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Research Article

Some New Identities on the Bernoulli and Euler Numbers

Dae San Kim,[1] Taekyun Kim,[2] Sang-Hun Lee,[3] D. V. Dolgy,[4] and Seog-Hoon Rim[5]

[1] *Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*
[2] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[3] *Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea*
[4] *Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea*
[5] *Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Republic of Korea*

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

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We give some new identities on the Bernoulli and Euler numbers by using the bosonic p-adic integral on $\mathbb{Z}_p$ and reflection symmetric properties of Bernoulli and Euler polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu\left(x + p^N\mathbb{Z}_p\right) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{1.1}$$

From (1.1), we note that

$$I(f_1) = I(f) + f'(0), \quad \text{where } f_1(x) = f(x+1), \tag{1.2}$$

see [1]. As is well known, the ordinary Bernoulli polynomials are defined by the generating function as follows:

$$F(t,x) = \frac{t}{e^t - 1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.3}$$

see [1–19], where we use the technical notation by replacing $B^n(x)$ by $B_n(x)(n \geq 0)$, symbolically. In the special case, $x = 0$, $B_n(0) = B_n$ are called the n-th ordinary Bernoulli numbers. That is, the generating function of ordinary Bernoulli numbers is given by

$$F(t) = F(t,0) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{1.4}$$

see [1–19]. From (1.4), we can derive the following relation:

$$B_0 = 1, \qquad (B+1)^n - B_n = \delta_{1,n}, \tag{1.5}$$

see [1, 10], where $\delta_{1,n}$ is the Kronecker symbol.

By (1.3) and (1.4), we easily get

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l. \tag{1.6}$$

By (1.2) and (1.3), we easily get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.7}$$

see [1, 10]. From (1.7), we can derive Witt's formula for the n-th Bernoulli polynomials as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x), \quad \text{where } n \in \mathbb{Z}_+, \tag{1.8}$$

see [11]. By (1.1) and (1.8), we easily see that

$$\int_{\mathbb{Z}_p} (y+1-x)^n d\mu(y) = (-1)^n \int_{\mathbb{Z}_p} (y+x)^n d\mu(y). \tag{1.9}$$

Thus, by (1.8) and (1.9), we get reflection symmetric relation for the Bernoulli polynomials as follows:

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{where } n \in \mathbb{Z}_+. \tag{1.10}$$

The ordinary Euler polynomials are defined by the generating function as follows:

$$F_e(t,x) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.11}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [8, 9]). In the special case, $x=0$, $E_n(0)=E_n$ are called the n-th Euler numbers (see [8, 9]).

From (1.11), we note that

$$\frac{2}{e^t+1}e^{xt} = \frac{2}{1+e^{-t}}e^{-(1-x)t} = \sum_{n=0}^{\infty}(-1)^n E_n(1-x)\frac{(t)^n}{n!}, \tag{1.12}$$

By comparing the coefficients on both sides of (1.11) and (1.12), we obtain the following reflection symmetric relation for Euler polynomials as follows:

$$E_n(x) = (-1)^n E_n(1-x), \quad \text{where } n \in \mathbb{Z}_+. \tag{1.13}$$

The equations (1.10) and (1.13) are useful in deriving our main results in this paper.

For $n,k \in \mathbb{Z}_+$, the Bernstein polynomials are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.14}$$

see [13]. By (1.14), we easily get $B_{k,n}(x) = B_{n-k,n}(1-x)$.

In this paper we consider the p-adic integrals for the Bernoulli and Euler polynomials. From those p-adic integrals, we derive some new identities on the Bernoulli and Euler numbers.

2. Identities on the Bernoulli and Euler Numbers

First, we consider the p-adic integral on $\mathbb{Z}_p$ for the nth ordinary Bernoulli polynomials as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} B_n(x)d\mu(x) = \sum_{l=0}^{n}\binom{n}{l}B_{n-l}\int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{l=0}^{n}\binom{n}{l}B_{n-l}B_l, \quad \text{where } n \in \mathbb{Z}_+. \end{aligned} \tag{2.1}$$

On the other hand, by (1.3) and (1.10), one gets

$$I_1 = (-1)^n \int_{\mathbb{Z}_p} B_n(1-x)d\mu(x). \tag{2.2}$$

From (1.5), (1.6), (1.8), and (2.2), one notes that

$$\begin{aligned} I_1 &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu(x) \\ &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} (l + B_l + \delta_{1,l}) \\ &= (-1)^n n B_{n-l}(1) + (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l + (-1)^n n B_{n-l}. \end{aligned} \tag{2.3}$$

Equating (2.1) and (2.3), one gets

$$\begin{aligned} \left(1 + (-1)^{n+1}\right) \sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l &= (-1)^n n (\delta_{1,n-l} + B_{n-1}) + (-1)^n n B_{n-1} \\ &= 2(-1)^n n B_{n-l} + (-1)^n n \delta_{1,n-1}. \end{aligned} \tag{2.4}$$

Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$. Then, by (2.4), one has

$$\sum_{l=0}^{2n-1} \binom{2n-1}{l} B_{2n-1-l} B_l = -(2n-1) B_{2n-2}. \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$, one has*

$$\left(1 + (-1)^{n+1}\right) \sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l = 2(-1)^n n B_{n-1} + (-1)^n n \delta_{1,n-1}. \tag{2.6}$$

In particular,

$$\sum_{l=0}^{2n-1} \binom{2n-1}{l} B_{2n-1-l} B_l = -(2n-1) B_{2n-2}. \tag{2.7}$$

By the same motivation, let us also consider the p-adic integral on $\mathbb{Z}_p$ for Euler polynomials as follows:

$$\begin{aligned} I_2 &= \int_{\mathbb{Z}_p} E_n(x) d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l, \quad \text{where } n \in \mathbb{Z}_+. \end{aligned} \tag{2.8}$$

On the other hand, by (1.12) and (1.13), one gets

$$\begin{aligned} I_2 &= (-1)^n \int_{\mathbb{Z}_p} E_n(1-x)d\mu(x) = (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu(x) \\ &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l}(l + B_l + \delta_{1,l}) \\ &= n(-1)^n E_{n-l}(1) + (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l + (-1)^n n E_{n-l}. \end{aligned} \tag{2.9}$$

From (1.12) and the definition of Euler numbers, one has

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} x^l = (E+x)^n, \tag{2.10}$$

$$E_0 = 1, \qquad (E+1)^n + E_n = 2\delta_{0,n}, \tag{2.11}$$

see [8, 9] with the usual convention of replacing E^n by E_n. By (2.9), (2.10), and (2.11), one gets

$$I_2 = n(-1)^n (2\delta_{0,n-1} - E_{n-1}) + (-1)^n n E_{n-1} + (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l. \tag{2.12}$$

Equating (2.8) and (2.12), one has

$$\left(1 + (-1)^{n-1}\right) \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l = 2n(-1)^n \delta_{0,n-1}. \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N} \cup \{0\}$, one has*

$$\left(1 + (-1)^{n-1}\right) \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l = 2(-1)^n n \delta_{0,n-1}. \tag{2.14}$$

In particular,

$$\sum_{l=0}^{2n+1} \binom{2n+1}{l} E_{2n+1-l} B_l = 0, \quad \text{for } n \in \mathbb{N}. \tag{2.15}$$

Let us consider the following p-adic integral on $\mathbb{Z}_p$ for the product of Bernoulli and Euler polynomials as follows:

$$\begin{aligned} I_3 &= \int_{\mathbb{Z}_p} B_m(x)E_n(x)d\mu(x) \\ &= \sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}B_{m-k}E_{n-\ell}\int_{\mathbb{Z}_p} x^{k+\ell}(x)d\mu(x) \\ &= \sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}B_{m-k}E_{n-\ell}B_{k+\ell}. \end{aligned} \tag{2.16}$$

On the other hand, by (1.10) and (1.13), one gets

$$\begin{aligned} I_3 &= (-1)^{m+n}\int_{\mathbb{Z}_p} B_m(1-x)E_n(1-x)d\mu(x) \\ &= (-1)^{m+n}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}B_{m-k}E_{n-\ell}\int_{\mathbb{Z}_p}(1-x)^{k+\ell}d\mu(x) \\ &= (-1)^{m+n}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{\mathrm{k}}\binom{n}{\ell}B_{m-k}E_{n-\ell}(k+\ell+B_{k+\ell}+\delta_{1,k+\ell}) \\ &= (-1)^{m+n}mB_{m-1}(1)E_n(1)+(-1)^{m+n}nB_m(1)E_{n-1}(1) \\ &\quad + (-1)^{m+n}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}B_{m-k}E_{n-\ell}B_{k+\ell}+(-1)^{m+n}(mB_{m-1}E_n+nB_mE_{n-1}). \end{aligned} \tag{2.17}$$

Equating (2.16) and (2.17), one gets

$$\begin{aligned} &\left((-1)^{m+n+1}+1\right)\sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}B_{m-k}E_{n-\ell}B_{k+\ell} \\ &\quad = (-1)^{m+n}m(B_{m-1}+\delta_{1,m-1})(2\delta_{0,n}-E_n) \\ &\qquad + (-1)^{m+n}n(B_m+\delta_{1,m})(2\delta_{0,n-1}-E_{n-1})+(-1)^{m+n}(nB_mE_{n-1}+mB_{m-1}E_n). \end{aligned} \tag{2.18}$$

For $n \in \mathbb{N}$, by (2.18), one gets

$$\begin{aligned} &\left((-1)^{m+1}+1\right)\sum_{k=0}^{m}\sum_{\ell=0}^{2n}\binom{m}{k}\binom{2n}{\ell}B_{m-k}E_{2n-\ell}B_{k+\ell} \\ &\quad = (-1)^{m+1}2n(B_m+\delta_{1,m})E_{2n-1}+(-1)^{m}(2nB_mE_{2n-1}) \\ &\quad = (-1)^{m+1}2n\delta_{1,m}E_{2n-1}. \end{aligned} \tag{2.19}$$

Therefore, by (2.19), one obtains the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$, one has*

$$\left((-1)^{m+1}+1\right)\sum_{k=0}^{m}\sum_{\ell=0}^{2n}\binom{m}{k}\binom{2n}{\ell}B_{m-k}E_{2n-\ell}B_{k+\ell}=(-1)^{m+1}2n\delta_{1,m}E_{2n-1}. \tag{2.20}$$

In particular, for $m \in \mathbb{N}$, one has

$$\sum_{k=0}^{2m+1}\sum_{\ell=0}^{2n}\binom{2m+1}{k}\binom{2n}{\ell}B_{2m+1-k}E_{2n-\ell}B_{k+\ell}=0. \tag{2.21}$$

By the same motivation, we consider the p-adic integral on $\mathbb{Z}_p$ for the product of Bernoulli and Bernstein polynomials as follows:

$$I_4=\int_{\mathbb{Z}_p}B_m(x)B_{k,n}(x)d\mu(x) \quad \text{where } m,n,k \in \mathbb{N}\cup\{0\}. \tag{2.22}$$

From (1.6) and (1.14), one gets

$$\begin{aligned}
I_4&=\sum_{\ell=0}^{m}\binom{m}{\ell}B_{m-\ell}\int_{\mathbb{Z}_p}x^{\ell}B_{k,n}(x)d\mu(x)\\
&=\binom{n}{k}\sum_{\ell=0}^{m}\binom{m}{\ell}B_{m-\ell}\int_{\mathbb{Z}_p}x^{k+\ell}(1-x)^{n-k}d\mu(x)\\
&=\binom{n}{k}\sum_{\ell=0}^{m}\sum_{j=0}^{n-k}(-1)^{j}\binom{m}{\ell}\binom{n-k}{j}B_{m-\ell}B_{k+\ell+j}.
\end{aligned} \tag{2.23}$$

On the other hand,

$$\begin{aligned}
I_4&=(-1)^{m}\int_{\mathbb{Z}_p}B_m(1-x)B_{n-k,n}(1-x)d\mu(x)\\
&=(-1)^{m}\binom{n}{k}\sum_{\ell=0}^{m}\sum_{j=0}^{k}(-1)^{j}\binom{m}{\ell}\binom{k}{j}B_{m-\ell}\left(n-k+j+\ell+B_{n-k+\ell+j}+\delta_{1,n-k+\ell+j}\right)\\
&=(-1)^{m}\binom{n}{k}(n-k)B_m(1)\delta_{0,k}+(-1)^{m}\binom{n}{k}mB_{m-1}(1)\delta_{0,k}-(-1)^{m}\binom{n}{k}mB_m(1)k\delta_{0,k-1}\\
&\quad+(-1)^{m}\binom{n}{k}\sum_{\ell=0}^{m}\sum_{j=0}^{k}(-1)^{j}\binom{m}{\ell}\binom{k}{j}B_{m-\ell}B_{n-k+\ell+j}\\
&\quad+(-1)^{m}\binom{n}{k}(mB_{m-1}-kB_m)\delta_{n,k}+(-1)^{m}\binom{n}{k}B_m\delta_{n,k+1}.
\end{aligned} \tag{2.24}$$

Equating (2.23) and (2.24), one gets

$$\begin{aligned}
&(-1)^m\sum_{\ell=0}^{m}\sum_{j=0}^{n-k}(-1)^j\binom{m}{\ell}\binom{n-k}{j}B_{m-\ell}B_{k+\ell+j}\\
&\quad=((n-k)B_m(1)+mB_{m-1}(1))\delta_{0,k}-kB_m(1)\delta_{0,k-1}+(mB_{m-1}-kB_m)\delta_{n,k}\\
&\qquad+B_m\delta_{n,k+1}+\sum_{\ell=0}^{m}\sum_{j=0}^{k}(-1)^j\binom{m}{\ell}\binom{k}{j}B_{m-\ell}B_{n-k+\ell+j}.
\end{aligned}\tag{2.25}$$

By (2.25), we obtain the following theorem.

Theorem 2.4. *For $n, m \in \mathbb{N}$, one has*

$$\sum_{\ell=0}^{2m}\sum_{j=0}^{2n}(-1)^j\binom{2m}{\ell}\binom{2n}{j}B_{2m-\ell}B_{\ell+j}=2nB_{2m}+\sum_{\ell=0}^{2m}\binom{2m}{\ell}B_{2m-\ell}B_{2n+\ell}.\tag{2.26}$$

Now, we consider the p-adic integral on $\mathbb{Z}_p$ for the product of Euler and Bernstein polynomials as follows:

$$\begin{aligned}
I_5&=\int_{\mathbb{Z}_p}E_m(x)B_{k,n}(x)d\mu(x)\\
&=\sum_{\ell=0}^{m}\binom{m}{\ell}E_{m-\ell}\int_{\mathbb{Z}_p}x^{\ell}B_{k,n}(x)d\mu(x)\\
&=\binom{n}{k}\sum_{\ell=0}^{m}\sum_{j=0}^{n-k}(-1)^j\binom{m}{\ell}\binom{n-k}{j}E_{m-\ell}B_{k+\ell+j}.
\end{aligned}\tag{2.27}$$

On the other hand, by (1.13) and (1.14), one gets

$$\begin{aligned}
I_5&=(-1)^m\int_{\mathbb{Z}_p}B_{n-k,n}(1-x)E_m(1-x)d\mu(x)\\
&=(-1)^m\binom{n}{k}\sum_{\ell=0}^{m}\sum_{j=0}^{k}(-1)^j\binom{m}{\ell}\binom{k}{j}E_{m-\ell}\int_{\mathbb{Z}_p}(1-x)^{n-k+\ell+j}d\mu(x)\\
&=(-1)^m\binom{n}{k}\sum_{\ell=0}^{m}\sum_{j=0}^{k}(-1)^j\binom{m}{\ell}\binom{k}{j}\left(n-k+\ell+j+B_{n-k+\ell+j}+\delta_{1,n-k+\ell+j}\right)E_{m-\ell}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^m (n-k)\binom{n}{k} E_m(1)\delta_{0,k} + (-1)^m \binom{n}{k} m E_{m-1}(1)\delta_{0,k} - (-1)^m \binom{n}{k} E_m(1) k \delta_{0,k-1} \\
&\quad + (-1)^m \binom{n}{k} \sum_{\ell=0}^{m}\sum_{j=0}^{k} (-1)^j \binom{m}{\ell}\binom{k}{j} E_{m-\ell} B_{n-k+\ell+j} \\
&\quad + (-1)^m \binom{n}{k} \left(\delta_{n,k+1} E_m + \delta_{n,k}\left(m E_{m-1} - k E_m\right)\right).
\end{aligned}
\tag{2.28}
$$

Equating (2.27) and (2.28), one gets

$$
\begin{aligned}
&(-1)^m \sum_{\ell=0}^{m}\sum_{j=0}^{n-k} (-1)^j \binom{m}{\ell}\binom{n-k}{j} E_{m-\ell} B_{k+\ell+j} \\
&\quad = (n-k) E_m(1)\delta_{0,k} + m\delta_{0,k} E_{m-1}(1) - k E_m(1)\delta_{0,k-1} \\
&\qquad + \sum_{\ell=0}^{m}\sum_{j=0}^{k} (-1)^j \binom{m}{\ell}\binom{k}{j} E_{m-\ell} B_{n-k+\ell+j} \\
&\qquad + \delta_{n,k+1} E_m + \left(m E_{m-1} - k E_m\right)\delta_{n,k}.
\end{aligned}
\tag{2.29}
$$

Therefore, by (2.11) and (2.29), we obtain the following theorem.

Theorem 2.5. *For $n, m \in \mathbb{N}$, one has*

$$
\sum_{\ell=0}^{2m}\sum_{j=0}^{2n} (-1)^j \binom{2m}{\ell}\binom{2n}{j} E_{2m-\ell} B_{\ell+j} = -2m E_{2m-1} + B_{2m+2n}.
\tag{2.30}
$$

Finally, we consider the p-adic integral on $\mathbb{Z}_p$ for the product of Euler, Bernoulli, and Bernstein polynomials as follows:

$$
\begin{aligned}
I_6 &= \int_{\mathbb{Z}_p} B_r(x) E_s(x) B_{k,n}(x)\, d\mu(x) \\
&= \binom{n}{k} \sum_{\ell=0}^{r}\sum_{j=0}^{s} \binom{r}{\ell}\binom{s}{j} B_{r-\ell} E_{s-j} \int_{\mathbb{Z}_p} x^{k+\ell+j}(1-x)^{n-k}\, d\mu(x) \\
&= \binom{n}{k} \sum_{\ell=0}^{r}\sum_{j=0}^{s}\sum_{i=0}^{n-k} (-1)^i \binom{r}{\ell}\binom{s}{j}\binom{n-k}{i} B_{r-\ell} E_{s-j} B_{k+\ell+i+j}.
\end{aligned}
\tag{2.31}
$$

On the other hand, by (1.10), (1.13), and (1.14), one gets

$$
\begin{aligned}
I_6 &= (-1)^{r+s} \int_{\mathbb{Z}_p} B_r(1-x) E_s(1-x) B_{n-k,n}(1-x)\, d\mu(x) \\
&= (-1)^{r+s} \binom{n}{k} \sum_{\ell=0}^{r}\sum_{j=0}^{s}\sum_{i=0}^{k} (-1)^i \binom{r}{\ell}\binom{s}{j}\binom{k}{i} B_{r-\ell} E_{s-j} \int_{\mathbb{Z}_p} (1-x)^{n-k+\ell+i+j}\, d\mu(x).
\end{aligned}
\tag{2.32}
$$

Equating (2.31) and (2.32), we easily see that

$$
\begin{aligned}
&(-1)^{r+s}\sum_{\ell=0}^{r}\sum_{j=0}^{s}\sum_{i=0}^{n-k}(-1)^{i}\binom{r}{\ell}\binom{s}{j}\binom{n-k}{i}B_{r-\ell}E_{s-j}B_{k+\ell+i+j}\\
&\quad=\sum_{\ell=0}^{r}\sum_{j=0}^{s}\sum_{i=0}^{k}(-1)^{i}\binom{r}{\ell}\binom{s}{j}\binom{k}{i}\left(n-k+\ell+i+j+B_{n-k+\ell+i+j}+\delta_{1,n-k+\ell+i+j}\right)B_{r-\ell}E_{s-j}\\
&\quad=(n-k)B_r(1)E_s(1)\delta_{0,k}+rB_{r-1}(1)\delta_{0,k}E_s(1)+sB_r(1)E_{s-1}(1)\delta_{0,k}\\
&\qquad-kB_r(1)E_s(1)\delta_{0,k-1}+\sum_{\ell=0}^{\mathrm{r}}\sum_{j=0}^{s}\sum_{i=0}^{k}(-1)^{i}\binom{r}{\ell}\binom{s}{j}\binom{k}{i}B_{r-\ell}E_{s-j}B_{n-k+\ell+i+j}\\
&\qquad+\delta_{n,k+1}B_rE_s+\left(rB_{r-1}E_s+sB_rE_{s-1}-kB_rE_s\right)\delta_{n,k}.
\end{aligned}
\tag{2.33}
$$

Therefore, by (1.5) and (2.11), we obtain the following theorem.

Theorem 2.6. *For $r, n, s \in \mathbb{N}$, one has*

$$
\begin{aligned}
&\sum_{\ell=0}^{2r}\sum_{j=0}^{2s}\sum_{i=0}^{2n}(-1)^{i}\binom{2r}{\ell}\binom{2s}{j}\binom{2n}{i}B_{2r-\ell}E_{2s-j}B_{\ell+i+j}\\
&\quad=-2sB_{2r}E_{2s-1}+\sum_{\ell=0}^{r}\binom{2r}{2l}B_{2r-2l}B_{2n+2l+2s}-r\sum_{j=1}^{s}\binom{2s}{2j-1}E_{2s-2j+1}B_{2n+2r+2j-2}.
\end{aligned}
\tag{2.34}
$$

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