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Research Article

Dynamic Analysis of an Impulsively Controlled Predator-Prey Model with Holling Type IV Functional Response

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The dynamic behavior of a predator-prey model with Holling type IV functional response is investigated with respect to impulsive control strategies. The model is analyzed to obtain the conditions under which the system is locally asymptotically stable and permanent. Existence of a positive periodic solution of the system and the boundedness of the system is also confirmed. Furthermore, numerical analysis is used to discover the influence of impulsive perturbations. The system is found to exhibit rich dynamics such as symmetry-breaking pitchfork bifurcation, chaos, and nonunique dynamics.

1. Introduction

In recent years, impulsive control strategies in predator-prey models have become a major field of inquiry. Many authors have studied the dynamics of predator-prey models with impulsive control strategies [1–10]. Much research has also been done on three-species food chain systems with impulsive perturbations [11–17]. Holling-type functional responses are well known, and therefore many authors have studied the dynamics of predator-prey models with different Holling-type functional responses with respect to an impulsive control strategy. For example, the dynamics of a predator-prey model with Holling type I functional response with respect to an impulsive control strategy have been reported in [18]. The dynamics of a predator-prey model with Holling type II functional response with an impulsive control strategy were presented in [19, 20]. In [21, 22], the results of studies of the dynamics of a predator-prey model with Holling type VI functional response with respect to an impulsive control strategy were presented. The following predator-prey model with Holling type VI functional response with respect to an impulsive control strategy was proposed in [23]:

$$\begin{aligned}
&\left.\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(a-bx(t)) - \frac{c_1x(t)y(t)}{1+e_1(x(t))^2},\\
\frac{dy(t)}{dt} &= \frac{c_2x(t)y(t)}{1+e_1(x(t))^2} - d_1y(t),
\end{aligned}\right\} \quad t \neq (n+\tau-1)T,\ t \neq nT,\\
&\left.\begin{aligned}
x(t^+) &= (1-p_1)x(t),\\
y(t^+) &= (1-p_2)y(t),
\end{aligned}\right\} \quad t = (n+\tau-1)T,\\
&\left.\begin{aligned}
x(t^+) &= x(t),\\
y(t^+) &= y(t)+q,
\end{aligned}\right\} \quad t = nT,\\
&(x(0^+), y(0^+)) = (x_0, y_0),
\end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the respective densities of prey and predator at time t, the constant a is the intrinsic growth rate of the prey population, b is the coefficient of intraspecies competition, c_1 is the per-capita rate of predation of the predator, e_1 is the half-saturation constant, which depicts a critical concentration of the nutrient composition to maintain the normal growth of the predator, d_1 is the death rate of the predator, and c_2 is the rate of conversion of a consumed prey to a predator.

More recently, the author of [24] has developed the following three-species Holling type IV system by introducing spraying of pesticides and periodic constant release of mid-level predators at different fixed times:

$$\begin{aligned}
&\left.\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(a-bx(t)) - \frac{c_1x(t)y(t)}{1+e_1(x(t))^2},\\
\frac{dy(t)}{dt} &= \frac{c_2x(t)y(t)}{1+e_1(x(t))^2} - \frac{c_3y(t)z(t)}{1+e_2(y(t))^2} - d_1y(t),\\
\frac{dz(t)}{dt} &= \frac{c_4y(t)z(t)}{1+e_2(y(t))^2} - d_2z(t),
\end{aligned}\right\} \quad t \neq (n+\tau-1)T,\ t \neq nT,\\
&\left.\begin{aligned}
\Delta x(t) &= -p_1x(t),\\
\Delta y(t) &= -p_2y(t),\\
\Delta z(t) &= -p_3z(t),
\end{aligned}\right\} \quad t = (n+\tau-1)T,\\
&\left.\begin{aligned}
x(t^+) &= x(t),\\
y(t^+) &= y(t)+q,\\
z(t^+) &= z(t)
\end{aligned}\right\} \quad t = nT,\\
&(x(0^+), y(0^+)) = (x_0, y_0),
\end{aligned} \tag{1.2}$$

where c_3 is the per-capita rate of predation of the top predator, e_2 is the half-saturation constant, d_2 is the death rate of the top predator, and c_4 is the rate of conversion of consumed mid-level predators to the top predator.

Assume that the top predator also eats the prey, or in other words, that the relationship between the top predator and the mid-level predator is not only that of predator and prey, but also that of competitors. To represent this, a predator-prey model with Holling type

IV functional response with respect to an impulsive control strategy can be constructed as follows:

$$\begin{aligned}
&\frac{dx(t)}{dt}=x(t)(a-bx(t))-\frac{c_1x(t)y(t)}{1+e_1(x(t))^2}-\frac{c_5x(t)z(t)}{1+e_3(x(t))^2},\\
&\frac{dy(t)}{dt}=\frac{c_2x(t)y(t)}{1+e_1(x(t))^2}-\frac{c_3y(t)z(t)}{1+e_2(y(t))^2}-d_1y(t), \qquad t\neq nT,\\
&\frac{dz(t)}{dt}=\frac{c_4y(t)z(t)}{1+e_2(y(t))^2}+\frac{c_6x(t)z(t)}{1+e_3(x(t))^2}-d_2z(t),\\
&\Delta x(t)=0,\\
&\Delta y(t)=-qy(t)+p, \quad t=nT,\\
&\Delta z(t)=0,
\end{aligned} \tag{1.3}$$

where $x(t)$, $y(t)$, and $z(t)$ are the respective densities of the prey, top predator, and mid-level predator at time t, c_5 is the per-capita rate of predation of the top predator, e_3 is the half-saturation constant, and c_6 is the rate of conversion of consumed prey to the top predator. The meanings of the other parameters are the same as in (1.1) and (1.2). $\Delta x(t)=x(t^+)-x(t)$, $\Delta y(t)=y(t^+)-y(t)$, and $\Delta z(t)=z(t^+)-z(t)$, $0\leq q<1$, represent the fractions of prey and predator which die because of harvesting or for other reasons, p is the magnitude of immigration or stocking of the predator, and T is the period of impulsive immigration or stocking of the predator. Here it is assumed that the rate of conversion of consumed prey to predator is smaller than the per-capita rate of predation of the predator.

The rest of this paper is organized as follows. Section 2 presents a mathematical analysis of the model. Section 3 describes some numerical simulations. In the last section, a brief discussion is provided, and conclusions are drawn.

2. Mathematical Analysis

First, some useful notations and statements will be provided for use in subsequent proofs. The following definitions will be useful.

Let $R_+=[0,+\infty)$, $R_+^3=\{X=(x(t),y(t),z(t))\in R^3 \mid X\geq 0\}$. Denote by $f=(f_1,f_2,f_3)$ the map defined by the right-hand sides of the first, second, third, and fourth equations of system (1.3). Let$V_0=\{V:R_+\times R_+^3\rightarrow R_+\}$; then V is continuous on $(nT,(n+1)T]\times R_+^3, n\in N$, $\lim_{(t,\mu)\rightarrow(nT^+,X)}V(t,\mu)=V(nT^+,X)$ exists, and V is locally Lipschitzian in X.

Definition 2.1. Let $V\in V_0$; then for $(nT,(n+1)T]\times R_+^3$, the upper right derivative of $V(t,X)$ with respect to the impulsive differential system (1.3) can be defined as

$$D^+V(t,X)=\lim_{h\rightarrow 0^+}\sup\frac{1}{h}\left[V(t+h,X+hf(t,X))-V(t,X)\right]. \tag{2.1}$$

The solution of system (1.3) is a piecewise continuous function $X:R_+\times R_+^3$, $X(t)$ is continuous on $(nT,(n+1)T], n\in N$, and $X(nT^+)=\lim_{t\rightarrow nT^+}X(T)$ exists. Obviously, the smoothness properties of f guarantee the global existence and uniqueness of a solution of system (1.3); for details, see [25, 26].

Definition 2.2. System (1.3) is permanent if there exists $M \geq m > 0$ such that, for any solution $(x(t), y(t), z(t))$ of system (1.3) with a positive initial value X_0, $m \leq \lim_{t\to\infty} \inf x(t) \leq \lim_{t\to\infty} \sup x(t) \leq M$, $m \leq \lim_{t\to\infty} \inf y(t) \leq \lim_{t\to\infty} \sup y(t) \leq M$, and $m \leq \lim_{t\to\infty} \inf z(t) \leq \lim_{t\to\infty} \sup z(t) \leq M$.

Lemma 2.3. *Assume that $X(t)$ is a solution of system* (1.3) *with $X(0^+) \geq 0$; then $X(t) \geq 0$ for all $t \geq 0$. Furthermore, $X(t) > 0$, $t > 0$ if $X(0^+) > 0$.*

Lemma 2.4 (see [23]). *Let $V \in V_0$ and assume that*

$$\begin{aligned} D^+V(t,X) &\leq g(t, V(t,X)), \quad t \neq nT, \\ V(t, X(t^+)) &\leq \Phi_n(V(t, X(t))), \quad t = nT, \end{aligned} \tag{2.2}$$

where $g : R_+ \times R_+ \to R$ is continuous on $(nT, (n+1)T]$ for $u \in R_+^2$, $n \in N$, $\lim_{(t,y)\to(nT^+)} g(t,v) = g(nT^+, u)$ exists, and $\phi_n^i (i = 1,2) : R_+ \to R_+$ is nondecreasing. Let $r(t)$ be a maximal solution of the scalar impulsive differential equation:

$$\begin{aligned} \frac{du(t)}{dt} &= g(t, u(t)), \quad t \neq nT, \\ u(t^+) &= \Phi_n(u(t)), \quad t = nT, \\ u(0^+) &= u_0 \end{aligned} \tag{2.3}$$

existing on $(0, +\infty]$. Then $V(0^+, X_0) \leq u_0$ implies that $V(t, X(t)) \leq r(t)$, $t \geq 0$, where $X(t)$ is any solution of system (1.3). *If certain smoothness conditions on g can be established to guarantee the existence and uniqueness of solutions for* (2.3), *then $r(t)$ is exactly a unique solution of* (2.3).

Now consider a special case of Lemma 2.4. Let $\mathrm{PC}(R^+, R)(\mathrm{PC}^1(R^+, R))$ denote the class of real piecewise continuous functions defined on R_+.

Lemma 2.5 (see [23]). *Let $u(t) \in PC^1(R^+, R)$ and let it satisfy the inequalities:*

$$\begin{aligned} u'(t) &\leq f(t)u(t) + h(t), \quad t \neq \tau_k, \ t > 0, \\ u(\tau_k^+) &\leq \alpha_k u(\tau_k) + \beta_k, \quad k \geq 0, \\ u(0^+) &\leq u_0, \end{aligned} \tag{2.4}$$

where $f, h \in PC^1(R_+, R)$ and $\alpha_k \geq 0$, β_k and u_0 are constants, and $\tau_k (k \geq 0)$ is a strictly increasing sequence of positive real numbers. Then, for $t > 0$,

$$\begin{aligned} u(t) \leq{}& u_0 \left(\prod_{0<\tau_k<t} \alpha_k \right) \exp\left(\int_0^t f(s)ds \right) + \int_0^t \left(\prod_{s\leq\tau_k<t} \alpha_k \right) \exp\left(\int_s^t f(\gamma) \right) h(s)ds \\ &+ \sum_{0<\tau_k<t} \left(\prod_{\tau_k<\tau_j<t} \alpha_j \right) \exp\left(\int_{\tau_k}^t f(\gamma)d\gamma \right) \beta_k. \end{aligned} \tag{2.5}$$

For convenience, some basic properties can be defined for the following subsystems of system (1.3):

$$\begin{gathered}\frac{dy(t)}{dt}=-d_1y(t),\quad t\neq nT,\\ y(t^+)=(1-q)y(t)+p,\quad t=nT,\\ y(0^+)=y_0.\end{gathered}\tag{2.6}$$

Then the following lemma results.

Lemma 2.6. *For a positive periodic solution $y^*(t)$ of system* (2.6) *and a solution $y(t)$ of system* (2.6) *with initial value $y_0=y(0^+)\geq 0$,*

$$\left|y(t)-y^*(t)\right|\longrightarrow 0,\quad t\longrightarrow\infty,\tag{2.7}$$

where

$$\begin{gathered}y^*(t)=\frac{p\exp(-d_1(t-nT))}{1-(1-q)\exp(-d_1T)},\quad t\in(nT,(n+1)T],\ n\in N,\\ y^*(0^+)=\frac{p}{1-(1-q)\exp(-d_1T)},\\ y(t)=(1-q)^{n+1}\left(y(0^+)-\left(\frac{p}{1-(1-q)\exp(-d_1T)}\right)\right)\exp(-d_1T)+y^*(t).\end{gathered}\tag{2.8}$$

Next, some main theorems will be proposed.

Theorem 2.7 (boundedness). *There exists a constant M such that $x(t)\leq M$, $y(t)\leq M$, and $z(t)\leq M$ for each solution $X=(x(t),y(t),z(t))$ of system* (1.3) *for all sufficiently large t.*

Proof. Let $(x(t),y(t),z(t))$ be a solution of system (1.3); let $u(t)=x(t)+y(t)+z(t),(t\geq 0)$. Then

$$\begin{aligned}u'(t)=x(t)(a-bx(t))+(c_2-c_1)\frac{x(t)y(t)}{1+e_1(x(t))^2}+(c_4-c_3)\frac{y(t)z(t)}{1+e_2(y(t))^2}\\ +(c_6-c_5)\frac{x(t)z(t)}{1+e_3(x(t))^2}-d_1y(t)-d_2z(t).\end{aligned}\tag{2.9}$$

From the first part, it is known that $c_2-c_1<0, c_4-c_3<0, c_6-c_5<0$, so $u'(t)\leq x(t)(a-bx(t))-d_1y(t)-d_2z(t)$. Choosing $0<C<\min\{d_1,d_2\}$, then $u'(t)+Cu(t)\leq -b(x(t))^2+(a+C)x(t)<\beta_0,(\beta_0=(a+C)^2/4b)$. The following can then be obtained:

$$\begin{gathered}u'(t)+Cu(t)\leq\beta_0,\quad t\neq nT,\\ u(nT^+)\leq(1-q)u(nT)+p.\end{gathered}\tag{2.10}$$

From Lemma 2.5,

$$\begin{aligned}
u(t) &\le u_0(1-q)^{[t/kT]}\exp\left(\int_0^t -C ds\right) + \int_0^t (1-q)^{[(t-s)/kT]}\exp\left(\int_s^t -C d\gamma\right)\beta_0 ds \\
&\quad + \sum_{j=1}^{[t/kT]} (1-q)^{[(t-s)/kT]}\exp\left(\int_{kT}^t -C d\gamma\right)p \\
&\le u(0^+)\exp(-Ct) + \frac{\beta_0}{C}\left(1-\exp(-Ct)\right) + \frac{p\exp(CT)}{\exp(CT)-1}.
\end{aligned} \tag{2.11}$$

When $t \to \infty$, (2.11) $\to \beta_0/C + p\exp(CT)/(\exp(CT)-1) < \infty$. It is clear that $u(t)$ is bounded for sufficiently large t. Therefore, $x(t)$, $y(t)$, and $z(t)$ are bounded. This completes the proof. □

Next the stability of a prey and top-predator eradication periodic solution will be examined.

Theorem 2.8. *The solution* $(0, y^*(t), 0)$ *is locally asymptotically stable if* $T < c_1\Gamma/a$ *and* $T > (c_4/d_2)((\theta_1-\theta_2)/d_1\sqrt{e_2})$, *where* $\Gamma = p(1-\exp(-d_1T))/d_1(1-(1-q)\exp(-d_1T))$, $\theta_1 = e_2p/\Lambda$, $\theta_2 = (e_2p/\Lambda)\exp(-d_1T)$, *and* $\Lambda = \sqrt{e_2}(1-(1-q)\exp(-d_1T))$.

Proof. The local stability of the periodic solution $(0, y^*(t), 0, 0)$ may be determined by considering the behavior of small-amplitude perturbations of the solution. Define

$$x(t) = u(t), \quad y(t) = v(t) + y^*(t), \quad z(t) = w(t). \tag{2.12}$$

Substituting (2.12) into system (1.3), it is possible to obtain a linearization of the system as follows:

$$\begin{aligned}
&\left.\begin{aligned}
\frac{du(t)}{dt} &= (a - c_1 y^*(t))u(t), \\
\frac{dv(t)}{dt} &= c_2 y^*(t)u(t) - d_1 v(t) - \frac{c_3 y^*(t)}{1+e_2(y^*(t))^2}w(t) \\
\frac{dw(t)}{dt} &= \left(-d_2 + \frac{c_4 y^*(t)}{1+e_2(y^*(t))^2}\right)w(t),
\end{aligned}\right\} \quad t \ne nT, \\
&\left.\begin{aligned}
\Delta u(t) &= 0, \\
\Delta v(t) &= -qy(t) + p \\
\Delta w(t) &= 0
\end{aligned}\right\} \quad t = nT,
\end{aligned} \tag{2.13}$$

which can be written as

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \phi(t)\begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix}, \quad 0 \le t \le T, \tag{2.14}$$

where the $\phi(t)$ satisfies

$$\frac{d\phi(t)}{dt}=\begin{pmatrix} a-c_1y^*(t) & 0 & 0 \\ c_2y^*(t) & -d_1 & -\dfrac{c_3y^*(t)}{1+e_2(y^*(t))^2} \\ 0 & 0 & -d_2+\dfrac{c_4y^*(t)}{1+e_2(y^*(t))^2} \end{pmatrix} \tag{2.15}$$

with $\phi(0)=I$, where I is the identity matrix and

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \\ w(nT^+) \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u(nT) \\ v(nT) \\ w(nT) \end{pmatrix}. \tag{2.16}$$

Hence, the stability of the periodic solution $(0,y^*(t),0)$ is determined by the eigenvalues of

$$\theta=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\phi(t). \tag{2.17}$$

If the absolute values of all eigenvalues are less than one, the periodic solution $(0,y^*(t),0)$ is locally stable. Then all eigenvalues of ϕ can be denoted by λ_1,λ_2, and λ_3:

$$\begin{aligned} \lambda_1&=\exp\left(\int_0^T\left(a-c_1y^*(t)dt\right)\right), \\ \lambda_2&=\exp(-d_1T)<1, \\ \lambda_3&=\exp\left(\int_0^T\left(-d_2+\frac{c_4y^*(t)}{1+e_2(y^*(t))^2}\right)dt\right). \end{aligned} \tag{2.18}$$

Therefore,

$$\int_0^T y^*(t)dt=\frac{p(1-\exp(-d_1T))}{d_1(1-(1-q)\exp(-d_1T))}\triangleq\Gamma, \qquad \int_0^T\frac{y^*(t)}{1+e_2(y^*(t))^2}dt=\frac{1}{d_1\sqrt{e_2}}(\theta_1-\theta_2), \tag{2.19}$$

where $\theta_1=\tan^{-1}(e_2p/\Lambda)$, $\theta_2=\tan^{-1}((e_2p/\Lambda)\exp(-d_1T))$, and $\Lambda=\sqrt{e_2}(1-(1-q)\exp(-d_1T))$. Clearly, $|\lambda_2|<1$ and $|\lambda_1|<1$ only if $T<c_1\Gamma/a$, and $|\lambda_3|<1$ only if $T>(c_4/d_2)((\theta_1-\theta_2)/d_1\sqrt{e_2})$. According to the Floquet theory of impulsive differential equations, the periodic solution $(0,y^*(t),0)$ is locally stable. This completes the proof. □

Theorem 2.9. *The solution $(0,y^*(t),0)$ is said to be globally stable if $T>c_4\Gamma/d_2$.*

Proof. By Theorem 2.7, there exists a constant $M > 0$ such that $x(t) \le M$, $y(t) \le M$, $z(t) \le M$ for each solution $X = (x(t), y(t), z(t))$ of system (1.3) for all sufficiently large t. From the first equation in system (1.3), it is possible to obtain

$$x'(t) \le ax(t) - \frac{c_1 x(t) y(t)}{1 + e_1 M^2} = \left(a - \frac{c_1 y(t)}{1 + e_1 M^2}\right) x(t). \tag{2.20}$$

Then,

$$\begin{aligned} \frac{dy}{dt} = \frac{c_2 x(t) y(t)}{1 + e_1 (x(t))^2} - \frac{c_3 y(t) z(t)}{1 + e_2 (y(t))^2} - d_1 y(t) \le -(d_1 - c_2 M) y(t), \quad t \neq nT \\ \Delta y(t) = -q y(t) + p, \quad t = nT \end{aligned} \tag{2.21}$$

and using (2.21),

$$y(t) \ge y_1^*(t) - \varepsilon = \frac{p \exp(-(d_1 - c_2 M)(t - nT))}{1 - (1 - q) \exp(-(d_1 - c_2 M) T)} - \varepsilon. \tag{2.22}$$

Denoting λ as: $\lambda = (p \exp(-(d_1 - c_2 M)(t - nT)))/(1 - (1 - q) \exp(-(d_1 - c_2 M)T)) - \varepsilon$, by Lemmas 2.4 and 2.6, there exists a $t_1 > 0$, and it is possible to select $\varepsilon > 0$. If ε is small enough, then $y(t) \ge y_1^*(t) - \varepsilon$ for all $t \ge t_1$ and satisfying $a < c_1 \lambda/(1 + e_1 M^2)$. Therefore, (2.20) $\le (a - c_1 \lambda/(1 + e_1 M^2)) x(t)$; if a $< c_1 \lambda/(1 + e_1 M^2)$, then (2.20) < 0. As $t \to \infty$, $x(t) \to 0$; this implies that there exist values of ε_1 and T_1 such that $x(t) < \varepsilon_1$ for $t \ge T_1$, and when $T > c_4 \Gamma / d_2$, these satisfy $\delta = \exp(-d_2 T + c_4 \Gamma_{\varepsilon_1} + c_4 \varepsilon_1 T + c_6 \varepsilon_1 T) < 1$. From the second equation in system (1.3), $y'(t) \le y(t)(-d_1 + c_2 x(t)) \le y(t)(-d_1 + c_2 \varepsilon_1)$. Let $y_1(t)$ be the solution of the following equation:

$$\begin{gathered} y_1'(t) = -(d_1 - c_2 \varepsilon_1) y_1(t), \quad t \neq nT, \\ y_1(t^+) = (1 - q) y_1(t) + p, \quad t = nT, \\ y_1(0^+) = y_0. \end{gathered} \tag{2.23}$$

From Lemmas 2.4 and 2.6, and the third equation in system (1.3),

$$z'(t) \le z(t)\left(-d_2 + c_4 y_1(t) + c_6 x(t)\right) \le z(t)\left(-d_2 + c_4 y_1^*(t) + c_6 \varepsilon_1\right), \tag{2.24}$$

where $y_1^*(t)$ is the periodic solution of (2.23):

$$y^*(t) = \frac{p \exp(-(d_1 - c_2 \varepsilon_1)(t - nT))}{1 - (1 - q) \exp(-(d_1 - c_2 \varepsilon_1) T)}. \tag{2.25}$$

Integrating (2.24) over $(nT, (n+1)T]$,

$$z((n+1)T) \le z(nT^+) \exp \int_{nT}^{(n+1)T} \left(-d_2 + c_4 y_1^*(t) + c_4 \varepsilon_1 + c_6 \varepsilon_1\right) dt \le z(nT)\ \delta, \tag{2.26}$$

where $\delta = \exp(-d_2 T + c_4 \Gamma_{\varepsilon_1} + c_4 \varepsilon_1 T + c_6 \varepsilon_1 T)$ and

$$\Gamma_{\varepsilon_1} = \frac{p(\exp(Td_1) - \exp(Tc_2\varepsilon_1))}{(1-q)(1-d_1)\exp(Tc_2\varepsilon_1) - (c_2\varepsilon_1 - d_1)\exp(Td_1)}. \tag{2.27}$$

When $\delta < 1$, $z((n+1)T) \le z(0)\delta^n \to 0$ as $n \to \infty$. For $t \in (nT, (n+1)T]$,

$$z(t) \le z(nT^+) \le z(nT) \int_{nT}^{t} (-d_2 + c_4 y_1^*(t) + c_4\varepsilon_1 + c_6\varepsilon_1)\, dt \le z(0)\delta^n, \tag{2.28}$$

which implies that $z(t) \to 0$ as $t \to \infty$. Therefore, it can be assumed that $z(t) < \varepsilon_2$ for $t > 0$. From the second equation in system (1.3), $y'(t) \ge y(t)(-d_1 - c_3 z(t)) \ge y(t)(-d_1 - c_3\varepsilon_2)$. Let $y_2(t)$ and $y_2^*(t)$ be solutions of the following equation:

$$\begin{aligned} y_2'(t) &= -(d_1 + c_3\varepsilon_2) y_2(t), \quad t \ne nT, \\ y_2(t^+) &= (1-q) y_2(t) + p, \quad t = nT, \\ y_2(0^+) &= y_0. \end{aligned} \tag{2.29}$$

From Lemmas 2.4 and 2.6, $y_2(t) \le y(t) \le y_1(t)$ and $y_1(t) \to y_1^*(t)$, $y_2(t) \to y_2^*(t)$ as $t \to \infty$. Also note that $y_1^*(t) \to y^*(t)$ and $y_2^*(t) \to y^*(t)$ as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Therefore, $y(t) \to y^*(t)$ as $t \to \infty$. This completes the proof. □

Theorem 2.10. *System* (1.3) *is permanent if*

$$\begin{aligned} T &> \frac{c_1\Gamma}{a - bM_0 - c_5 M_0} > \frac{c_1\Gamma}{a}, \\ T &< \frac{c_4}{d_2}\frac{\theta_1 - \theta_2}{d_1\sqrt{e_2}}. \end{aligned} \tag{2.30}$$

Proof. Let $(x(t), y(t), z(t))$ be a solution of system (1.3). From Theorem 2.7, there exists a constant $M > 0$ such that $x(t) \le M$, $y(t) \le M$, $z(t) \le M$ for each solution $X = (x(t), y(t), z(t))$ of system (1.3) for all sufficiently large t. Let $M_0 = \max\{M, M/c_3\}$; then $x(t) \le M_0$, $y(t) \le M_0$, $z(t) \le M_0$, and $y'(t) \ge -(d_1 + M_0) y(t)$.

By Lemmas 2.4 and 2.6, $y(t) \ge u^*(t) - \varepsilon$, $(\varepsilon > 0)$, where

$$u^*(t) = \frac{p\exp(-(d_1 + M_0)(t - nT))}{1 - (1-q)\exp(-(d_1 + M_0)T)}. \tag{2.31}$$

Then, $y(t) \ge (p\exp(-(d_1 + M_0)))/(1 - (1-q)\exp(-(d_1 + M_0)T)) - \varepsilon \equiv m_0$, for sufficiently large t. Therefore, it is necessary only to find an $m_2 > 0$ such that $x(t) \ge m_2$ and $z(t) \ge m_2$ for sufficiently large t. This can be done in the following two steps.

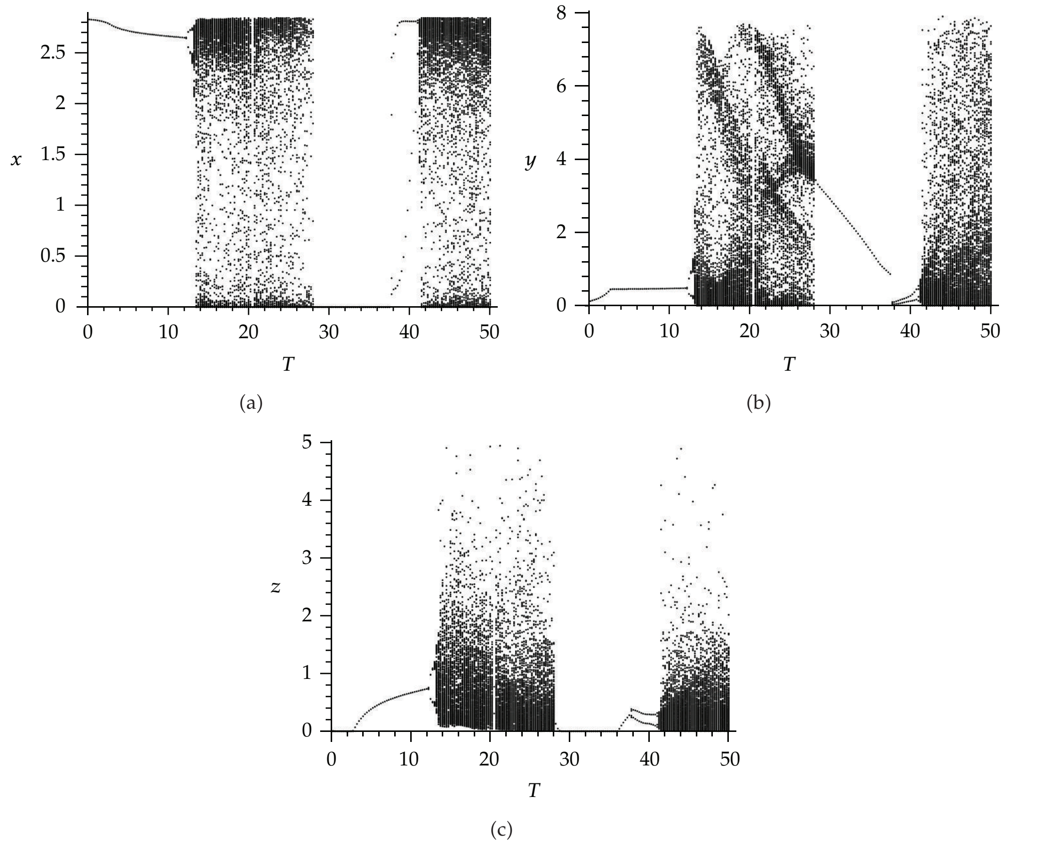

Figure 1: Bifurcation diagram of system (1.3) with initial conditions $x(0) = 0.1$, $y(0) = 0.1$, $z(0) = 0.1$, and $0 \leq T \leq 50$; $a = 1.85$, $b = 0.65$, $c_1 = 0.65$, $c_2 = 0.55$, $c_3 = 0.45$, $c_4 = 0.15$, $c_5 = 0.55$, $c_6 = 0.8$, $e_1 = 0.6$, $e_2 = 0.6$, $d_1 = 0.1$, $d_2 = 0.3$, $P = 0.1$, and $q = 0.8$.

Step 1. Choose $m_1 > 0, \varepsilon_1 > 0$ small enough that when (2.30) and (2.11) hold,

$$\begin{gathered}\sigma \equiv \exp(aT - bM_0T - c_1\Gamma_{\varepsilon_1} - c_3\varepsilon_1 T - c_5 M_0 T) < 1, \\ \rho \equiv \exp\left(-d_2T - c_4\varepsilon_1 T + \frac{c_4}{(d_1 + c_3 m_1)\sqrt{e_2}}\left(\theta_1^{\varepsilon_1} - \theta_2^{\varepsilon_1}\right)\right) < 1,\end{gathered} \tag{2.32}$$

where

$$\begin{gathered}\Gamma_{\varepsilon_1} = \frac{p\left(\exp(Td_1) - \exp(Tc_2\varepsilon_1)\right)}{(1-q)(1-d_1)\exp(Tc_2\varepsilon_1) - (d_1 + c_3 m_1)\exp(Td_1)}, \\ \theta_1^{\varepsilon_1} = \tan^{-1}\left(\frac{e_2\left(-p\exp(-(d_1 + c_3 m_1)(n-1)T) + \varepsilon_1\exp((d_1 + c_3 m_1)T) - (1-q)\right)}{\Lambda^{m_1}}\right), \\ \theta_2^{\varepsilon_1} = \tan^{-1}\left(\frac{e_2\left(-p - \varepsilon_1(1-q)\exp(-(d_1 + c_3 m_1)T)\right)}{\Lambda^{m_1}}\right), \\ \Lambda^{m_1} = \sqrt{e_2}\left(1 - (1-q)\exp(-(d_1 + c_3 m_1)T)\right).\end{gathered} \tag{2.33}$$

This step will show that $x(t_1) \geq m_1$ and $z(t_1) \geq m_1$ for some $t_1 > 0$. Assuming the contrary, the following system results:

$$\begin{aligned}
&u'(t) = (a - bM_0 - c_1 v(t) - c_5 M_0)u(t),\\
&v'(t) = (-d_1 - c_1 m_1)v(t), \qquad t \neq nT,\\
&w'(t) = \left(-d_2 + \frac{c_4 v(t)}{1 + e_2 (v(t))^2}\right) w(t)\\
&\Delta u(t) = 0,\\
&\Delta v(t) = -q v(t) + p, \qquad t = nT,\\
&\Delta w(t) = 0.
\end{aligned} \tag{2.34}$$

By Lemma 2.4, $x(t) \geq u(t)$, $y(t) \geq v(t)$, and $z(t) \geq w(t)$. By Lemma 2.6, $v^*(t) + \varepsilon_1 \geq v(t) \geq v^*(t) - \varepsilon_1$, $v^*(t) = (p \exp(-(d_1 + c_1 m_1)(t - nT)))/(1 - (1-q)\exp(-(d_1 + c_1 m_1)T))$,$(t \in (nT, (n+1)T])$. Thus,

$$\begin{gathered}
u'(t) \geq (a - bM_0 - c_5 M_0 - c_1 \varepsilon_1 - c_1 v^*(t))u(t),\\
w'(t) \geq \left(-d_2 + \frac{c_4 (v^*(t) - \varepsilon_1)}{1 + e_2 (v^*(t) - \varepsilon_1)^2}\right) w(t) \geq \left(-d_2 - c_4 \varepsilon_1 + \frac{c_4 v^*(t)}{1 + e_2 (v^*(t) - \varepsilon_1)^2}\right).
\end{gathered} \tag{2.35}$$

Integrating (2.35) over $(nT, (n+1)T]$ yields

$$\begin{gathered}
u((n+1)T) \geq u(nT^+) \exp \int_{nT}^{(n+1)T} (a - c_1 v^*(t) - c_1 \varepsilon_1 - bM_0 - c_5 M_0) dt \geq u(nT)\sigma,\\
w((n+1)T) \geq w(nT^+) \exp \int_{nT}^{(n+1)T} \left(-d_2 - c_4 \varepsilon_1 + \frac{c_4 v^*(t)}{1 + e_2 (v^*(t) - \varepsilon_1)^2}\right) dt \geq w(nT)\rho.
\end{gathered} \tag{2.36}$$

Therefore, $x((n+k)T) \geq u((n+k)T) \geq u(nT)\sigma^k$ and $w((n+k)T) \geq w((n+k)T) \geq w(nT)\sigma^k$.

Therefore, $\sigma^k \to \infty$ and $\rho^k \to \infty$ as $k \to \infty$. This implies that $x(t) \to \infty$ and $z(t) \to \infty$ as $t \to \infty$, which contradicts the boundedness of $x(t)$ and $z(t)$.

Step 2. If $x(t) \geq m_1$ and $z(t) \geq m_1$ for all $t \geq t_1$, then the proof is complete. If not, let $t^* = \inf_{t \geq t_1}\{x(t) < m_1, z(t) < m_1\}$; then $x(t) \geq m_1$ and $z(t) \geq m_1$ for $t \in [t_1, t^*)$ and $x(t^*) = m_1$,$z(t^*) = m_1$. By Step 1, there exists a $t' > t^*$ such that $z(t') \geq m_1$. Set $t_2 = \inf_{t > t^*}\{x(t) \geq m_1, z(t) \geq m_1\}$; then $x(t) < m_1$ and $z(t) < m_1$ for $t \in (t^*, t_2)$, and $z(t_2) = m_1$. This process can be continued by repeating Step 1. If this process stops after a finite number of repetitions, the proof is complete. If not, there exists an interval sequence $[t_n, t_{n+1}], (n \in N)$, such that $x(t) \geq m_1$ and $z(t) \geq m_1$, $t \in [t_n, t_{n+1}], (n \in N)$. Let $T' = \sup\{t_{n+1} - t_n \mid n \in N\}$. If $T' = \infty$, there must exist

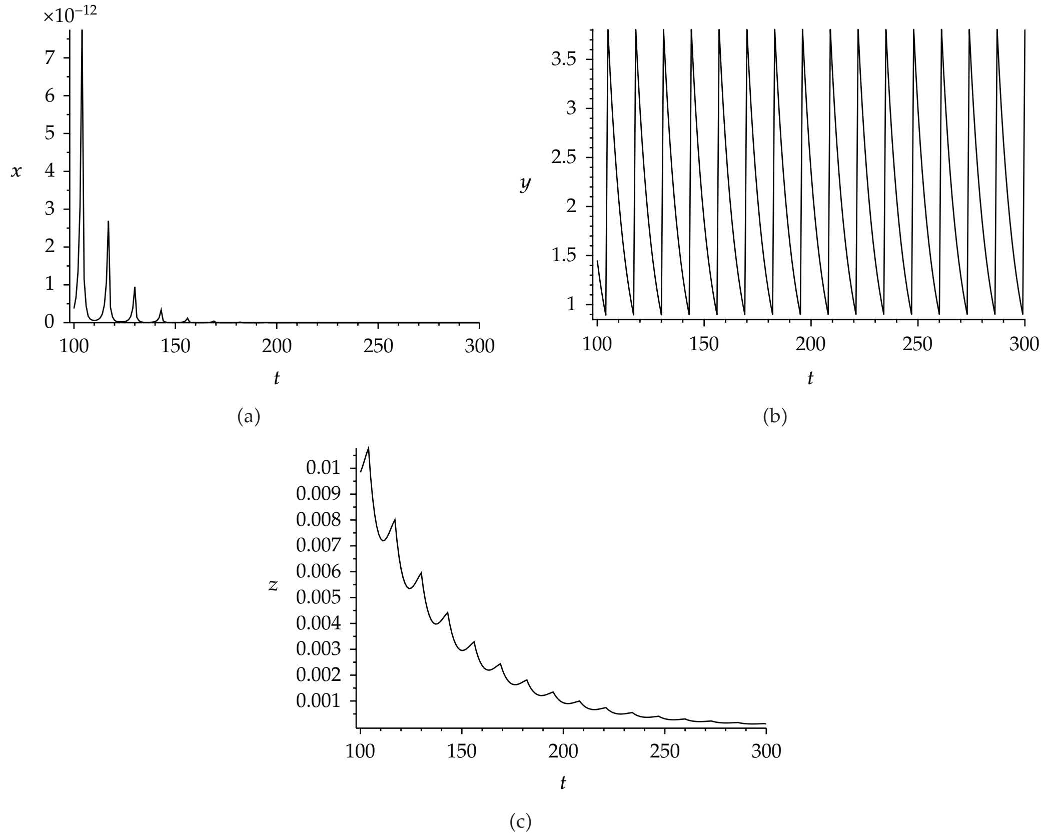

Figure 2: Dynamics of system (1.3) with $a = 1.72$, $b = 0.52$, $c_1 = 0.83$, $c_2 = 0.64$, $c_3 = 0.23$, $c_4 = 0.32$, $c_5 = 0.76$, $c_6 = 0.48$, $e_1 = 0.56$, $e_2 = 0.62$, $d_1 = 0.12$, $d_2 = 0.25$, $T = 13$, $p = 4.2$, and $q = 0.9$. (a) time series of prey, (b) time series of intermediate predator, and (c) time series of top predator.

a subsequence $\{t_{n_i}\}$ such that $t_{n_i+1} - t_{n_i} \to \infty$ as $n_i \to \infty$. From Step 1, this can lead to a contradiction with the boundedness of $x(t)$ and $z(t)$. Therefore, $T' < \infty$. Then,

$$x(t) \geq x(t_n) \exp \int_{t_n}^{t} (a - c_1 v^*(s) - c_1 \varepsilon_1 - bM_0 - c_5 M_0) ds \geq m_1 \exp\left(-(bM_0 + c_5 M_0)T'\right) \equiv m_2,$$

$$z(t) \geq z(t_n) \exp \int_{t_n}^{t} \left(-d_2 - c_4 \varepsilon_1 + \frac{c_4 v^*(s)}{1 + e_2 (v^*(s) - \varepsilon_1)^2}\right) ds \geq m_1 \exp\left(-d_2 T'\right) \equiv m_3. \tag{2.37}$$

Let $m_4 = \min\{m_2, m_3\}$; then $\liminf_{t\to\infty} x(t) \geq m_4$, and $\liminf_{t\to\infty} z(t) \geq m_4$. This completes the proof. □

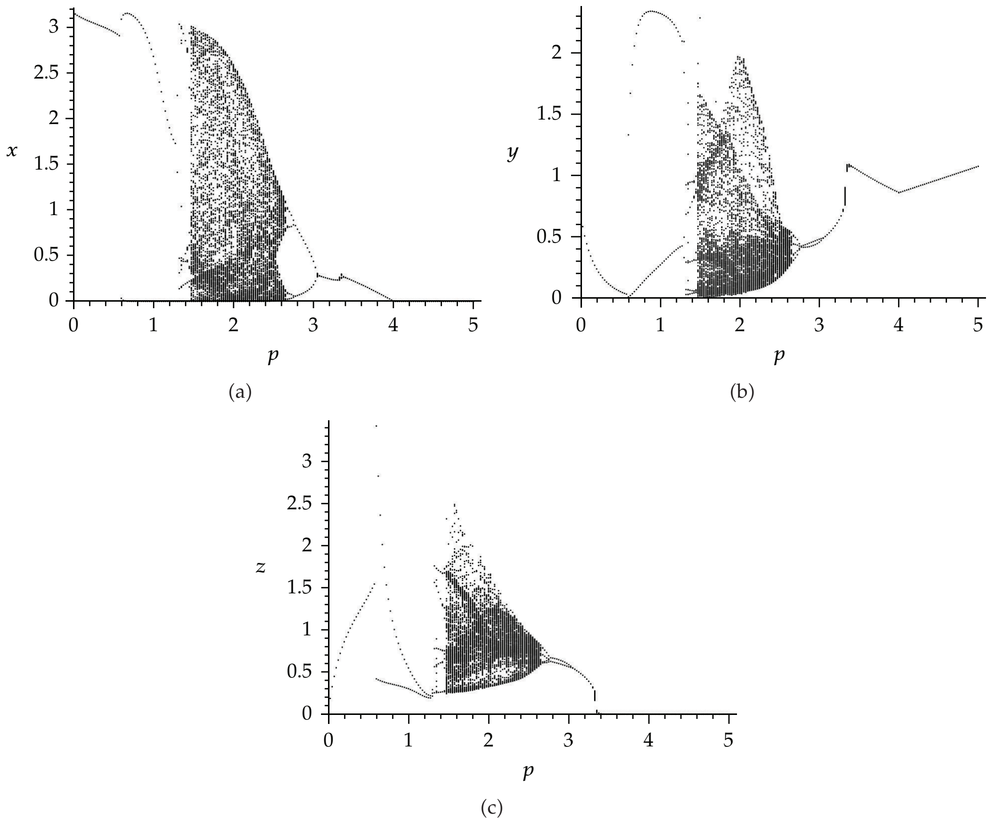

Figure 3: Bifurcation diagram of system (1.3) with initial conditions $x(0) = 0.1$, $y(0) = 0.1$, $z(0) = 0.1$, and $0 \leq p \leq 5$; $a = 1.72$, $b = 0.52$, $c_1 = 0.83$, $c_2 = 0.64$, $c_3 = 0.23$, $c_4 = 0.32$, $c_5 = 0.76$, $c_6 = 0.48$, $e_1 = 0.56$, $e_2 = 0.62$, $d_1 = 0.12$, $d_2 = 0.25$, $T = 13$, and $q = 0.9$.

3. Numerical Analysis

3.1. Bifurcation

To study the dynamics of system (1.3), a period T was chosen, and the impulsive control parameter p was defined as the bifurcation parameter. The bifurcation diagram provides a summary of the essential dynamic behavior of the system [27–29].

First, the influence of the period T will be investigated. The bifurcation diagrams are shown in Figure 1. The results are in accordance with Theorem 2.8. Next, the influence of the impulsive control parameter p will be examined. A time series of the system response is shown in Figure 2. It is apparent that the solution $(0, y^*(t), 0)$, which is said to be globally stable, is in agreement with Theorem 2.9. The bifurcation diagrams are shown in Figure 3.

To see the dynamics of system (1.3) clearly, a phase diagram with a different value of the parameter p corresponding to the bifurcation diagrams in Figure 3 is shown in Figure 4.

Figures 1 and 3 illustrate the complex dynamics of system (1.3), such as period-doubling cascades, symmetry-breaking pitchfork bifurcations, chaos, and nonunique dynamics. Because all the bifurcation diagrams are similar, only one of them will be explained. Part (a) of Figure 3 will be taken as an example. When $p \in [0, 0.414]$, the dynamics of the

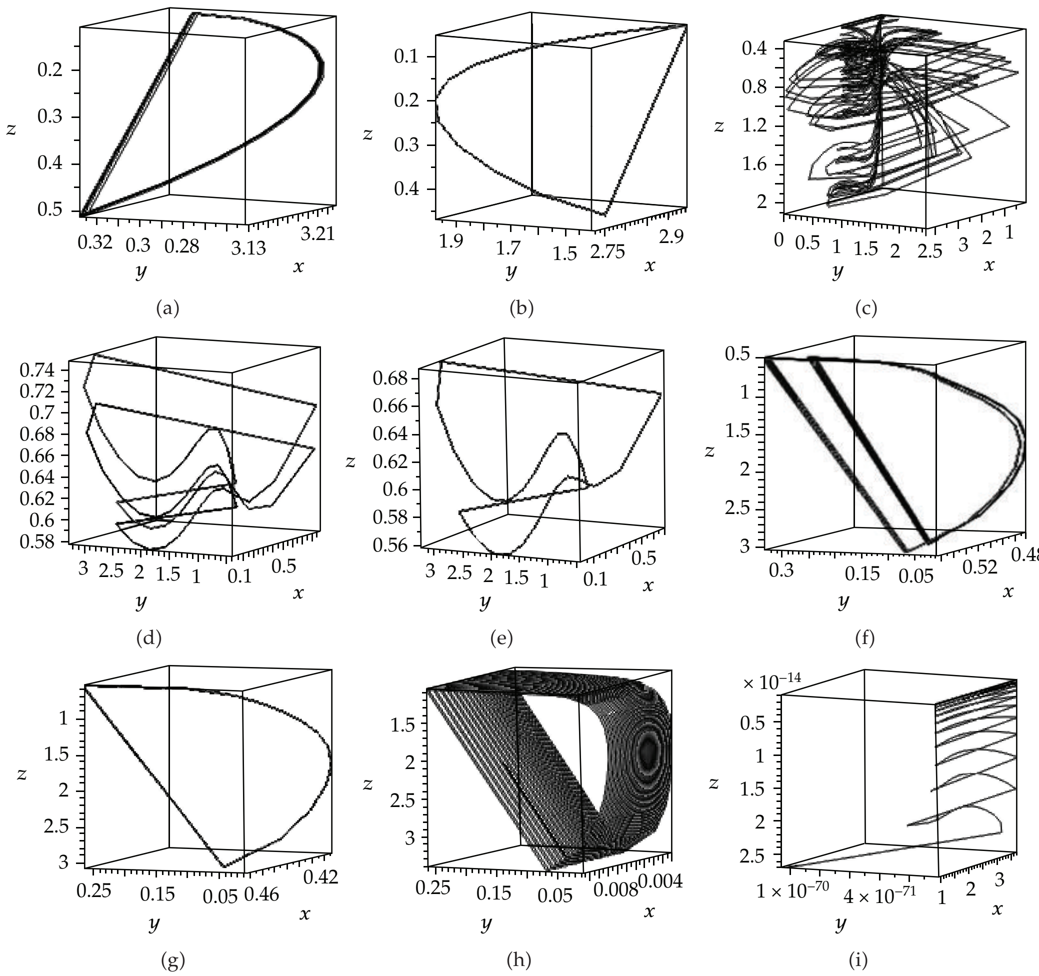

Figure 4: Periodic behavior and chaos corresponding to Figure 3: (a) phase diagram for $p = 0.05$, (b) phase diagram for $p = 0.5$, (c) phase diagram for $p = 2$, (d) phase diagram for $p = 2.75$, (e) phase diagram for $p = 2.85$, (f) phase diagram for $p = 3.05$, (g) phase diagram for $p = 3.15$, (h) phase diagram for $p = 3.4$, and (i) phase diagram for $p = 4.3$.

system are not obvious, but as p increases, the dynamics become more evident. The system enters into periodic windows with a chaotic band, as shown in Figures 4(a) (not obviously chaotic) and 4(b) (T-periodic solution). When p is between 0.414 and 2.689, chaos is intense, as shown in Figure 4(c). When p moves beyond 2.689, the chaotic behavior disappears, and periodic windows appear, as shown in Figures 4(d) ($4T$-periodic solution) and Figure 4(e) ($2T$-periodic solution). When $p \in [3.019, 3.076]$, the chaotic attractor increases in strength, and chaos reappears, as shown in Figure 4(f). For p greater than 3.076, periodic windows appear, as shown in Figure 4(g). For p in the interval 3.258 to 3.486, chaos reemerges, as shown in Figure 4(h). As the value of p increases further, the system enters into a stable state, as shown in Figure 4(i).

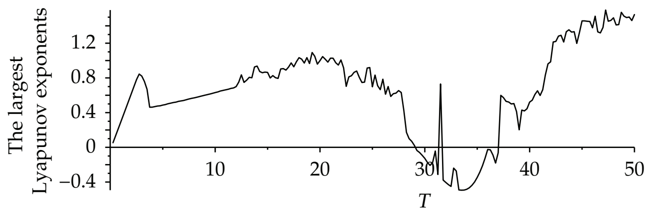

Figure 5: The largest Lyapunov exponents (LLE) corresponding to Figure 1.

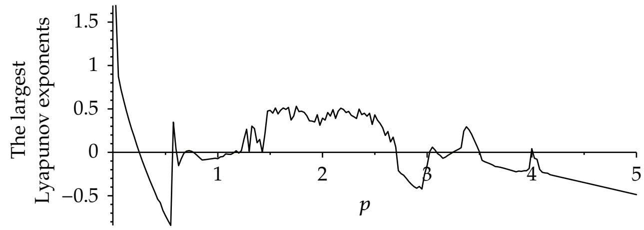

Figure 6: The largest Lyapunov exponents (LLE) corresponding to Figure 3.

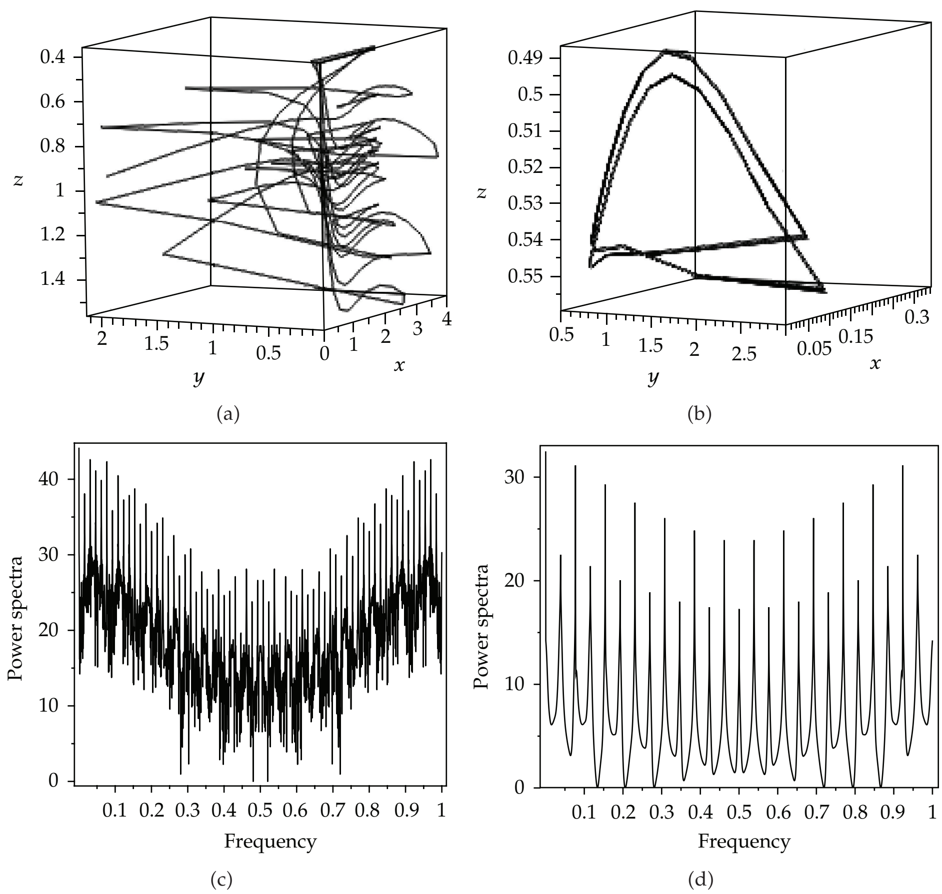

Figure 7: Strange attractors and power spectra: (a) strange attractor when $p = 2.2$, (b) strange attractor when $p = 3.039$, (c) power spectrum of attractor (a), (d) power spectrum of attractor (b).

3.2. The Largest Lyapunov Exponent

The largest Lyapunov exponent is always calculated to detect whether a system is exhibiting chaotic behavior. The largest Lyapunov exponent takes into account the average exponential rates of divergence or convergence of nearby orbits in phase space [30]. A positive largest Lyapunov exponent indicates that the system is chaotic. If the largest Lyapunov exponent is negative, then periodic windows or a stable state must exist. Using the largest Lyapunov exponent, it is possible to see when the system is chaotic, at what time periodic windows disappear, and when the system is stable. The largest Lyapunov exponents corresponding to Figures 1 and 3 were calculated and are plotted in Figures 5 and 6.

3.3. Strange Attractors and Power Spectra

To understand the qualitative nature of strange attractors, power spectra can be used [31]. From the discussion in Section 3.2, it is known that the largest Lyapunov exponent for strange attractor (a) is 0.475 and that for strange attractor (b) is 0.0144. This means that these are both chaotic attractors, and the fact that the exponent of (a) is larger than that of (b) means that the chaotic dynamics of (a) are more intense than those of (b). Considering the power spectra of these attractors, the spectrum of strange attractor (b) is composed of strong broadband components and sharp peaks, as shown in Figure 7(d). By contrast, in the spectrum of the strong chaotic attractor (a), it is not easy to distinguish any sharp peaks, as can be seen in Figure 7(c). By means of power spectra, it is possible to determine that (b) originates in a strong limit cycle, but that (a) comes from a set of weak limit cycles.

4. Conclusions and Remarks

In this paper, the dynamic behavior of a predator-prey model with Holling type IV functional response with respect to an impulsive control strategy has been investigated. The conditions for locally asymptotically stable and globally stable periodic solutions and for system permanence have been determined. It has been determined that an impulsive control strategy changes the dynamic behavior of the model. Complex dynamic patterns also have been observed in continuous-time predator-prey or three-species food-chain models [32–34]. Numerical simulations were performed to obtain bifurcation diagrams with respect to the period T and the impulsive control parameter p. Using computer-based simulation, phase diagrams were generated to reveal details of the bifurcation behavior, such as period-doubling cascades, symmetry-breaking pitchfork bifurcations, chaos, and nonunique dynamics. The largest Lyapunov exponents were also simulated to verify the chaotic dynamics of the system. In addition, power spectra were used to understand the qualitative nature of strange attractors. On the basic of numerical simulations, the main difference of this paper from other papers is that the top predator z cannot be persistent although the mid-level predator y is persistent. This is because the mid-level predator y is easy to escape the top predator z, which is a variable to the competitive. It can be concluded that an impulsive control strategy is an effective method to control the system dynamics of a predator-prey model.

Acknowledgments

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