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*Research Article*

# Oscillation and Nonoscillation Criteria for Nonlinear Dynamic Systems on Time Scales

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We consider the nonlinear dynamic system $x^{\Delta}(t) = a(t)g(y(t))$, $y^{\Delta}(t) = -f(t, x^{\sigma}(t))$. We establish some necessary and sufficient conditions for the existence of oscillatory and nonoscillatory solutions with special asymptotic properties for the system. We generalize the known results in the literature. Some examples are included to illustrate the results.

## 1. Introduction

In this paper we investigate the nonlinear two-dimensional dynamic system:

$$x^{\Delta}(t) = a(t)g(y(t)), \qquad y^{\Delta}(t) = -f(t, x^{\sigma}(t)), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $a(t)$ is a nonnegative, rd-continuous function which is defined for $t \in [t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$. Here, $\mathbb{T}$ is a time scale unbounded from above. We assume throughout that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with $ug(u) > 0$ for $u \neq 0$, and $f(t, u) : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous as a function of $u \in \mathbb{R}$ with sign property $uf(t, u) > 0$ for $u \neq 0$ and $t \in [t_0, \infty)_{\mathbb{T}}$.

By the solution of system (1.1), we mean a pair of nontrivial real-valued functions $(x(t), y(t))$ which has property $x, y \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and satisfies system (1.1) for $t \in [t_0, \infty)_{\mathbb{T}}$. Our attention is restricted to those solutions $(x(t), y(t))$ of system (1.1) which exist on some half-line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| + |y(t)| : t \geq t_x\} > 0$ for any $t_x \geq t_0$. As usual, a continuous real-valued function defined on $[T_0, \infty)$ is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. A solution $(x(t), y(t))$ of system (1.1) is called oscillatory if both $x(t)$ and $y(t)$ are oscillatory functions, and otherwise it will be called nonoscillatory. System (1.1) is called oscillatory if its solutions are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. thesis in 1990 in order to unify continuous and discrete analysis (see [1]). Not only can this theory of the so-called “dynamic equations” unify the theories of differential equations and difference equations, but also extend these classical cases to cases “in between”, for example, to the so-called $q$-difference equations and can be applied on other different types of time scales. Since Hilger formed the definition of derivatives and integral on time scales, several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [2] and references cited therein. A book on the subject of time scales (see [3]) summarizes and organizes much of time scale calculus. The reader is referred to Chapter 1 in [3] for the necessary time scale definitions and notations used throughout this paper.

The system (1.1) includes two-dimensional linear/nonlinear differential and difference systems, which were investigated in the literature, see for example [4–9] and the references therein.

On the other hand, the system (1.1) reduces to some important second-order dynamic equations in the particular case, for example

$$\begin{gathered}\left[\frac{x^{\Delta}(t)}{a(t)}\right]^{\Delta}+b(t)f(x^{\sigma}(t))=0,\\ x^{\Delta\Delta}(t)+b(t)|x^{\sigma}(t)|^{\lambda-1}x^{\sigma}(t)=0,\quad \lambda>0,\end{gathered}\tag{1.2}$$

where $b(t)$ is rd-continuous on $[t_0,\infty)_{\mathbb{T}}$. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equation (1.2) on time scales. We refer the reader to the recent papers [10–13] and the references therein. However, most of previous studies for the system (1.1) have been restricted to the case where $f(t,u)=b(t)f(u)$, for example [4–8, 14–18] and the references therein. Erbe and Mert [14, 17] obtained some oscillation results for the system (1.1). Fu and Lin [15] obtained some oscillation and nonoscillation criteria for the linear dynamic system (1.1).

Since there are few works about oscillation and nonoscillation of dynamic systems on time scales (see [15]), motivated by [9, 14, 15], in this paper we investigate oscillatory and nonoscillatory properties for the system (1.1) in the case of general $f(t,u)$ in which $t$ and $u$ are not necessarily separable. In the next section, by means of appropriate hypotheses on $f(t,u)$ and fixed point theorem, we establish some new sufficient and necessary conditions for the existence of nonoscillatory solutions with special asymptotic properties for the system (1.1). In Section 3, we obtain sufficient and necessary conditions for all solutions of the system (1.1) to be oscillatory via the results in Section 2 and some inequality techniques without using Riccati transformation. Our results not only unify the known results of differential and difference systems but also extend and improve the existing results of dynamic systems on time scales in the literature.

## 2. Nonoscillation Results

In this section, we generalize and improve some results of [7–9, 15, 18]. Some necessary and sufficient conditions are given for the system (1.1) to admit the existence of nonoscillatory solutions with special asymptotic properties. These results will be used for the next section. Additional hypotheses on $g(u)$ and $f(t,u)$ are needed for this purpose.

($H_1$) For any positive constant $l$ and $L$ with $l < L$, there exist positive constants $h$ and $H$, depending possibly on $l$ and $L$, such that $l \le |u| \le L$ implies

$$hf(t,l) \le \left|f(t,u)\right| \le Hf(t,L), \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.1}$$

($H_2$) There exists a positive constant $k$ such that $g(uv) \ge kg(u)g(v)$ for $uv > 0$.

($H_3$) For any positive constant $l$ and $L$ with $l < L$, there exist positive constants $h$ and $H$, depending possibly on $l$ and $L$, such that $l \le |u| \le L$ implies

$$hf(t,l\theta(t)) \le \left|f(t,u\theta(t))\right| \le Hf(t,L\theta(t)), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.2}$$

where $\theta(t)$ is a positive nondecreasing function.

For convenience, we will employ the following notation:

$$A(s,t) = \int_s^t a(\tau)\Delta\tau, \quad s,t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.3}$$

**Theorem 2.1.** *Assume that $g$ is nondecreasing and that ($H_1$) holds. Then system* (1.1) *has a nonoscillatory solution $(x(t),y(t))$ such that $\lim_{t\to\infty}x(t) = \alpha \neq 0$ and $\lim_{t\to\infty}y(t) = 0$ if and only if for any positive constant $d$*

$$\int_{t_0}^{\infty} a(t)g\left(d\int_t^{\infty}\left|f(s,c)\right|\Delta s\right)\Delta t < \infty \quad \textit{for some} \quad c \neq 0. \tag{2.4}$$

*Proof.* Suppose that $(x(t),y(t))$ is a nonoscillatory solution of (1.1) such that $\lim_{t\to\infty}x(t) = \alpha \neq 0$ and $\lim_{t\to\infty}y(t) = 0$. Without loss of generality, we assume that $\alpha > 0$. Then there exist $t_1 \in [t_0,\infty)_{\mathbb{T}}$ and positive constants $l$ and $L$ such that $l \le x(t) \le L$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Condition ($H_1$) implies that

$$f(t,x^{\sigma}(t)) \ge hf(t,l) \tag{2.5}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$ and some constant $h > 0$. It follows from the second equation in (1.1) that

$$y(s) - y(t) = -\int_t^s f(\tau,x^{\sigma}(\tau))\Delta\tau. \tag{2.6}$$

Let $s \to \infty$ and noting that $\lim_{s\to\infty}y(s) = 0$, we have

$$y(t) = \int_t^{\infty} f(\tau,x^{\sigma}(\tau))\Delta\tau, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.7}$$

Thus, from (2.5), (2.7) and the first equation in (1.1), we obtain that

$$\begin{aligned}\infty > \lim_{t\to\infty} x(t) - x(t_1) &= \int_{t_1}^{\infty} a(s) g(y(s))\Delta s \\ &= \int_{t_1}^{\infty} a(s) g\left(\int_{s}^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau\right)\Delta s \\ &\geq \int_{t_1}^{\infty} a(s) g\left(h\int_{s}^{\infty} f(\tau, l)\Delta\tau\right)\Delta s,\end{aligned} \tag{2.8}$$

which implies that (2.4) holds.

Conversely, suppose that (2.4) holds, we may assume that $c > 0$. In view of $(\mathrm{H}_1)$, there is a constant $H > 0$ such that $c/2 \leq x(t) \leq c$ implies

$$f(t, x^{\sigma}(t)) \leq H f(t, c), \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.9}$$

Since (2.4) holds, we can choose $t_1 \in [t_0, \infty)_{\mathbb{T}}$ large enough such that

$$\int_{t_1}^{\infty} a(s) g\left(H\int_{s}^{\infty} f(\tau, c)\Delta\tau\right)\Delta s \leq \frac{c}{2}. \tag{2.10}$$

Let $BC[t_0, \infty)_{\mathbb{T}}$ be the Banach space of all real-valued rd-continuous functions on $[t_0, \infty)_{\mathbb{T}}$ endowed with the norm $\|x\| = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} |x(t)| < \infty$. We defined a bounded, convex, and closed subset $\Omega$ of $BC[t_0, \infty)_{\mathbb{T}}$ as

$$\Omega = \left\{x \in BC[t_0, \infty)_{\mathbb{T}} : \frac{c}{2} \leq x(t) \leq c\right\}. \tag{2.11}$$

Define an operator $\Gamma : \Omega \to BC[t_0, \infty)_{\mathbb{T}}$ as follows:

$$(\Gamma x)(t) = \begin{cases} c - \displaystyle\int_{t}^{\infty} a(s) g\left(\int_{s}^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau\right)\Delta s, & t \in [t_1, \infty)_{\mathbb{T}}, \\ c - \displaystyle\int_{t_1}^{\infty} a(s) g\left(\int_{s}^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau\right)\Delta s, & t \in [t_0, t_1]_{\mathbb{T}}. \end{cases} \tag{2.12}$$

Now we show that $\Gamma$ satisfies the assumptions of Schauder's fixed-point theorem (see [19, Corollary 6]).

(i) We will show that $\Gamma x \in \Omega$ for any $x \in \Omega$. In fact, for any $x \in \Omega$ and $t \in [t_1, \infty)_{\mathbb{T}}$, in view of (2.10), we get

$$\begin{aligned} c \geq (\Gamma x)(t) &= c - \int_t^{\infty} a(s) g\left(\int_s^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau\right) \Delta s \\ &\geq c - \int_{t_1}^{\infty} a(s) g\left(H \int_s^{\infty} f(\tau, c) \Delta\tau\right) \Delta s \\ &\geq c - \frac{c}{2} = \frac{c}{2}. \end{aligned} \tag{2.13}$$

Similarly, we can prove that $c/2 \leq (\Gamma x)(t) \leq c$ for any $x \in \Omega$ and $t \in [t_0, t_1)_{\mathbb{T}}$. Hence, $\Gamma x \in \Omega$ for any $x \in \Omega$.

(ii) We prove that $\Gamma$ is a completely continuous mapping. First, we consider the continuity of $\Gamma$. Let $x_n \in \Omega$ and $\|x_n - x\| \to 0$ as $n \to \infty$, then $x \in \Omega$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [t_0, \infty)_{\mathbb{T}}$. Consequently, by the continuity of $g$ and $f$, for any $t \in [t_1, \infty)_{\mathbb{T}}$, we have

$$\lim_{n\to\infty} \left| a(t) \left[ g\left(\int_t^{\infty} f(\tau, x_n^{\sigma}(\tau)) \Delta\tau\right) - g\left(\int_t^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau\right) \right] \right| = 0. \tag{2.14}$$

From (2.9), we obtain that

$$a(t) \left| g\left(\int_t^{\infty} f(\tau, x_n^{\sigma}(\tau)) \Delta\tau\right) - g\left(\int_t^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau\right) \right| \leq 2a(t) g\left(H \int_t^{\infty} f(\tau, c) \Delta\tau\right). \tag{2.15}$$

On the other hand, from (2.12) we have

$$|(\Gamma x_n)(t) - (\Gamma x)(t)| \leq \int_{t_1}^{\infty} a(s) \left| g\left(\int_s^{\infty} f(\tau, x_n^{\sigma}(\tau)) \Delta\tau\right) - g\left(\int_s^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau\right) \right| \Delta s \tag{2.16}$$

for $t \in [t_0, t_1]_{\mathbb{T}}$ and

$$|(\Gamma x_n)(t) - (\Gamma x)(t)| \leq \int_{t}^{\infty} a(s) \left| g\left(\int_s^{\infty} f(\tau, x_n^{\sigma}(\tau)) \Delta\tau\right) - g\left(\int_s^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau\right) \right| \Delta s \tag{2.17}$$

for $t \in [t_1, \infty]_{\mathbb{T}}$. Therefore, from (2.16) and (2.17), we have

$$\|\Gamma x_n - \Gamma x\| \leq \int_{t_1}^{\infty} a(s) \left| g\left(\int_s^{\infty} f(\tau, x_n^{\sigma}(\tau)) \Delta\tau\right) - g\left(\int_s^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau\right) \right| \Delta s. \tag{2.18}$$

Referring to Chapter 5 in [20], we see that the Lebesgue dominated convergence theorem holds for the integral on time scales. Then, from (2.14) and (2.15), (2.18) yields $\lim_{n\to\infty} \|\Gamma x_n - \Gamma x\| = 0$, which implies that $\Gamma$ is continuous on $\Omega$.

Next, we show that $\Gamma\Omega$ is uniformly cauchy. In fact, for any $\varepsilon > 0$, take $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and $t_2 > t_1$ such that

$$\int_{t_2}^{\infty} a(s) g\left( H \int_{s}^{\infty} f(\tau, c) \Delta\tau \right) \Delta s \leq \varepsilon. \tag{2.19}$$

Then for any $x \in \Omega$ and $t, r \in [t_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} |(\Gamma x)(t) - (\Gamma x)(r)| &\leq \left| \int_{t}^{\infty} a(s) g\left( H \int_{s}^{\infty} f(\tau, c) \Delta\tau \right) \Delta s \right| \\ &\quad + \left| \int_{r}^{\infty} a(s) g\left( H \int_{s}^{\infty} f(\tau, c) \Delta\tau \right) \Delta s \right| \\ &\leq 2 \int_{t_2}^{\infty} a(s) g\left( H \int_{s}^{\infty} f(\tau, c) \Delta\tau \right) \Delta s \leq 2\varepsilon. \end{aligned} \tag{2.20}$$

This means that $\Gamma\Omega$ is uniformly cauchy.

Finally, we prove that $\Gamma\Omega$ is equicontinuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we set $t_2 > t_1$. For any $x \in \Omega$, we have $|(\Gamma x)(t) - (\Gamma x)(r)| \equiv 0$ for $t, r \in [t_0, t_1]_{\mathbb{T}}$ and

$$\begin{aligned} |(\Gamma x)(t) - (\Gamma x)(r)| &= \left| \int_{t}^{\infty} a(s) g\left( \int_{s}^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau \right) \Delta s - \int_{r}^{\infty} a(s) g\left( \int_{s}^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau \right) \Delta s \right| \\ &\leq \left| \int_{t}^{r} a(s) g\left( H \int_{s}^{\infty} f(\tau, c) \Delta\tau \right) \Delta s \right| \end{aligned} \tag{2.21}$$

for $t, r \in [t_1, t_2]_{\mathbb{T}}$.

Now, we see that for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $t$, $r \in [t_0, t_2]_{\mathbb{T}}$ with $|t - r| < \delta$, $|(\Gamma x)(t) - (\Gamma x)(r)| < \varepsilon$ for any $x \in \Omega$. This means that $\Gamma\Omega$ is equicontinuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. By Arzela-Ascoli theorem (see [19, Lemma 4]), $\Gamma\Omega$ is relatively compact. From the above, we have proved that $\Gamma$ is a completely continuous mapping.

By Schauder's fixed point theorem, there exists $x \in \Omega$ such that $\Gamma x = x$. Therefore, we have

$$x(t) = c - \int_{t}^{\infty} a(s) g\left( \int_{s}^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau \right) \Delta s, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.22}$$

Set

$$y(t) = \int_{t}^{\infty} f(\tau, x^{\sigma}(\tau)) \Delta\tau, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.23}$$

Then $\lim_{t\to\infty} y(t) = 0$ and $y^{\Delta}(t) = -f(t, x^{\sigma}(t))$. On the other hand,

$$x(t) = c - \int_t^{\infty} a(s) g(y(s)) \Delta s, \tag{2.24}$$

which implies that $\lim_{t\to\infty} x(t) = c$ and $x^{\Delta}(t) = a(t)g(y(t))$. The proof is complete. □

**Corollary 2.2.** *Suppose that $g$ is nondecreasing and that ($H_1$) and ($H_2$) hold. Then system* (1.1) *has a nonoscillatory solution* $(x(t), y(t))$ *such that* $\lim_{t\to\infty} x(t) = \alpha \neq 0$ *and* $\lim_{t\to\infty} y(t) = 0$ *if and only if for some $c \neq 0$*

$$\int_{t_0}^{\infty} a(t) g\left(\int_t^{\infty} |f(s, c)| \Delta s\right) \Delta t < \infty. \tag{2.25}$$

**Theorem 2.3.** *Suppose that* $\lim_{t\to\infty} A(t_0, t) = \infty$ *and $g$ is nondecreasing. Suppose further that ($H_3$) holds. Then system* (1.1) *has a nonoscillatory solution* $(x(t), y(t))$ *such that* $\lim_{t\to\infty}(x(t)/A(t_0, t)) = \alpha \neq 0$ *and* $\lim_{t\to\infty} y(t) = \beta \neq 0$ *if and only if for some $c \neq 0$*

$$\int_{t_0}^{\infty} |f(t, cA(t_0, \sigma(t)))| \Delta t < \infty. \tag{2.26}$$

*Proof.* Suppose that $(x(t), y(t))$ is a nonoscillatory solution of (1.1) such that $\lim_{t\to\infty}(x(t)/A(t_0, t)) = \alpha \neq 0$ and $\lim_{t\to\infty} y(t) = \beta \neq 0$. We may assume that $\alpha > 0$. Hence, there exist $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and positive constant $l$, $L$ such that $lA(t_0, t) \leq x(t) \leq LA(t_0, t)$ and $y^{\Delta}(t) < 0$, $y(t) > \beta$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By condition ($H_3$), there exists a constant $h > 0$ such that $f(t, x^{\sigma}(t)) \geq hf(t, lA(t_0, \sigma(t)))$ for $t \in [t_1, \infty)_{\mathbb{T}}$. According to the second equation in (1.1), we have

$$\infty > y(t_1) - \beta = \int_{t_1}^{\infty} f(t, x^{\sigma}(t)) \Delta t \geq h \int_{t_1}^{\infty} f(t, lA(t_0, \sigma(t))) \Delta t, \tag{2.27}$$

which implies that (2.26) holds with $c = l$.

Conversely, Let (2.26) holds for some $c = 2p$, where $p > 0$. By ($H_3$), there exists a constant $H > 0$ such that $p \leq u \leq 2p$ implies $f(t, uA(t_0, \sigma(t))) \leq Hf(t, 2pA(t_0, \sigma(t)))$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Take $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so large that

$$H \int_{t_1}^{\infty} f(t, cA(t_0, \sigma(t))) \Delta t \leq d, \tag{2.28}$$

where $d = g^{-1}(c)/2$. We introduce $BC[t_1, \infty)_{\mathbb{T}}$ be the partially ordered Banach space of all real-valued and rd-continuous functions $x(t)$ with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}}(|x(t)|/A(t_1, t))$, and the usual pointwise ordering $\leq$.

Define

$$\Omega = \{x \in BC[t_1, \infty)_{\mathbb{T}} : g(d)A(t_1, t) \leq x(t) \leq g(2d)A(t_1, t)\}. \tag{2.29}$$

It is easy to see that $\Omega$ is a bounded, convex, and closed subset of $BC[t_1,\infty)_{\mathbb{T}}$. Let us further define an operator $\Gamma:\Omega \to BC[t_1,\infty)_{\mathbb{T}}$ as follows:

$$(\Gamma x)(t) = \int_{t_1}^{t} a(s) g\left(d + \int_{s}^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau\right)\Delta s, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.30}$$

Since it can be shown that $\Gamma$ is continuous and sends $\Omega$ into a relatively compact subset of $\Omega$, the Schauder's fixed point theorem ensures that the existence of an $x \in \Omega$ such that $x = \Gamma x$, this is

$$x(t) = \int_{t_1}^{t} a(s) g\left(d + \int_{s}^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau\right)\Delta s, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.31}$$

Set

$$y(t) = d + \int_{t}^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.32}$$

Then $\lim_{t\to\infty} y(t) = d$ and $y^{\Delta}(t) = -f(t, x^{\sigma}(t))$. On the other hand, by L'Hôpital's Rule (see [15, Lemma 2.11]), we have

$$\begin{aligned} \lim_{t\to\infty} \frac{x(t)}{A(t_0,t)} &= \lim_{t\to\infty} \frac{a(t) g\left(d + \int_t^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau\right)}{a(t)} \\ &= \lim_{t\to\infty} g\left(d + \int_{t}^{\infty} f(\tau, x^{\sigma}(\tau))\Delta\tau\right) = g(d) \neq 0, \end{aligned} \tag{2.33}$$

and $x^{\Delta}(t) = a(t) g(y(t))$. The proof is complete. □

*Remark 2.4.* Theorems 2.1 and 2.3 extend and improve essentially the known results of [7–9, 15, 18].

## 3. Oscillation Results

In this section, we need some additional conditions to guarantee that the system (1.1) has oscillatory solutions.

($H_4$) There exists a continuous nondecreasing function $\varphi:\mathbb{R} \to \mathbb{R}$ such that

$$\operatorname{sgn}\varphi(u) = \operatorname{sgn} u, \qquad \int^{\pm\infty} \frac{du}{g(\varphi(u))} < \infty, \tag{3.1}$$

and $|f(t,u)| \geq |f(t,l)|\,|\varphi(u)|$, $t \in [t_0,\infty)_{\mathbb{T}}$, $|u| \geq u_0$ for some constants $u_0 > 0$ and $l \neq 0$ with $\operatorname{sgn} l = \operatorname{sgn} u$.

($H_5$) There exists a continuous nondecreasing function $\varphi : [-M, M] \to \mathbb{R}$, $M > 0$ being a constant, such that

$$\operatorname{sgn}\varphi(v) = \operatorname{sgn} v, \qquad \int_0^{\pm M} \frac{dv}{\varphi(g(v))} < \infty, \tag{3.2}$$

and $|f(t,uv)| \geq k|f(t,u)||\varphi(v)|$, $t \in [t_0,\infty)_{\mathbb{T}}$, $u \neq 0$, $0 < |v| < v_0$ for some positive constant $k > 0$ and $v_0 > 0$.

**Theorem 3.1.** *Suppose that* $\lim_{t\to\infty} A(t_0,t) = \infty$ *and* $g$ *is nondecreasing. Suppose further that* ($H_1$), ($H_2$) *and* ($H_4$) *hold. Then system* (1.1) *is oscillatory if and only if for all* $c \neq 0$

$$\int_{t_0}^{\infty} a(t) g\left(\int_t^{\infty} \left|f(s,c)\right| \Delta s\right) \Delta t = \infty. \tag{3.3}$$

*Proof.* If (3.3) does not hold, by Theorem 2.1, system (1.1) has a nonoscillatory solution $(x(t), y(t))$ such that $\lim_{t\to\infty} x(t) = \alpha \neq 0$ and $\lim_{t\to\infty} y(t) = 0$.

Conversely, suppose that (3.3) holds and that (1.1) has a nonoscillatory solution $(x(t), y(t))$ for $t \in [t_0,\infty)_{\mathbb{T}}$. We may assume that $x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Since $\lim_{t\to\infty} A(t_0,t) = \infty$, it is easy to show that $y(t) > 0$, $t \in [t_1,\infty)_{\mathbb{T}}$. From the second equation in (1.1), we have $y^{\Delta}(t) < 0$, $t \in [t_1,\infty)_{\mathbb{T}}$. Hence, $\lim_{t\to\infty} y(t) \geq 0$. It follows from the first equation in (1.1) that $x^{\Delta}(t) > 0$, $t \in [t_1,\infty)_{\mathbb{T}}$, and $\lim_{t\to\infty} x(t) = \infty$ by Theorem 2.1. Integrating the second equation in (1.1) from $t$ to $\infty$ yields that

$$y(t) \geq \int_t^{\infty} f(s, x^{\sigma}(s)) \Delta s, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.4}$$

By (3.4), ($H_2$) and in view of nondecreasing $\varphi$, it follows that

$$\frac{x^{\Delta}(t)}{g(\varphi(x^{\sigma}(t)))} = \frac{a(t)g(y(t))}{g(\varphi(x^{\sigma}(t)))} \geq \frac{a(t)g\left(\int_t^{\infty} f(s,x^{\sigma}(s))\Delta s\right)}{g(\varphi(x^{\sigma}(t)))} \geq ka(t)g\left(\int_t^{\infty} \frac{f(s,x^{\sigma}(s))}{\varphi(x^{\sigma}(s))} \Delta s\right) \tag{3.5}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$.

Since ($H_4$) holds and $\lim_{t\to\infty} x(t) = \infty$, there is $t_2 \in [t_1,\infty)_{\mathbb{T}}$ and $l > 0$ such that

$$\frac{f(t,x^{\sigma}(t))}{\varphi(x^{\sigma}(t))} \geq f(t,l) \tag{3.6}$$

for $t \in [t_2,\infty)_{\mathbb{T}}$. From (3.5) and (3.6), we get

$$\frac{x^{\Delta}(t)}{g(\varphi(x^{\sigma}(t)))} \geq ka(t)g\left(\int_t^{\infty} f(s,l)\Delta s\right). \tag{3.7}$$

Integrating (3.7) from $t_2$ to $t$, we have

$$\int_{t_2}^{t} \frac{x^{\Delta}(s)}{g(\varphi(x^{\sigma}(s)))}\Delta s \geq k\int_{t_2}^{t} a(s)g\left(\int_{s}^{\infty} f(\tau,l)\Delta\tau\right)\Delta s. \tag{3.8}$$

Since $g$, $\varphi$, and $x$ are nondecreasing, we obtain

$$\int_{t_2}^{t} \frac{x^{\Delta}(s)}{g(\varphi(x^{\sigma}(s)))}\Delta s \leq \int_{t_2}^{t} \frac{x^{\Delta}(s)}{g(\varphi(x(s)))}\Delta s. \tag{3.9}$$

By ($H_4$), (3.8) and (3.9), we get

$$k\int_{t_2}^{t} a(s)g\left(\int_{s}^{\infty} f(\tau,l)\Delta\tau\right)\Delta s \leq \int_{x(t_2)}^{x(t)} \frac{du}{g(\varphi(u))} < \infty, \tag{3.10}$$

which contradicts (3.3) when $t \to \infty$. The proof is complete. □

**Theorem 3.2.** *Suppose that* $\lim_{t\to\infty} A(t_0,t) = \infty$ *and* ($H_5$) *holds. Suppose further that* $f(t,u)$ *is nondecreasing in* $u$ *for each fixed* $t \in [t_0,\infty)_{\mathbb{T}}$ *and* $g$ *is nondecreasing. Then system* (1.1) *is oscillatory if and only if for all* $c \neq 0$

$$\int_{t_0}^{\infty} \left|f(t,cA(t_0,\sigma(t)))\right|\Delta t = \infty. \tag{3.11}$$

*Proof.* If (3.11) does not hold, by Theorem 2.3, system (1.1) has a nonoscillatory solution $(x(t),y(t))$ such that $\lim_{t\to\infty}(x(t)/A(t_0,t)) = \alpha \neq 0$ and $\lim_{t\to\infty} y(t) = \beta \neq 0$.

Conversely, suppose that (3.11) holds and that system (1.1) has a nonoscillatory solution $(x(t),y(t))$ for $t \in [t_0,\infty)_{\mathbb{T}}$. We assume that $x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then by same argument in the proof of Theorem 3.1, we have $x^{\Delta}(t) > 0$, $y^{\Delta}(t) < 0$, $y(t) > 0$ eventually. We claim that (3.11) implies $\lim_{t\to\infty} y(t) = 0$. In fact, if $\lim_{t\to\infty} y(t) = \beta > 0$, then $y(t) \geq \beta$ for $t \in [t_1,\infty)_{\mathbb{T}}$. According to the first equation in (1.1), we get

$$\begin{aligned} x^{\sigma}(t) &= x(t_1) + \int_{t_1}^{\sigma(t)} a(s)g(y(s))\Delta s \\ &\geq \int_{t_1}^{t} a(s)g(y(s))\Delta s + \int_{t}^{\sigma(t)} a(s)g(y(s))\Delta s \\ &\geq g(y(t))\left[A(t_1,t) + \mu(t)a(t)\right] = g(y(t))A(t_1,\sigma(t)) \geq g(\beta)A(t_1,\sigma(t)). \end{aligned} \tag{3.12}$$

Integrating the second equation in (1.1) from $t_1$ to $\infty$, we have

$$\beta - y(t_1) = -\int_{t_1}^{\infty} f(s,x^{\sigma}(s))\Delta s \leq -\int_{t_1}^{\infty} f(s,g(\beta)A(t_1,\sigma(s)))\Delta s = -\infty, \tag{3.13}$$

which is a contradiction. Hence, $\lim_{t\to\infty} y(t) = 0$.

By ($H_5$), we have

$$kf(t, A(t_1, \sigma(t))) \leq kf\left(t, \frac{x^\sigma(t)}{g(y(t))}\right) \leq \frac{f(t, x^\sigma(t))}{\varphi(g(y(t)))} = \frac{-y^\Delta(t)}{\varphi(g(y(t)))}. \tag{3.14}$$

From (3.14), it follows

$$\int_{t_1}^{t} kf(s, A(t_1, \sigma(s)))\Delta s \leq -\int_{t_1}^{t} \frac{y^\Delta(s)}{\varphi(g(y(s)))}\Delta s. \tag{3.15}$$

Hence,

$$k\int_{t_1}^{t} f(s, A(t_1, \sigma(s)))\Delta s \leq -\int_{y(t_1)}^{y(t)} \frac{du}{\varphi(g(u))}. \tag{3.16}$$

In view of ($H_5$) and (3.11), this is a contradiction. The proof is complete. □

*Remark 3.3.* Theorems 3.1 and 3.2 improve the existing results of [15, 18].

*Example 3.4.* Consider the system:

$$x^\Delta(t) = \left|y(t)\right|^{1/\alpha-1} y(t), \qquad y^\Delta(t) = -\frac{t^v |x^\sigma(t)|^{\gamma-1} x^\sigma(t)}{1 + t^u |x^\sigma(t)|^m}, \tag{3.17}$$

where $\mathbb{T} = a\mathbb{N} = \{an \mid n \in \mathbb{N}\}$, $a$, $m$, $\gamma$, $u$, $\alpha > 0$ and $v$ are constants as well as $\gamma > m$.

Let

$$a(t) = 1, \qquad g(y) = \left|y\right|^{(1/\alpha)-1} y, \qquad f(t, x) = \frac{t^v |x|^{\gamma-1} x}{1 + t^u |x|^m}. \tag{3.18}$$

It is easy to see that $g(y)$ is increasing and for $0 < l \leq x \leq L$, $\gamma \geq m$,

$$\begin{gathered} f(t, l) \leq f(t, x) \leq f(t, L), \\ f(t, lt) \leq f(t, xt) \leq f(t, Lt). \end{gathered} \tag{3.19}$$

For $u > v + 2\alpha$, we have

$$\begin{aligned}
\int_{t_0}^{\infty} a(t) g\left(\int_t^{\infty} |f(s,c)| \Delta s\right) \Delta t &= \int_{t_0}^{\infty} \left(\int_t^{\infty} \frac{s^v |c|^\gamma}{1 + s^u |c|^m} \Delta s\right)^{1/\alpha} \Delta t \\
&\le |c|^{(\gamma - m)/\alpha} \int_{t_0}^{\infty} \left(\int_t^{\infty} s^{v-u} \Delta s\right)^{1/\alpha} \Delta t \\
&\le |c|^{(\gamma - m)/\alpha} a^{(v-u+\alpha+1)/\alpha} \sum_{n=n_0}^{\infty} \sum_{r=n}^{\infty} r^{(v-u)/\alpha} \\
&= |c|^{(\gamma - m)/\alpha} a^{(v-u+\alpha+1)/\alpha} \sum_{n=n_0}^{\infty} n^{(v-u+\alpha)/\alpha} < \infty,
\end{aligned} \tag{3.20}$$

that is, (2.25) holds. By Corollary 2.2, system (3.17) has a nonoscillatory solution $(x(t), y(t))$ such that $\lim_{t\to\infty} x(t) \ne 0$ and $\lim_{t\to\infty} y(t) = 0$.

On the other hand, For $u + m > v + \gamma + 1$, we obtain

$$\begin{aligned}
\int_a^{\infty} |f(t, cA(a, \sigma(t)))| \Delta t &= \int_a^{\infty} \frac{t^{v+\gamma} |c|^\gamma}{1 + t^{u+m} |c|^m} \Delta t \\
&\le |c|^{\gamma - m} \int_a^{\infty} t^{v+\gamma-u-m} \Delta t \\
&= |c|^{\gamma - m} a^{v+\gamma-u-m+1} \sum_{n=1}^{\infty} n^{v+\gamma-u-m} < \infty.
\end{aligned} \tag{3.21}$$

Hence, (2.26) holds. By Theorem 2.3, system (3.17) has a nonoscillatory solution $(x(t), y(t))$ such that $\lim_{t\to\infty}(x(t)/t) \ne 0$ and $\lim_{t\to\infty} y(t) \ne 0$.

*Example 3.5.* Consider the system:

$$x^\Delta(t) = \frac{1}{t} y^5(t), \qquad y^\Delta(t) = -\frac{t^3 |x^\sigma(t)|^{4/3} x^\sigma(t)}{1 + t^3 (x^\sigma(t))^2}, \tag{3.22}$$

where $\mathbb{T} = a^{\mathbb{N}_0}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $a > 1$.

Let

$$a(t) = \frac{1}{t}, \qquad g(y) = y^5, \qquad f(t,x) = \frac{t^3 |x|^{4/3} x}{1 + t^3 x^2}. \tag{3.23}$$

Obviously, $f(t,x)$ is increasing in $x$ for fixed $t$, and taking $\varphi(u) = |u|^{-2/3} u$, we have

$$\begin{gathered}
|f(t,x)| \ge |f(t, \operatorname{sgn} x)| |x|^{1/3}, \quad |x| \ge 1, \\
\int^{\pm\infty} \frac{du}{g(\varphi(u))} = \int^{\pm\infty} \frac{du}{u^{5/3}} < \infty.
\end{gathered} \tag{3.24}$$

On the other hand, we obtain

$$\begin{aligned}\int_1^\infty a(t)g\left(\int_t^\infty \left|f(s,c)\right|\Delta s\right)\Delta t &= \int_1^\infty \frac{1}{t}\left(\int_t^\infty \frac{s^3|c|^{7/3}}{1+s^3|c|^2}\Delta s\right)^5 \Delta t \\ &\le (a-1)^6|c|^{35/3}\sum_{n=0}^{\infty}\left(\sum_{r=n}^{\infty}\frac{a^{4r}}{1+a^{3r}|c|^2}\right)^5 = \infty,\end{aligned} \tag{3.25}$$

that is, (3.3) holds. Hence, system (3.22) is oscillatory by Theorem 3.1.

*Example 3.6.* Consider the system:

$$x^\Delta(t) = y(t), \qquad y^\Delta(t) = -b(t)|x^\sigma(t)|^\lambda \operatorname{sgn} x^\sigma(t) \tag{3.26}$$

on a time scale $\mathbb{T}$ which contains only isolated points and is unbounded above. Here, $a(t) = 1$, $g(y) = y$, $0 < \lambda < 1$, $f(t,x) = b(t)|x|^\lambda \operatorname{sgn} x$, $b(t)$ is a nonnegative rd-continuous function on $[t_0,\infty)_{\mathbb{T}}$.

We take $\varphi(x) = |x|^{\lambda-1}x$, $0 < |x| \le 1$, then all conditions of Theorem 3.2 are satisfied. Hence, system (3.26) is oscillatory if and only if

$$\int_{t_0}^\infty \sigma^\lambda(t)b(t)\Delta t = \infty. \tag{3.27}$$

On the other hand, system (3.26) can be written in the Emden-Fowler equation:

$$x^{\Delta\Delta}(t) + b(t)|x^\sigma(t)|^\lambda \operatorname{sgn} x^\sigma(t) = 0. \tag{3.28}$$

Since we do not assume that $\lambda$ is a quotient of odd positive integers, (3.28) includes the equation studied in [21]. Theorem 3.2 generalizes and improves Theorem 7 of [21].

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