Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 232794, 20 pages
doi:10.1155/2012/232794

Research Article

Exponential Synchronization for Impulsive Dynamical Networks

Lijun Pan[1,2] and Jinde Cao[1]

[1] *Department of Mathematics, Southeast University, Nanjing 210096, China*
[2] *School of Mathematics, Jiaying University, Meizhou, Guangdong 514015, China*

Correspondence should be addressed to Jinde Cao, jdcao@seu.edu.cn

Received 29 November 2011; Accepted 20 June 2012

Academic Editor: Cengiz Çinar

Copyright © 2012 L. Pan and J. Cao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is devoted to exponential synchronization for complex dynamical networks with delay and impulsive effects. The coupling configuration matrix is assumed to be irreducible. By using impulsive differential inequality and the Kronecker product techniques, some criteria are obtained to guarantee the exponential synchronization for dynamical networks. We also extend the delay fractioning approach to the dynamical networks by constructing a Lyapunov-Krasovskii functional and comparing to a linear discrete system. Meanwhile, numerical examples are given to demonstrate the theoretical results.

1. Introduction

In the past two decades, complex dynamical networks have attracted lot of attention in different areas, such as physical science, engineering, mathematics, biology, and sociology [1–3]. The synchronization of all dynamical nodes is an important and interesting phenomena mostly because the synchronization can well explain many natural phenomena. Consequently, the synchronization has been actively investigated due to past physics and potential engineering applications. Recently, there has been an increasing interest in the investigation of synchronization of complex dynamical networks, then many synchronization results have been derived for complex dynamical networks [4–9].

Impulsive effects widely exist in the networks. Such systems are described by impulsive differential systems which have been used efficiently in modelling many practical problems that arise in the fields of engineering, physics, and science as well. So the theory of impulsive differential equations is also attracting much attention in recent years [10–13]. Correspondingly, based on the theory of impulsive differential equations, a lot of synchronization results of dynamical networks with impulsive effects have been obtained [13–20].

As is well known, two kinds of impulses in terms of synchronization in complex dynamical networks are considered. One is desynchronizing impulse, the other is synchronizing impulse. An impulsive sequence is said to be desynchronizing if the impulsive effect can suppress the synchronization of complex dynamical networks. An impulsive sequence is said to be synchronizing if a corresponding impulsive effect can enhance the synchronization of the complex dynamical networks. According to the previous literature, complex dynamical networks with delay and impulses can reach synchronization provided that delayed dynamical networks are synchronized. In this paper, by impulsive differential inequality [21], the Lyapunov functional method and the Kronecker product techniques, some sufficient conditions are derived for the globally exponential synchronization of dynamical networks. We also extend the delay fractioning method [22, 23] to dynamical networks by constructing Lyapunov-Krasovskii functional and comparing to a linear discrete system. Meanwhile, numerical simulations are given to show that our derived criteria can easily be used to make judgements on synchronization for the delayed dynamical networks with impulsive effects and show that impulsive effects play an important role in the delay dynamical networks. The rest of this paper is organized as follows. In Section 2, the network model is presented, together with some definitions and lemmas. In Section 3, some synchronization criteria are derived for general dynamical networks with delay and impulsive effects. In Section 4, two numerical examples are given to demonstrate that our results are relevant to not only linear coupling but also delay and impulsive effects. Finally, some conclusions are given in Section 5.

Notations. Throughout this paper, the superscript T represents the transpose. I_n stands for the identity matrix of order n. For $x = (x_1, x_2, \dots, x_n)^T \in R^n$, the norm is defined as $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. For matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of matrix A, respectively. For real symmetric matrices X and Y, the notation $X \le Y$ (resp., $X < Y$) means that the matrix $X - Y$ is negative semidefinite (resp., negative definite). For a sequence $\{t_k, k = 0, 1, \dots\}$ satisfying $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, let $\Delta_k \triangleq t_{k+1} - t_k$, $\Delta_{\sup} = \sup_{k \ge 0}\{\Delta_k\}$, $\Delta_{\inf} = \inf_{k \ge 0}\{\Delta_k\}$.

2. Model Description and Preliminaries

We consider a delayed complex dynamical network consisting of N-coupled identical nodes. Each node is an n-dimensional dynamical system composed of linear term and nonlinear term. The ith node can be described as follows:

$$\dot{x}_i = Cx_i + B_1 f(x_i(t)) + B_2 g(x_i(t - \tau(t))), \quad i = 1, 2, \dots, N, \tag{2.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T$ is the state vector of the ith node at time t, $C, B_1, B_2 \in R^{n \times n}$; $0 < \tau(t) \le \tau$, $\tau'(t) \le \sigma < 1$, $\tau > 0$, $f(x), g(x) \in C(R^n, R^n)$, $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \dots, f_n(x_{in}(t)))^T$, $g(x_i(t - \tau(t))) = (g_1(x_{i1}(t - \tau(t))), g_2(x_{i2}(t - \tau(t))), \dots, g_n(x_{in}(t - \tau(t))))^T$.

The dynamical behavior of the dynamical network with delay can be described by the following linearly coupled systems:

$$\begin{aligned}\dot{x}_i &= Cx_i + B_1 f(x_i(t)) + B_2 g(x_i(t-\tau(t))) \\ &\quad + c \sum_{j=1, j\neq i}^{N} a_{ij}\Gamma\left(x_j(t) - x_i(t)\right), \quad i = 1, 2, \ldots, N,\end{aligned} \tag{2.2}$$

where $\Gamma = \operatorname{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is the inner coupling positive definite matrix between two connected nodes i and j, c is the coupling strength, and a_{ij} is defined as follows: if there is a connection from node j to node $i (j \neq i)$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$.

In the process of signal transmission, due to the impulsive effects, the states $x_i(t), i = 1, 2, \ldots, N$ are suddenly changed in the form of impulses at discrete times t_k. That is, $x_i(t_k^+) = d_k x_i(t_k)$. Let $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$. Thus, the dynamical network with delay and impulsive effects can be obtained by the following form:

$$\begin{aligned}\dot{x}_i &= Cx_i + B_1 f(x_i(t)) + B_2 g(x_i(t-\tau(t))) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t), \quad t \geq t_0, \ t \neq t_k, \\ x_i(t_k^+) &= d_k x_i(t_k), \quad k = 1, 2, \ldots, \\ x_i(t) &= \varphi_i(t), \quad t \in [t_0 - \tau, t_0], \ i = 1, 2, \ldots, N,\end{aligned} \tag{2.3}$$

where $x_i(t_k^+) = \lim_{h\to 0^+} x_i(t_k + h), x_i(t_k) = \lim_{h\to 0^-} x_i(t_k + h), t_k \geq 0$ are impulsive moments satisfying $t_k < t_{k+1}$ and $\lim_{k\to+\infty} t_k = +\infty$, $d_k, k = 1, 2, \ldots$ are the impulsive gains at t_k for ith unit, $A = (a_{ij})_{N\times N}$ is the Laplacian matrix of the corresponding network.

By a solution $x_i = x_i(t)$ of system (2.3), we mean a real function on $[t_0 - \tau, \infty)$ such that $x_i(t_0) = \varphi_i(t)$ for $t \in [t_0 - \tau, t_0]$, and $x_i(t)$ satisfies system (2.3) for $t \geq t_0$, and $x_i(t)$ is continuous everywhere except for some t_k and left continuous at $t = t_k$, and the right limit $x(t_k^+)$ $k = 1, 2, \ldots$ exists. Here, we always assume that system (2.3) has a unique solution.

Remark 2.1. If $|d_k| < 1$, the impulsive sequence is of synchronizing impulse, which may enhance the synchronization of the networks. But if $|d_k| > 1$, the impulsive sequence can suppress the synchronization, which is said to be desynchronizing impulse.

Definition 2.2. The dynamical networks (2.3) are said to be globally exponentially synchronized if there exist $\eta > 0$ and $M > 0$ such that for any initial values $\varphi_i(t)$ $(i = 1, 2, \ldots, N)$:

$$\left\|x_i(t) - x_j(t)\right\| \leq Me^{-\eta(t-t_0)} \tag{2.4}$$

hold all $t > t_0$, and for any $i, j = 1, 2, \ldots, N$.

Definition 2.3. For $A = (a_{ij})_{m\times n} \in R^{m\times n}$, $B = (b_{ij})_{p\times q} \in R^{p\times q}$, the Kronecker product between two matrices is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in R^{mp\times nq}. \tag{2.5}$$

Assumption 2.4. There exist constants $l_i, l_i' > 0$ $(i = 1,2,\ldots,N)$ such that $|f_i(x_1) - f_i(x_2)| \leq l_i|x_1 - x_2|$ and $|g_i(x_1) - g_i(x_2)| \leq l_i'|x_1 - x_2|$ hold for any $x_1, x_2 \in R$.

Assumption 2.5. The coupling configuration matrix A is irreducible, and the real parts of the eigenvalues of A are all negative except an eigenvalue 0 with multiplicity 1.

To derive our main results, we need the following lemmas.

Lemma 2.6 (see [24]). *If an irreducible matrix A with nonnegative offdiagonal elements satisfies $a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}, i = 1,2,\ldots,N$, then the following propositions are obtained:*

(1) *if λ is an eigenvalue of A and $\lambda \neq 0$, then* $\mathrm{Re}(\lambda) < 0$;

(2) *A has an eigenvalue 0 with multiplicity* 1 *and the right eigenvector* $(1,1,\ldots,1)^T$;

(3) *suppose that $\xi = (\xi_1,\xi_2,\ldots,\xi_N)^T \in R^N$ satisfying $\sum_{i=1}^{N}\xi_i = 1$ is the normalized left eigenvector of A corresponding to eigenvalue 0. Then, $\xi_i > 0$ hold for all $i = 1,2,\ldots,N$;*

(4) *furthermore, if A is symmetric, then we have $\xi_i = 1/N$ for $i = 1,2,\ldots,N$.*

Lemma 2.7 (see [21]). *Let p, q, τ, $d_k, k = 1,2,\ldots$ be constants and $q \geq 0$, $\tau > 0$, $d_k \geq 0$ and assume that $u(t)$ is a piece continuous nonnegative function satisfying:*

$$\begin{aligned} D^+u(t) &\leq pu(t) + q\overline{u}(t) \quad t \geq t_0,\ t \neq t_k, \\ u(t_k^+) &\leq d_k(u(t_k)), \quad k = 1,2,\ldots, \\ u(t) &= \phi(t), \quad t \in [t_0 - \tau, t_0]. \end{aligned} \tag{2.6}$$

If there exist α such that for $k = 1,2,\ldots$

$$\begin{aligned} \frac{\ln d_k}{t_k - t_{k-1}} &\leq \alpha, \\ p + dq + \alpha &< 0. \end{aligned} \tag{2.7}$$

Then

$$u(t) \leq d\left(\sup_{t_0-\tau\leq t\leq t_0} |\phi|\right)e^{-\lambda(t-t_0)}, \tag{2.8}$$

where $\overline{u}(t) = \sup_{t-\tau\leq\sigma\leq t} x(\sigma)$, $d = \sup_{1\leq k<+\infty}\{e^{\alpha(t_k-t_{k-1})}, 1/e^{\alpha(t_k-t_{k-1})}\}$, λ is an unique positive solution of $\lambda + p + dqe^{\lambda\tau} + \alpha = 0$.

Remark 2.8. The condition of Lemma 2.7 does not need $-p > q$ due to the effects α, which implies that the above inequality is less conservative than the results in [25].

Lemma 2.9. *For any vectors $x, y \in R^n$, scalar $\epsilon > 0$, and positive definite matrix $Q \in R^{n\times n}$, the following inequality holds:*

$$2x^T y \le \epsilon x^T Q x + \epsilon^{-1} y^T Q^{-1} y. \tag{2.9}$$

Lemma 2.10. *Let $A \in R^{n\times n}$ be a positive definite matrix, then for $x \in R^n$,*

$$\lambda_{\min}(A) x^T x \le x^T A x \le \lambda_{\max}(A) x^T x. \tag{2.10}$$

3. Synchronization Analysis

In this section, the globally exponential synchronization will be analyzed for delayed dynamical networks with impulsive effects. We assume that the network topology is strongly connected, then the corresponding Laplacian coupling matrix A is irreducible.

Let $x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t))^T$, $F(x(t)) = (f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t)))^T$, $G(x(t)) = (g^T(x_1(t)), g^T(x_2(t)), \ldots, g^T(x_N(t)))^T$ and $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t))^T$. Then, the delayed dynamical network (2.3) can be rewritten in the following Kronecker product form:

$$\begin{aligned}
\dot{x} &= (I_N \otimes C)x(t) + (I_N \otimes B_1)F(x(t)) \\
&\quad + (I_N \otimes B_2)G(x(t-\tau(t))) + c(A\otimes\Gamma)x(t), \quad t \ge t_0,\ t \ne t_k, \\
x(t_k^+) &= d_k x(t_k), \quad k = 1, 2, \ldots, \\
x(t) &= \varphi(t), \quad t \in [t_0 - \tau, t_0].
\end{aligned} \tag{3.1}$$

Suppose that $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$ is the left eigenvector of the configuration coupling matrix A with respect to eigenvalue 0 satisfying $\sum_{i=1}^N \xi_i = 1$. Since the coupling configuration matrix A is irreducible, by Lemma 2.6, we can see that $\xi_i > 0$ for $i = 1, 2, \ldots, N$. Let $\Xi = \operatorname{diag}\{\xi_1, \xi_2, \ldots, \xi_N\} > 0$, $L = \operatorname{diag}\{l_1, l_2, \ldots, l_n\}$, $L' = \operatorname{diag}\{l_1', l_2', \ldots, l_n'\}$, $W = \Xi - \xi\xi^T$ and $\overline{A} = \Xi A + A^T \Xi$.

Theorem 3.1. *Suppose that Assumptions 2.4 and 2.5 hold. Also suppose that there exist a diagonal positive-definite matrix P and scalars $\eta > 0$, $\varepsilon > 0$, $\gamma > 0$, $\mu_2 \ge 0$, μ_1, δ such that*

(H_1) $\Theta_1 = PC + C^T P + \varepsilon P B_1 B_1^T P + \gamma P B_2 B_2^T P + \varepsilon^{-1} L^2 - c\eta P\Gamma - \mu_1 P \le 0$;

(H_2) $\Theta_2 = \gamma^{-1} L'^2 - \mu_2 P \le 0$;

(H_3) *for all* $k = 1, 2, \ldots$, $2\ln|d_k|/(t_k - t_{k-1}) \le \delta$;

(H_4) $\mu_1 + d\mu_2 + \delta < 0$;

(H_5) $\eta\lambda_{\max}(W) + \lambda_2(\overline{A}) \le 0$.

Then the complex dynamical networks (3.1) *are exponentially synchronized, where* $d = \sup_{1\le k<+\infty}\{e^{\delta(t_k - t_{k-1})}, 1/e^{\delta(t_k - t_{k-1})}\}$, *$\lambda_2(\overline{A})$ is defined to be the second largest eigenvalue of $\overline{A}$.*

Proof. We define a Lyapunov function $V(t) = x^T(t)(W \otimes P)x(t)$. Since $W = \Xi - \xi\xi^T$, we have $w_{ij} = -\xi_i\xi_j$ for $i \neq j$ and $w_{ii} = \xi_i - \xi_i^2$. In view of $\sum_{j=1}^N \xi_j = 1$, it follows that $\sum_{j=1}^N w_{ij} = \xi_i - \sum_{j=1}^N \xi_i\xi_j = 0$. Therefore, we can conclude that $V(t) = \sum_{i=1}^N \sum_{j=1, j\neq i}^N -(1/2)w_{ij}(x_i(t) - x_j(t))^T P(x_i(t) - x_j(t))$. Calculating the Dini derivative of $V(t)$ along the trajectories of the systems (3.1), we have for $t \neq t_k, k = 1, 2, \ldots$:

$$\begin{aligned} D^+V(t) = 2x^T(t)(W \otimes P) \times (I_N \otimes C)x(t) \\ + 2x^T(W \otimes P) \times (I_N \otimes B_1)F(x(t)) \\ + 2x^T(W \otimes P) \times (I_N \otimes B_2)G(x(t-\tau(t))) \\ + 2cx^T(t)(W \otimes P) \times (A \otimes \Gamma)x(t). \end{aligned} \tag{3.2}$$

By adding $-cx^T(t)(W \otimes \eta P\Gamma)x(t) + cx^T(t)(W \otimes \eta P\Gamma)x(t)$ to (3.2) and noting that $WA = (\Xi - \xi\xi^T)A = \Xi A - \xi(\xi^T A) = \Xi A$, we can obtain that

$$\begin{aligned} D^+V(t) \leq & -\sum_{i=1}^N \sum_{j=1, j\neq i}^N w_{ij}\bigg[(x_i(t) - x_j(t))^T\left(PC - \frac{1}{2}c\eta P\Gamma\right)(x_i(t) - x_j(t)) \\ & + (x_i(t) - x_j(t))^T PB_1(f(x_i(t)) - f(x_j(t))) \\ & + (x_i - x_j)^T PB_2(g(x_i(t-\tau(t))) - g(x_j(t-\tau(t))))\bigg] \\ & + cx^T(t) \times \left[\left(\Xi A + A^T\Xi\right) \otimes P\Gamma + W \otimes \eta P\Gamma\right]x(t). \end{aligned} \tag{3.3}$$

Since the matrix $\overline{A} = \Xi A + A^T\Xi$ has the following property:

$$\begin{aligned} &\overline{A} = \left(\overline{A}_{ij}\right)_{N\times N}, \quad \overline{A}_{ii} = 2\xi_i A_{ii} < 0, \quad i = 1, 2, \ldots, N, \\ &\overline{A}_{ij} = \xi_i A_{ij} + \xi_j A_{ji} = \overline{A}_{ji}, \quad i \neq j, \qquad \sum_{j=1}^N \overline{A}_{ij} = \xi_i \sum_{j=1}^N \overline{A}_{ij} + \sum_{j=1}^N \xi_j \overline{A}_{ji} = 0. \end{aligned} \tag{3.4}$$

By Perron-Frobenius theorem (see [24]), we can arrange the eigenvalues of matrix $\overline{A}$ as follows: $0 = \lambda_1(\overline{A}) > \lambda_2(\overline{A}) \geq \cdots \geq \lambda_N(\overline{A})$. Applying matrix decomposition theory (see [24]), there exists unitary matrix U, such that $\overline{A} = U\Lambda U^T$, where $\Lambda = \operatorname{diag}\{0, \lambda_2(\overline{A}), \ldots, \lambda_N(\overline{A})\}$ and $U = \{u_1, u_2, \ldots, u_N\}$ with $u_1 = (1/\sqrt{N}, 1/\sqrt{N}, \ldots, 1/\sqrt{N})^T$ and $U^T U = I_N$.

Let $y(t) = (U^T \otimes I_n)x(t)$, where $y(t) = (y_1^T(t), y_2^T(t), \ldots, y_N^T(t))^T$, $y_i(t) \in R^n, i = 1, 2, \ldots, N$. Then we have $x(t) = (U \otimes I_n)y(t)$. Thus, we have

$$\begin{aligned} x^T(t)\left[\left(\Xi A + A^T\Xi\right) \otimes P\Gamma\right]x(t) &= y^T(t)\left(U^T \otimes I_n\right)\left(\overline{A} \otimes P\Gamma\right)(U \otimes I_n)y(t) \\ &= \sum_{i=2}^N \lambda_i\left(\overline{A}\right)y_i^T(t)P\Gamma y_i(t) \leq \lambda_2\left(\overline{A}\right)\sum_{i=2}^N y_i^T(t)P\Gamma y_i(t). \end{aligned} \tag{3.5}$$

In view of matrix W is a zero row sum irreducible symmetric matrix with negative off-diagonal elements, we see that $\lambda_{\max}(W) > 0$ and $Wu_1 = (0, 0, \dots, 0))^T$. Hence by Lemma 2.10, we have

$$\begin{aligned} x^T(t)(W \otimes \eta P\Gamma)x(t) &= \eta y^T(t)\left(U^T W U \otimes P\Gamma\right)y(t) \\ &\leq \eta\lambda_{\max}(W)\sum_{i=2}^{N} y_i^T(t)P\Gamma y_i(t). \end{aligned} \tag{3.6}$$

It follows from condition $\lambda_2(\overline{A}) + \eta\lambda_{\max}(W) \leq 0$ that

$$\begin{aligned} &cx^T(t)\left[\left(\Xi A + A^T\Xi\right) \otimes P\Gamma + W \otimes \eta P\Gamma\right]x(t) \\ &\quad \leq c\left(\lambda_2\left(\overline{A}\right) + \eta\lambda_{\max}(W)\right)\sum_{i=2}^{N} y_i^T(t)Py_i(t) \leq 0. \end{aligned} \tag{3.7}$$

By Assumption 2.4 and Lemma 2.10, there exists $\varepsilon > 0$ such that

$$\begin{aligned} &2\left(x_i(t) - x_j(t)\right)^T PB_1\left(f\left(x_i(t)\right) - f\left(x_j(t)\right)\right) \\ &\quad \leq \varepsilon\left(x_i(t) - x_j(t)\right)^T PB_1B_1^T P\left(x_i(t) - x_j(t)\right) \\ &\qquad + \varepsilon^{-1}\left(f\left(x_i(t)\right) - f\left(x_j(t)\right)\right)^T\left(f\left(x_i(t)\right) - f\left(x_j(t)\right)\right) \\ &\quad \leq \varepsilon\left(x_i(t) - x_j(t)\right)^T PB_1B_1^T P\left(x_i(t) - x_j(t)\right) \\ &\qquad + \varepsilon^{-1}\left(x_i(t) - x_j(t)\right)^T L^2\left(x_i(t) - x_j(t)\right) \\ &\quad = \left(x_i(t) - x_j(t)\right)^T\left(\varepsilon PB_1B_1^T P + \varepsilon^{-1}L^2\right)\left(x_i(t) - x_j(t)\right). \end{aligned} \tag{3.8}$$

Similarly, we have the following estimation:

$$\begin{aligned} &2\left(x_i(t) - x_j(t)\right)^T B_2\left(g\left(x_i(t - \tau(t))\right) - g\left(x_j(t - \tau(t))\right)\right) \\ &\quad \leq \left(x_i(t) - x_j(t)\right)^T \gamma PB_2B_2^T P\left(x_i(t) - x_j(t)\right) \\ &\qquad + \left(x_i(t - \tau(t)) - x_j(t - \tau(t))\right)^T\left(\gamma^{-1}L'^2\right)\left(x_i(t - \tau(t)) - x_j(t - \tau(t))\right), \end{aligned} \tag{3.9}$$

where $\gamma > 0$. Substituting these into (3.3), we have for $t \neq t_k$

$$\begin{aligned}
\dot{V}(t) \leq & -\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{1}{2}w_{ij} \\
& \times\left[(x_i(t)-x_j(t))^T\left(PC+C^TP+\varepsilon PB_1B_1^TP+\gamma PB_2B_2^TP+\varepsilon^{-1}L^2-c\eta P\Gamma\right)(x_i(t)-x_j(t))\right] \\
& -\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{1}{2}w_{ij} \\
& \times\left[(x_i(t-\tau(t))-x_j(t-\tau(t)))^T\left(\gamma^{-1}L'^2\right)(x_i(t-\tau(t))-x_j(t-\tau(t)))\right] \\
= & -\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{1}{2}w_{ij} \\
& \times\left[(x_i(t)-x_j(t))^T\left(PC+C^TP+\varepsilon PB_1B_1^TP+\gamma PB_2B_2^TP+\varepsilon^{-1}L^2-c\eta P\Gamma-\mu_1 P\right)\right. \\
& \quad\left.\times(x_i(t)-x_j(t))\right] \\
& -\mu_1\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{1}{2}w_{ij}(x_i(t)-x_j(t))^TP(x_i(t)-x_j(t)) \\
& -\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{1}{2}w_{ij} \\
& \times\left[(x_i(t-\tau(t))-x_j(t-\tau(t)))^T\left(\gamma^{-1}L'^2-\mu_2 P\right)(x_i(t-\tau(t))-x_j(t-\tau(t)))\right] \\
& -\mu_2\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{1}{2}w_{ij}\left[(x_i(t-\tau(t))-x_j(t-\tau(t)))^TP(x_i(t-\tau(t))-x_j(t-\tau(t)))\right] \\
\leq & \ \mu_1 V(t)+\mu_2 V(t-\tau(t)).
\end{aligned} \tag{3.10}$$

For $t = t_k$, we have

$$\begin{aligned}
V(t_k^+) &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}w_{ij}(x_i(t_k^+)-x_j(t_k^+))^TP(x_i(t_k^+)-x_j(t_k^+)) \\
&= -\frac{d_k^2}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}w_{ij}(x_i(t_k)-x_j(t_k))^TP(x_i(t_k)-x_j(t_k)) \\
&= d_k^2 V(t_k).
\end{aligned} \tag{3.11}$$

By Lemma 2.7, there exist $M > 0$ such that

$$V(t) \leq M\left(\sup_{-\tau\leq s\leq 0} V(t_0 + s)\right)e^{-\eta(t-t_0)}, \tag{3.12}$$

which implies that

$$\begin{aligned}\frac{1}{2}\xi_i\xi_j\lambda_{\min}(P)\|x_i(t) - x_j(t)\|^2 &\leq \frac{1}{2}\sum_{i=1,j=1}^{N}\xi_i\xi_j\left(x_i(t) - x_j(t)\right)^T P\left(x_i(t) - x_j(t)\right)\\ &= V(t) = O\left(e^{-\eta(t-t_0)}\right).\end{aligned} \tag{3.13}$$

Consequently, the complex dynamical network (3.1) can reach globally exponential synchronization. □

Remark 3.2. When the impulsive effects are desynchronizing, that is, $|d_k| > 1$, the condition (H_4) in Theorem 3.1 yields $-\mu_1 > \mu_2$, which means that the delayed complex networks without impulsive effects of (2.2) is exponentially synchronized. But when the impulsive effects are synchronizing, that is, $|d_k| < 1$, we do not need the condition $-\mu_1 > \mu_2$ due to the effect of impulses.

Theorem 3.3. *Suppose that Assumptions 2.4 and 2.5 hold and $\Delta_{\sup} < \infty$. Also suppose that there exist a diagonal positive definite matrix P and scalars $\overline{\eta} > 0$, $\overline{\varepsilon} > 0$, $\overline{\gamma} > 0$, $\overline{\mu}_1 > 0$, $\overline{\mu}_2 \geq 0$ such that*

- (H_1) $\Theta_1 = PC + C^T P + \overline{\varepsilon}PB_1B_1^T P + \overline{\gamma}PB_2B_2^T P + \overline{\varepsilon}^{-1}L^2 - c\overline{\eta}P\Gamma - \overline{\mu}_1 P \leq 0$;
- (H_2) $\Theta_2 = \overline{\gamma}^{-1}L'^2 - \overline{\mu}_2 P \leq 0$;
- (H_3) *for all* $k = 1, 2, \ldots, |d_k| < 1$;
- (H_4) *there exists a integer $m \geq 1$ such that $t_{k-m} \leq t_k - \tau \leq t_{k+1-m}$ for all $k \geq m$, and the discrete system:*

$$\theta(k+1) = J_k(m)\theta(k) \tag{3.14}$$

is globally exponentially stable with decay $\lambda > 0$, where

$$J_k(m) \triangleq \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ \beta_{k+1-m} & \beta_{k+2-m} & \beta_{k+3-m} & \cdots & \beta_{k-1} & \alpha_{k-1}\end{pmatrix}, \tag{3.15}$$

$\zeta = \mu_1 + \mu_2/(1-\sigma)$, $\alpha_k = d_k^2 e^{\zeta\Delta_{k-1}} + \beta_{k-1}$, $\beta_{k-j} = (\overline{\beta}/(1-\sigma))\Delta_{k-j}e^{\zeta\Delta_{k-j}}$, $j = 1, 2, \ldots, m-1$;

(H_5) *there exists a constant* $\mathcal{T}_0$ *such that the average dwell time* $\mathcal{T}_a$ *satisfies*

$$\mathcal{N}[t_0, t] \geq -\mathcal{T}_0 + \frac{t - t_0}{\mathcal{T}_a}, \quad t \geq t_0, \tag{3.16}$$

where $\mathcal{N}[t_0, t]$ *is the number of impulsive times of the impulsive sequence on the interval* $[t_0, t]$*;*

(H_6) $\overline{\eta}\lambda_{\max}(W) + \lambda_2(\overline{A}) \leq 0$.

Then the complex dynamical networks (3.1) *are exponentially synchronized with decay rate* $\lambda/2\mathcal{T}_a$.

Proof. Consider a Lyapunov-Krasovskii functional:

$$V(t) = V_1(t) + V_2(t), \tag{3.17}$$

with

$$V_1(t) = x^T(t)(W \otimes P)x(t), \qquad V_2(t) = \frac{\overline{\mu}_2}{1-\sigma}\int_{t-\tau(t)}^{t} x^T(s)(W \otimes P)x(s)ds. \tag{3.18}$$

Similar to the proof of Theorem 3.1, for $t \in (t_k, t_{k+1}]$, we get

$$D^+V_1(t) \leq \overline{\mu}_1 x^T(t)(W \otimes P)x(t) + \overline{\mu}_2 x^T(t-\tau(t))(W \otimes P)x(t-\tau(t)). \tag{3.19}$$

For $t \in (t_k, t_{k+1}]$, we have

$$D^+V_2(t) \leq \frac{\overline{\mu}_2}{1-\sigma}x^T(t)(W \otimes P)x(t) - \overline{\mu}_2 x^T(t-\tau(t))(W \otimes P)x(t-\tau(t)). \tag{3.20}$$

Then

$$D^+V(t) = D^+V_1(t, x(t)) + D^+V_2(t) \leq \left(\overline{\mu}_1 + \frac{\overline{\mu}_2}{1-\sigma}\right)V_1(t) \leq \zeta V(t). \tag{3.21}$$

Thus

$$V(t) \leq V\left(t_k^+\right)e^{\zeta(t-t_k)}, \quad t \in (t_k, t_{k+1}]. \tag{3.22}$$

By (3.11), for $t = t_k$, we have

$$V_1\left(t_k^+\right) \leq d_k^2 V_1(t_k). \tag{3.23}$$

It follows from condition (iii) that there exists some $\widehat{t}_{k-j+1} \in (t_{k-j}, t_{k-j+1}]$ such that

$$\begin{aligned} V_2(t_k^+) &\leq \frac{\overline{\mu}_2}{1-\sigma}\int_{t_k-\tau(t_k)}^{t_k} V_1(s)ds \leq \frac{\overline{\mu}_2}{1-\sigma}\int_{t_{k-m}}^{t_k} V_1(s)ds \\ &= \frac{\overline{\mu}_2}{1-\sigma}\sum_{j=1}^{m}\int_{t_{k-j}^+}^{t_{k-j+1}} V_1(s)ds = \frac{\overline{\mu}_2}{1-\sigma}\sum_{j=1}^{m}\Delta_{k-j}V_1\left(\widehat{t}_{k-j+1}\right). \end{aligned} \tag{3.24}$$

Then from (3.22), we have

$$V_2(t_k^+) \leq \frac{\overline{\mu}_2}{1-\sigma}\sum_{j=1}^{m}\Delta_{k-j}V\left(\widehat{t}_{k-j+1}\right) \leq \frac{\overline{\mu}_2}{1-\sigma}\sum_{j=1}^{m}\Delta_{k-j}e^{\zeta\Delta_{k-j}}V\left(t_{k-j}^+\right). \tag{3.25}$$

Together with (3.22), (3.23) and the above inequality, we have

$$\begin{aligned} V(t_k^+) &\leq \left(d_k^2 + \frac{\overline{\mu}_2}{1-\sigma}\Delta_{k-1}\right)e^{\zeta\Delta_{k-1}}V(t_{k-1}^+) + \frac{\overline{\mu}_2}{1-\sigma}\sum_{j=2}^{m}\Delta_{k-j}e^{\zeta\Delta_{k-j}}V\left(t_{k-j}^+\right) \\ &\triangleq \alpha_{k-1}V(t_{k-1}^+) + \sum_{j=1}^{m-1}\beta_{k-j-1}V\left(t_{k-j-1}^+\right). \end{aligned} \tag{3.26}$$

Set $Z(k) = (z_1(k), z_2(k), \ldots, z_m(k))^T$ and $z_1(k) = V(t_{k+1}^+)$, $z_2(k) = V(t_{k+2}^+), \ldots, z_m(k) = V(t_{k+m}^+)$. Then

$$\begin{pmatrix} z_1(k+1-m) \\ z_2(k+1-m) \\ \vdots \\ z_m(k+1-m) \end{pmatrix} \leq J_k(m)\begin{pmatrix} z_1(k-m) \\ z_2(k-m) \\ \vdots \\ z_m(k-m) \end{pmatrix}, \tag{3.27}$$

that is,

$$Z(k-m+1) \leq J_k(m)Z(k-m). \tag{3.28}$$

We consider the discrete system:

$$\theta(k+1) = J_k(m)\theta(k), \qquad \theta(m-1) = Z(-1). \tag{3.29}$$

Then, by the comparison principle, we see that for $k \geq m-1$

$$Z(k-m) \leq \theta(k). \tag{3.30}$$

Note that the system (3.29) is globally exponential stable with decay $\lambda > 0$, then there exists constant $M > 0$ such that

$$\|Z(k-m)\| \leq Me^{-\lambda(k-m+1)}\|Z(-1)\|, \quad k \geq m-1, \tag{3.31}$$

where $\|Z(-1)\| = [\sum_{j=0}^{m-1} V^2(t_j)]^{1/2}$, $\|Z(k-m)\| = [\sum_{j=1}^{m} V^2(t_{j+k-m})]^{1/2}$. From (3.17) and (3.22), we have

$$\begin{aligned} V_2\left(t_j^+\right) &= \frac{\overline{\mu}_2}{1-\sigma}\int_{t_j-\tau(t_j)}^{t_j} x^T(s)(W\otimes P)x(s)ds \leq \frac{\overline{\mu}_2}{1-\sigma}\int_{t_0-\tau}^{t_j} x^T(s)(W\otimes P)x(s)ds \\ &= \frac{\overline{\mu}_2}{1-\sigma}\int_{t_0-\tau}^{t_0} x^T(s)(W\otimes P)x(s)ds + \frac{\overline{\mu}_2}{1-\sigma}\sum_{s=0}^{j-1}\int_{t_s^+}^{t_{s+1}} x^T(s)(W\otimes P)x(s)ds \\ &\leq \frac{\overline{\mu}_2\tau}{1-\sigma}\sup_{-\tau\leq s\leq 0} V(t_0+s) + \frac{\overline{\mu}_2}{1-\sigma}\sum_{s=0}^{j-1}\Delta_s V(t_s^+)e^{\zeta\Delta_s}, \quad j = 0,1,\ldots,m-1. \end{aligned} \tag{3.32}$$

Furthermore, it follows that

$$\begin{aligned} V\left(t_j^+\right) &= V_1\left(t_j^+\right) + V_2\left(t_j^+\right) \\ &\leq \frac{\overline{\mu}_2\tau}{1-\sigma}\sup_{-\tau\leq s\leq 0} V(t_0+s) + d_j^2 e^{\zeta\Delta_{j-1}} V\left(t_{j-1}^+\right) + \frac{\overline{\mu}_2}{1-\sigma}\sum_{s=0}^{j-1}\Delta_s V(t_s^+)e^{\zeta\Delta_{s-1}} \\ &= \frac{\overline{\mu}_2\tau}{1-\sigma}\sup_{-\tau\leq s\leq 0} V(t_0+s) + \alpha_j V\left(t_{j-1}^+\right) + \sum_{s=0}^{j-2}\beta_s V(t_s^+), \quad j = 1,2,\ldots,m-1, \\ V(t_0) &\leq \left(1 + \frac{\overline{\mu}_2\tau}{1-\sigma}\right)\sup_{-\tau\leq s\leq 0} V(t_0+s). \end{aligned} \tag{3.33}$$

By induction, there exists a constant $\vartheta > 0$, which is dependent on τ, σ, $\overline{\mu}_1$, $\overline{\mu}_2$, Δ_j, $j = 0,1,\ldots,m-1$ such that

$$V\left(t_j^+\right) \leq \vartheta\|\xi\|^2, \tag{3.34}$$

which yields that

$$\|Z(-1)\| = \left[\sum_{j=0}^{m-1} V^2\left(t_j^+\right)\right]^{1/2} \leq \sqrt{m}\vartheta \sup_{-\tau\leq s\leq 0} V(t_0+s). \tag{3.35}$$

From (3.31) and the above inequality, we see that for all $k = 0,1,\ldots,$

$$V(t_k^+) \leq \|Z(k-m)\| \leq M\sqrt{m}\vartheta e^{-\lambda(k-m+1)}\sup_{-\tau\leq s\leq 0} V(t_0+s). \tag{3.36}$$

Therefore, by (3.17), (3.22), and (3.36), we conclude that for $t \in (t_k, t_{k+1}], k = 0, 1, \ldots,$

$$V(t) \leq e^{\zeta(t-t_k)} V\left(t_k^+\right) \leq \Upsilon e^{-\lambda k} \sup_{-\tau \leq s \leq 0} V(t_0 + s), \tag{3.37}$$

where $\Upsilon = M\sqrt{m}\vartheta e^{\lambda(m-1)+\zeta\Delta_k}$. For all $t \in (t_k, t_{k+1}], k = 0, 1, \ldots,$ we obtain that $\mathcal{N}[t_0, t] = k$. Then

$$V(t) \leq \Upsilon e^{\lambda \mathcal{T}_0} e^{-(\lambda/\mathcal{T}_a)(t-t_0)} \sup_{-\tau \leq s \leq 0} V(t_0 + s), \tag{3.38}$$

which means that

$$\begin{aligned} \frac{1}{2}\xi_i\xi_j\lambda_{\min}(P)\left\|x_i(t) - x_j(t)\right\|^2 &\leq \frac{1}{2}\sum_{i=1,j=1}^{N} \xi_i\xi_j\left(x_i(t) - x_j(t)\right)^T P\left(x_i(t) - x_j(t)\right) \\ &= V(t) = O\left(e^{-(\lambda/\mathcal{T}_a)(t-t_0)}\right). \end{aligned} \tag{3.39}$$

This completes the proof of the theorem. □

Remark 3.4. Theorem 3.3 presents a new delay-dependent exponential synchronization criterion for complex dynamical networks by using the Lyapunov-Krasovskii functional. Note that, for $d_k < 1$, $\sigma = 0$, the proposed result demonstrates its superiority to Theorem 3.1, which will be well illustrated via an example in the next section.

Corollary 3.5. *Suppose that Assumptions 2.4 and 2.5 hold and $\Delta_{\sup} < \infty$ and $\tau < t_k - t_{k-1}$ for all $k = 1, 2, \ldots$. If there exist positive definite matrix P and scalars $\overline{\eta} > 0$, $\overline{\varepsilon} > 0$, $\overline{\gamma} > 0$, $\overline{\mu}_1 > 0$, $\overline{\mu}_2 \geq 0$ such that (H_1)–(H_3) and (H_5)-(H_6) of Theorem 3.3 hold, and condition (H_4) of Theorem 3.3 is replaced by the following condition:*

(H_4') *there exists a constant $\lambda > 0$ such that*

$$\ln\left(d_k^2 + \frac{\overline{\mu}_2\tau}{1-\sigma}\right) + \zeta\Delta_{k-1} \leq -\lambda, \tag{3.40}$$

where $\zeta = \overline{\mu}1 + \overline{\mu}_2/(1-\sigma)$, then the complex dynamical networks (3.1) *are exponentially synchronized with decay rate $\lambda/2\mathcal{T}_a$.*

Proof. Choose a Lyapunov-Krasovskii functional candidate $V(x(t))$ as

$$V(x(t)) = V_1(x(t)) + V_2(x(t)), \tag{3.41}$$

with

$$V_1(x(t)) = x^T(t)(W \otimes P)x(t), \qquad V_2(x(t)) = \frac{\overline{\mu}_2}{1-\sigma}\int_{t-\tau(t)}^{t} x^T(s)(W \otimes P)x(s)ds. \tag{3.42}$$

By the proof of Theorem 3.3, for $t \in (t_k, t_{k+1}]$, we have

$$D^+V(t) \le \zeta V(t),$$
$$V(t) \le V(t_k^+)e^{\zeta(t-t_k)}, \quad t \in (t_k, t_{k+1}]. \tag{3.43}$$

Note that $\tau < t_k - t_{k-1}$, then there exists some $\hat{t}_k \in [t_k - \tau, t_k]$ such that

$$V_2(t_k^+) \le \frac{\overline{\mu}_2}{1-\sigma}\int_{t_k-\tau}^{t_k} V_1(s)ds = \frac{\overline{\mu}_2\tau}{1-\sigma}V_1\left(\hat{t}_k\right). \tag{3.44}$$

Thus

$$\begin{aligned} V(t_k^+) &\le d_k^2 V_1(t_k) + \frac{\overline{\mu}_2\tau}{1-\sigma}V_1\left(\hat{t}_k\right) \le d_k^2 e^{\zeta\Delta_{k-1}}V_1(t_{k-1}^+) + \frac{\overline{\mu}_2\tau}{1-\sigma}e^{\zeta\Delta_{k-1}}V_1(t_{k-1}^+) \\ &= e^{\ln(d_k^2+\overline{\mu}_2\tau/(1-\sigma))+\zeta\Delta_{k-1}}V(t_{k-1}^+). \end{aligned} \tag{3.45}$$

Then from condition (H_4'), we obtain

$$V(t_k^+) \le e^{-\lambda}V(t_{k-1}^+) \le \cdots \le e^{-\lambda k}V(t_0), \tag{3.46}$$

for all $k = 1, 2, \ldots$. The remainder proof of the theorem is similar to Theorem 3.3. □

Remark 3.6. By Corollary 3.5, under the case that $\tau < t_k - t_{k-1}$ for all $k = 1, 2, \ldots$, we see that the estimations of maximal time-delay τ' and maximal dwell time $\Delta_{\sup}$ as

$$\tau' < \sup_{k\ge 1}\left\{\frac{(1-\sigma)e^{-\zeta\Delta_{k-1}} - \lambda - d_k^2}{\zeta}\right\}, \qquad \Delta_{\sup} < \sup_{k\ge 1}\left\{\frac{-\lambda - \ln\left(d_k^2 + \overline{\mu}_2\tau/(1-\sigma)\right)}{\zeta}\right\}. \tag{3.47}$$

Remark 3.7. By Corollary 3.5, if we take the impulsive gains d_k as

$$0 < d_k < \sqrt{e^{-\zeta\Delta_{k-1}-\lambda} - \frac{\overline{\mu}_2\tau}{1-\sigma}}, \quad k = 1, 2, \ldots, \tag{3.48}$$

then network (3.1) achieves exponential synchronization.

Corollary 3.8. *Suppose that Assumptions 2.4 and 2.5 hold and $\Delta_{\sup} < \infty$. If there exist positive definite matrix P and scalars $\overline{\eta} > 0$, $\overline{\varepsilon} > 0$, $\overline{\gamma} > 0$, $\overline{\mu}_1 > 0$, $\overline{\mu}_2 \ge 0$ such that (H_1)–(H_3) and (H_5)-(H_6) of Theorem 3.3 hold and condition (H_4) of Theorem 3.3 is replaced by one of the following two conditions:*

(H_4'') *there exists a constant $m > 1$ such that $t_{k-m} < t_k - \tau \le t_{k+1-m}$ for all $k \ge m$, and the matrix $J(m)$ satisfies the spectral radius condition for some $\lambda > 0$*

$$\rho(J(m)) < e^{-\lambda}, \tag{3.49}$$

where

$$J(m) \triangleq \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 & \cdots & \epsilon_1 & \epsilon_1 + \epsilon_2 \end{pmatrix}, \tag{3.50}$$

$\epsilon_1 = (\overline{\mu}_2/(1-\sigma))\Delta_{\sup} e^{\zeta\Delta_{\sup}}$, $\epsilon_2 = de^{\zeta\Delta_{\sup}}$, $d = \sup_{k\geq 1}\{d_k^2\}$, $\zeta = \overline{\mu}_1 + \overline{\mu}_2/(1-\sigma)$;

(iii''') *there exists a positive integer* $m \geq 1$ *such that* $t_{k-m} < t_k - \tau \leq t_{k+1-m}$ *for all* $k \geq m$, *and there exists a constant* $0 < \varrho < 1$ *such that all roots* λ_j $(j = 1, 2, \ldots, m)$ *of the characteristic polynomial:*

$$\Psi_k(\lambda) \triangleq \lambda^m - \mu_{k-1}\lambda^{m-1} - \nu_{k-1}\lambda^{m-2} - \cdots - \nu_{k+2-m}\lambda - \nu_{k+1-m} \tag{3.51}$$

satisfy that $|\lambda_j| \leq \varrho < 1$,

then the complex dynamical networks (3.1) *are exponentially synchronized.*

Remark 3.9. From Theorems 3.1 and 3.3, when the delayed network dynamics are desynchronizing and the impulsive effects are synchronizing, in order to ensure synchronization, it should be naturally assumed that the frequency of impulses should not be too low. Usually, we always use condition $t_k - t_{k-1} \leq T_1$ $(T_1 > 0)$ to ensure that the frequency of impulses should not be too low. Conversely, when the delayed network dynamics are synchronizing but the impulsive effects are desynchronizing, the impulses should not occur too frequently in order to guarantee synchronization. To ensure that the impulses do not occur too frequently, we always assume that $t_k - t_{k-1} \geq T_2$ $(T_2 > 0)$.

4. Examples and Simulations

In this section, some examples and numerical simulations are provided to illustrate our results.

Example 4.1. Consider the following delayed neural networks [26]:

$$x'(t) = Cx(t) + B_1 f(x(t)) + B_2 g(x(t-1)), \tag{4.1}$$

where $x = (x_1, x_2)^T$, $f(x) = (f(x_1), f(x_2))^T$, $g(x) = (g(x_1), g(x_2))^T$, $f(x) = g(x) = \tanh(x)$, $C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 2 & -0.1 \\ -5.0 & 1.5 \end{pmatrix}$, and $B_2 = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -1 \end{pmatrix}$. These neural networks (4.1) are chaotic, and chaotic attractor is shown in Figure 1.

We consider the following linear coupled delayed networks:

$$x_i'(t) = Cx_i(t) + B_1 f(x_i(t)) + B_2 g(x_i(t-1)) + c\sum_{j=1}^{4} a_{ij}\Gamma x_j(t), \quad i = 1, 2, 3, 4, \tag{4.2}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$, $c = 1.4$, $\Gamma = \begin{pmatrix} 4.18 & 0 \\ 0 & 4.9 \end{pmatrix}$ and $A = \begin{pmatrix} -2 & 0.4 & 1 & 0.6 \\ 0.4 & -3 & 0 & 2.6 \\ 1 & 0 & -2.4 & 1.4 \\ 0.6 & 2.6 & 1.4 & -4.6 \end{pmatrix}$.

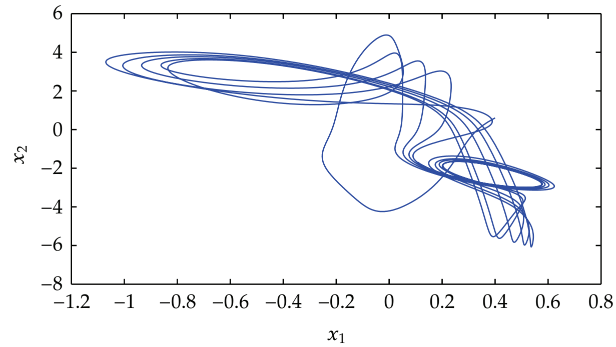

Figure 1: Chaotic and chaotic attractor.

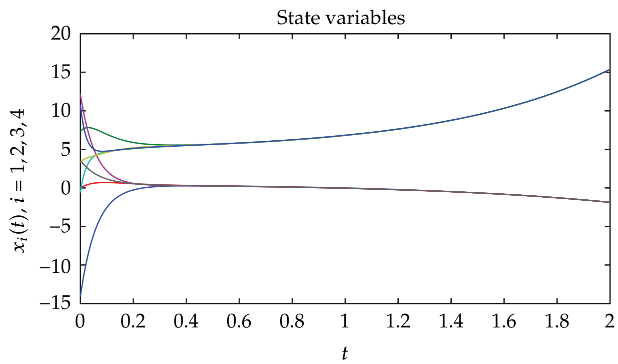

Figure 2: The state variables $x_{i1}(t)$ and $x_{i2}(t)$ without impulsive effects.

Figure 2 shows the synchronization of networks of (4.3).

At last, we consider the following linear coupled delayed networks with impulsive effects:

$$
\begin{aligned}
x_i'(t) &= Cx_i(t) + B_1 f(x_i(t)) + B_2 g(x_i(t-1)) \\
&\quad + c\sum_{j=1}^{4} a_{ij}\Gamma x_j(t), \quad t \geq 0,\ t \neq k, k = 1, 2, \ldots, \\
x_i(t_k^+) &= d_k x_i(t_k), \quad t = k,\ i = 1, 2, 3, 4,
\end{aligned}
\tag{4.3}
$$

where $d_k = 1.2, \lambda_{\max}(W) = 0.25$, and $\lambda_2(\overline{A}) = -1.0439$. Letting $L = L' = (1/2)I, \eta = 4.1756, \varepsilon = \gamma = \mu_2 = 1$ and solving the LMIs in (i), (ii) in Theorem 3.1, we get that $\mu_1 = -6.4461$ and $P = \mathrm{diag}\{0.8432, 0.8774\}$. By Theorem 3.1, we see that the complex dynamical networks (4.3) are exponentially synchronized. Figure 3 shows the synchronization of networks with delay and impulsive effects.

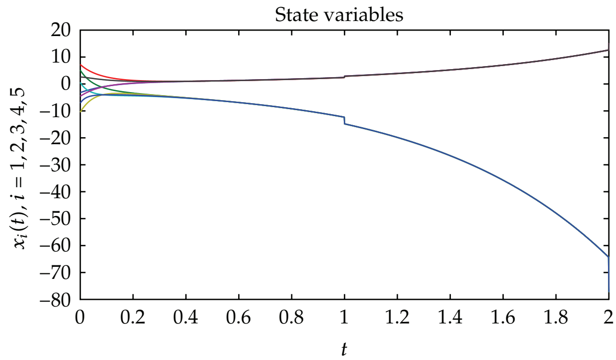

Figure 3: The state variables $x_{i1}(t)$ and $x_{i2}(t)$ with impulsive effects.

Example 4.2. Consider the following neural networks with delay and impulse:

$$\begin{aligned} x_i'(t) &= Cx_i(t) + B_1 f(x_i(t)) + B_2 g(x_i(t-\tau(t))) \\ &\quad + c\sum_{j=1}^{5} a_{ij}\Gamma x_j(t), \quad t \geq 0,\ t \neq k,\ k = 1,2,\ldots, \\ x_i(t_k^+) &= d_k x_i(t_k), \quad t = k,\ i = 1,2,\ldots,5, \end{aligned} \tag{4.4}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T$, $\tau(t) = 0.01, d_k = 0.25, c = 0.8, C = \mathrm{diag}\{-0.3, -0.6, -1\}$,

$$B_1 = \begin{pmatrix} 0.4 & -1.2 & 0.4 \\ 0.3 & -1 & -0.4 \\ 1.4 & 0.5 & -0.8 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0.6 & 0.7 & 0.2 \\ -0.3 & -0.3 & -0.4 \\ 1.1 & 0.5 & 0.4 \end{pmatrix},$$
$$A = \begin{pmatrix} -0.6 & 0.3 & 0 & 0.2 & 0.1 \\ 0.3 & -0.5 & 0 & 0.1 & 0.1 \\ 0 & 0 & -0.3 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.1 & -0.4 & 0 \\ 0.1 & 0.1 & 0.2 & 0 & -0.4 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1.1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 2.1 \end{pmatrix}, \tag{4.5}$$

$f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), f_3(x_{i3}(t)))^T, g(x_i(t)) = (g_1(x_{i1}(t)), g_2(x_{i2}(t)), g_3(x_{i3}(t)))^T$, $f_j(x) = x, g_j(x) = (1/10)(|x+1| - |x-1|), j = 1,2,3$. $\lambda_{\max}(W) = 0.56, \lambda_2(\overline{A}) = -0.4693$. Letting $L = I, L = (1/10)I, \eta = 0.838, \varepsilon = \gamma = \mu_2 = 1$ and solving the LMIs in (i), (ii) in Theorem 3.1, we get that $\mu_1 = 1.5626$ and $P = \mathrm{diag}\{1.3782, 0.9991, 1.4307\}$. We can verify that the synchronization criteria proposed by Theorem 3.1 are not satisfied. However, we conclude that the complex dynamical networks (4.4) are exponentially synchronized by Corollary 3.5. Figure 4 depicts the synchronization state variables $x_{i1}(t)$, $x_{i2}(t)$, and $x_{i3}(t)$ with impulsive effects. Figure 5 depicts the synchronization state variables $x_{i1}(t)$, $x_{i2}(t)$, and $x_{i3}(t)$ without impulsive effects.

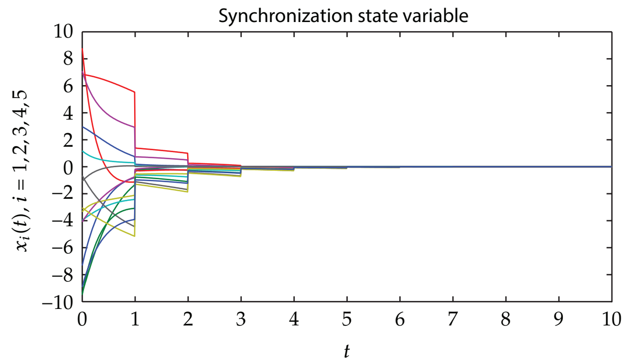

Figure 4: The state variables $x_{i1}(t)$, $x_{i2}(t)$, and $x_{i3}(t)$ with impulses.

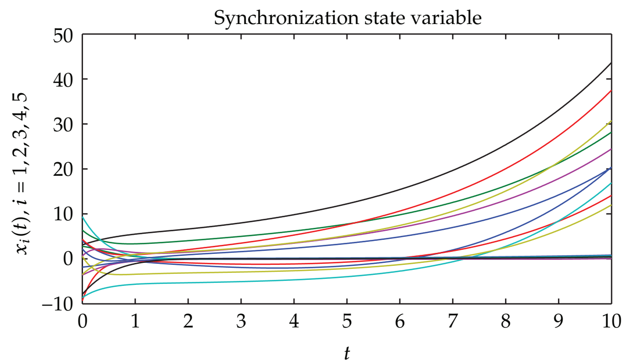

Figure 5: The state variables $x_{i1}(t)$, $x_{i2}(t)$, and $x_{i3}(t)$ without impulsive effects.

Remark 4.3. In Example 4.2, if we take $t_k - t_{k-1} = 0.1$ and $d_k = 0.2, \tau(t) = 0.9 \sin t$, it is easy to see that the synchronization criteria proposed by Corollary 3.8 are not satisfied. However, we conclude that the networks (4.4) are exponentially synchronized by Theorem 3.1.

5. Conclusions

In this paper, by establishing some lemmas of new impulsive differential inequality and by using the Lyapunov functional method and the Kronecker product techniques, exponential synchronization for impulsive dynamical networks with irreducible coupling matrix is derived. Some criteria are obtained not only relevant to delay but also to impulsive effects. In particular, the results can be extended to the case of one reducible coupling matrix A, which implies that the network topology may be a weakly connected graph containing a rooted spanning tree.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant 60874088, the Natural Science Foundation of Jiangsu Province of China under Grant BK2009271, and JSPS Innovation program under Grant CXZZ11_0132.

References

[1] D. J. Watts and S. H. Strogatz, "Collective dynamics of small-world networks," *Nature*, vol. 393, no. 6684, pp. 440–442, 1998.

[2] A. L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D. U. Hwang, "Complex networks: structure and dynamics," *Physics Reports*, vol. 424, no. 4-5, pp. 175–308, 2006.

[4] G. Rangarajan and M. Z. Ding, "Stability of synchronized chaos in coupled dynamical systems," *Physics Letters A*, vol. 296, no. 4-5, pp. 204–209, 2002.

[5] F. Sorrentino, M. Di Bernardo, and F. Garofalo, "Synchronizability and synchronization dynamics of weighed and unweighed scale free networks with degree mixing," *International Journal of Bifurcation and Chaos*, vol. 17, no. 7, pp. 2419–2434, 2007.

[6] J. Lü, X. Yu, and G. Chen, "Chaos synchronization of general complex dynamical networks," *Physica A*, vol. 334, no. 1-2, pp. 281–302, 2004.

[7] X. F. Wang and G. Chen, "Synchronization in scale-free dynamical networks: robustness and fragility," *IEEE Transactions on Circuits and Systems I*, vol. 49, no. 1, pp. 54–62, 2002.

[8] C. W. Wu, "Synchronization in networks of nonlinear dynamical systems coupled via a directed graph," *Nonlinearity*, vol. 18, no. 3, pp. 1057–1064, 2005.

[9] Z. Li, "Exponential stability of synchronization in asymmetrically coupled dynamical networks," *Chaos*, vol. 18, no. 2, Article ID 023124, 2008.

[10] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Singapore, 1989.

[11] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol. 14 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, World Scientific, Singapore, 1995.

[12] T. Yang, *Impulsive Control Theory*, vol. 272 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 2001.

[13] Q. Wu, J. Zhou, and L. Xiang, "Global exponential stability of impulsive differential equations with any time delays," *Applied Mathematics Letters*, vol. 23, no. 2, pp. 143–147, 2010.

[14] S. Cai, J. Zhou, L. Xiang, and Z. R. Liu, "Robust impulsive synchronization of complex delayed dynamical networks," *Physics Letters A*, vol. 372, no. 30, pp. 4990–4995, 2008.

[15] Y. Q. Yang and J. D. Cao, "Exponential synchronization of the complex dynamical networks with a coupling delay and impulsive effects," *Nonlinear Analysis*, vol. 11, no. 3, pp. 1650–1659, 2010.

[16] J. Zhou, L. Xiang, and Z. Liu, "Synchronization in complex delayed dynamical networks with impulsive effects," *Physica A*, vol. 384, no. 2, pp. 684–692, 2007.

[17] J. Lu, D. W. C. Ho, and J. D. Cao, "A unified synchronization criterion for impulsive dynamical networks," *Automatica*, vol. 46, no. 7, pp. 1215–1221, 2010.

[18] K. Li and C. H. Lai, "Adaptive-impulsive synchronization of uncertain complex dynamical networks," *Physics Letters A*, vol. 372, no. 10, pp. 1601–1606, 2008.

[19] Z. Yang and D. Xu, "Stability analysis of delay neural networks with impulsive effects," *IEEE Transactions on Circuits and Systems I*, vol. 52, no. 1, pp. 517–521, 2005.

[20] Z. H. Guan, Z. W. Liu, G. Feng, and Y. W. Wang, "Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control," *IEEE Transactions on Circuits and Systems I*, vol. 57, no. 8, pp. 2182–2195, 2010.

[21] L. Pan and J. Cao, "Anti-periodic solution for delayed cellular neural networks with impulsive effects," *Nonlinear Analysis*, vol. 12, no. 6, pp. 3014–3027, 2011.

[22] B. Liu and D. J. Hill, "Impulsive consensus for complex dynamical networks with nonidentical nodes and coupling time-delays," *SIAM Journal on Control and Optimization*, vol. 49, no. 2, pp. 315–338, 2011.

[23] S. H. Mou, H. J. Gao, W. Y. Qiang, and K. Chen, "New delay-dependent exponential stability for neural networks with time delay," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 38, no. 2, pp. 571–576, 2008.

[24] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1990.

[25] D. Y. Xu and Z. C. Yang, "Impulsive delay differential inequality and stability of neural networks," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 107–120, 2005.

[26] H. T. Lu, "Chaotic attractors in delayed neural networks," *Physics Letters A*, vol. 298, no. 2-3, pp. 109–116, 2002.

Advances in Operations Research

Advances in Decision Sciences

Mathematical Problems in Engineering

Algebra

Journal of Probability and Statistics

The Scientific World Journal

International Journal of Differential Equations

Hindawi

Submit your manuscripts at
http://www.hindawi.com

International Journal of Combinatorics

Advances in Mathematical Physics

Journal of Complex Analysis

Journal of Mathematics

International Journal of Stochastic Analysis

Abstract and Applied Analysis

Discrete Dynamics in Nature and Society

International Journal of Mathematics and Mathematical Sciences

Journal of Discrete Mathematics

Journal of Function Spaces

Journal of Applied Mathematics

Journal of Optimization