $g^{(k)}(\xi) \ - 1 \equiv 0$, then g is a polynomial with degree k. This contradicts $g \neq 0$.

If $g^{(k)}(\xi) - 1 \neq 0$, then by Hayman inequality (2.3) of Lemma 2.3, we know that g is a constant, which is impossible.

Hence, $g^{(k)}(\xi) - 1$ is a nonconstant meromorphic function and has at least one zero. By (2.7) and Hurwitz's theorem, we see that the zeros of $g^{(k)}(\xi) - 1$ are multiple.

Next, we prove that g has at most one distinct pole. By contraries, let ξ_0 and ξ_0^* be two distinct poles of g, and choose $\delta(> 0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$, where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (2.5), by Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$f_j^{-1}(z_j + \rho_j \xi_j) = 0, \qquad f_j^{-1}\left(z_j + \rho_j \xi_j^*\right) = 0. \tag{2.8}$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, f^{-1} and g^{-1} share 0 in D, we know that for any positive integer m

$$f_m^{-1}(z_j + \rho_j \xi_j) = 0, \qquad f_m^{-1}\left(z_j + \rho_j \xi_j^*\right) = 0. \tag{2.9}$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j \xi_j \to 0$, $z_j + \rho_j \xi_j^* \to 0$, then $f_m^{-1}(0) = 0$. Since the zeros of f_m^{-1} has no accumulation point, so.

$$z_j + \rho_j \xi_j = 0, \qquad z_j + \rho_j \xi_j^* = 0. \tag{2.10}$$

Hence $\xi_j = -z_j/\rho_j$, $\xi_j^* = -z_j/\rho_j$ This contradicts with $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$. So, g has at most one distinct pole.

For g applying Milloux inequality (2.4) of Lemma 2.3, we deduce

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g) + N\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-1}\right) + S(r,g) \\ &\leq \log r + \frac{1}{2}N\left(r,\frac{1}{g^{(k)}-1}\right) + S(r,g) \\ &\leq \frac{1}{2}T(r,g) + \log r + S(r,g). \end{aligned} \tag{2.11}$$

From (2.11), we know that g is a rational function of degree at most 2. Noting that $g \neq 0$, we can write $g(\xi) = 1/(a\xi + b)^m, 1 \leq m \leq 2$ where $a \neq 0$ and b are two finite complex numbers. Simple calculating shows that $g^{(k)}-1$ has at least $k+1$ distinct simple zeros. This is impossible.

The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. Similarly with the proof of Theorem 1.3, we have that (2.5) and (2.7) also hold. Moreover, $g(\xi)$ is an entire function.

If $g^{(k)}(\xi) - 1 \equiv 0$, then g is a polynomial with degree k. This contradicts $g \neq 0$.

If $g^{(k)}(\xi) - 1 \neq 0$, then by Hayman inequality (2.3) of Lemma 2.3 we know that g is a constant, which is impossible.

Hence, $g^{(k)}(\xi) - 1$ is a nonconstant entire function and has at least one zero.

Next, we prove that $g^{(k)}(\xi) - 1$ has one distinct zero. By contraries, let ξ_0 and ξ_0^* be two distinct zeros of $g^{(k)}(\xi) - 1$, and choose $\delta(> 0)$ small enough such that $D(\xi_0,\delta) \cap D(\xi_0^*,\delta) = \phi$, where $D(\xi_0,\delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*,\delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (2.7), by Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0,\delta)$, $\xi_j^* \in D(\xi_0^*,\delta)$ such that for sufficiently large j

$$H(f_j(z_j + \rho_j\xi_j)) - 1 = 0, \qquad H\left(f_j\left(z_j + \rho_j\xi_j^*\right)\right) - 1 = 0. \tag{2.12}$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $H(f) - 1$ and $H(g) - 1$ share 0 in D, we know that for any positive integer m

$$H(f_m(z_j + \rho_j\xi_j)) - 1 = 0, \qquad H\left(f_m\left(z_j + \rho_j\xi_j^*\right)\right) - 1 = 0. \tag{2.13}$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, then $f_m^{(k)}(0) - 1 = 0$. Since the zeros of $f_m^{(k)} - 1$ has no accumulation point, so

$$z_j + \rho_j\xi_j = 0, \qquad z_j + \rho_j\xi_j^* = 0. \tag{2.14}$$

Hence $\xi_j = -z_j/\rho_j$, $\xi_j^* = -z_j/\rho_j$ This contradicts with $\xi_j \in D(\xi_0,\delta)$, $\xi_j^* \in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta) \cap D(\xi_0^*,\delta) = \phi$. So, $g^{(k)}(\xi) - 1$ has one distinct zero.

For g applying Milloux inequality (2.4) of Lemma 2.3, we deduce

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g) + N\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-1}\right) + S(r,g) \\ &\leq \log r + S(r,g). \end{aligned} \tag{2.15}$$

From (2.15) and $g \neq 0$, we know that g is a polynomial of degree 1. We can write $g(\xi) = (c\xi + d)$, where $c \neq 0$ and d are two finite complex numbers. Hence, $g^{(k)} - 1$ has no only one distinct zero. This is impossible.

The proof of Theorem 1.4 is complete. □

Acknowlegment

This work was completed with the support of the NSF of China (no. 10771220) and Doctorial Point Fund of National Education Ministry of China (no. 200810780002).

References

[1] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, vol. 557 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.
[2] Y. X. Gu, X. C. Pang, and M. L. Fang, *On Rosenbloom's Fix-Points Theorem and Related Results*, Science Press, Beijing, China, 2007.
[3] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1964.
[4] L. Yang, *Value Distribution Theory*, Springer, Berlin, Germany, 1993.
[5] W. Bergweiler, "Bloch's principle," *Computational Methods and Function Theory*, vol. 6, no. 1, pp. 77–108, 2006.
[6] L. A. Rubel, "Four counterexamples to Bloch's principle," *Proceedings of the American Mathematical Society*, vol. 98, no. 2, pp. 257–260, 1986.
[7] D. Drasin, "Normal families and the Nevanlinna theory," *Acta Mathematica*, vol. 122, pp. 231–263, 1969.
[8] W. Schwick, "Normality criteria for families of meromorphic functions," *Journal d'Analyse Mathématique*, vol. 52, pp. 241–289, 1989.
[9] X. Pang and L. Zalcman, "Normal families and shared values," *The Bulletin of the London Mathematical Society*, vol. 32, no. 3, pp. 325–331, 2000.
[10] X. Pang and L. Zalcman, "Normality and shared values," *Arkiv för Matematik*, vol. 38, no. 1, pp. 171–182, 2000.
[11] W. J. Yuan, Z. R. Li, and B. Xiao, "Normal families of composite holomorphic functions and sharing a meromorphic function," *Scientia Sinica, Series A*, vol. 40, no. 5, pp. 429–436, 2010 (Chinese).
[12] Q. Zhang, "Normal families of meromorphic functions concerning shared values," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 545–551, 2008.
[13] Q. Zhang, "Some normality criteria of meromorphic functions," *Complex Variables and Elliptic Equations*, vol. 53, no. 8, pp. 791–795, 2008.
[14] J. L. Schiff, *Normal Families*, Universitext, Springer, New York, NY, USA, 1993.
[15] M. Fang and W. Yuan, "On the normality for families of meromorphic functions," *Indian Journal of Mathematics*, vol. 43, no. 3, pp. 341–351, 2001.
[16] L. Zalcman, "A heuristic principle in complex function theory," *The American Mathematical Monthly*, vol. 82, no. 8, pp. 813–817, 1975.
[17] L. Zalcman, "Normal families: new perspectives," *Bulletin of the American Mathematical Society*, vol. 35, no. 3, pp. 215–230, 1998.