Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 282908, 28 pages
doi:10.1155/2012/282908

Research Article

Hopf Bifurcation of a Predator-Prey System with Delays and Stage Structure for the Prey

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Received 22 August 2012; Revised 3 October 2012; Accepted 5 October 2012

Academic Editor: M. De la Sen

This paper is concerned with a Holling type III predator-prey system with stage structure for the prey population and two time delays. The main result is given in terms of local stability and bifurcation. By choosing the time delay as a bifurcation parameter, sufficient conditions for the local stability of the positive equilibrium and the existence of periodic solutions via Hopf bifurcation with respect to both delays are obtained. In particular, explicit formulas that can determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are established by using the normal form method and center manifold theorem. Finally, numerical simulations supporting the theoretical analysis are also included.

1. Introduction

Predator-prey dynamics continues to draw interest from both applied mathematicians and ecologists due to its universal existence and importance. Many kinds of predator-prey models have been studied extensively [1–6]. It is well known that there are many species whose individual members have a life history that takes them through immature stage and mature stage. To analyze the effect of a stage structure for the predator or the prey on the dynamics of a predator-prey system, many scholars have investigated predator-prey systems with stage structure in the last two decades [7–15]. In [7], Wang considered the following predator-prey system with stage structure for the predator and obtained the sufficient conditions for the global stability of a coexistence equilibrium of the system:

$$\frac{dx}{dt} = x(t)(r - ax(t)) - \frac{a_1 y_2(t) x(t)}{1 + mx(t)},$$

$$\frac{dy_1}{dt} = \frac{a_2 x(t) y_2(t)}{1 + m x(t)} - r_1 y_1(t) - D y_1(t),$$
$$\frac{dy_2}{dt} = D y_1(t) - r_2 y_2(t), \tag{1.1}$$

where $x(t)$ represents the density of the prey at time t. $y_1(t)$ and $y_2(t)$ represent the densities of the immature predator and the mature predator at time t, respectively. For the meanings of all the parameters in system (1.1), one can refer to [7]. Considering the gestation time of the mature predator, Xu [8] incorporated the time delay due to the gestation of the mature predator into system (1.1) and considered the effect of the time delay on the dynamics of system (1.1).

There has also been a significant body of work on the predator-prey system with stage structure for the prey. In [12], Xu considered a delayed predator-prey system with a stage structure for the prey:

$$\begin{aligned}
\frac{dx_1}{dt} &= a x_2(t) - r_1 x_1(t) - b x_1(t), \\
\frac{dx_2}{dt} &= b x_1(t) - r_2 x_2(t) - b_1 x_2^2(t) - \frac{a_1 x_2(t) y(t)}{1 + m x_2(t)}, \\
\frac{dy}{dt} &= \frac{a_2 x_2(t-\tau) y(t-\tau)}{1 + m x_2(t-\tau)} - r y(t),
\end{aligned} \tag{1.2}$$

where $x_1(t)$ and $x_2(t)$ denote the population densities of the immature prey and the mature prey at time t, respectively. $y(t)$ denotes the population density of the predator at time t. All the parameters in system (1.2) are assumed positive. a is the birth rate of the immature prey. b is the transformation rate from immature individual to mature individuals. b_1 is the intraspecific competition coefficient of the mature prey. r_1 and r_2 are the death rates of the immature and the mature prey, respectively. r is the death rate of the predator. a_1 and a_2 are the interspecific interaction coefficients between the mature prey and the predator, respectively. $a_1 x_2/(1 + m x_2)$ is the response function of the predator. And τ is a constant delay due to the gestation of the predator. In [12], Xu investigated the persistence of system (1.2) by means of the persistence theory on infinite dimensional systems, and sufficient conditions are obtained for the global stability of nonnegative equilibrium of the model by constructing appropriate Lyapunov function. But studies on the predator-prey system not only involve the persistence and stability, but also involve many other behaviors such as periodic phenomenon, attractivity, and bifurcation [16–19]. In particular, the properties of periodic solutions are of great interest [20–24]. Therefore, F. Li and H. W. Li [14] considered the property of periodic solutions of the following system:

$$\frac{dx_1}{dt} = a x_2(t) - r_1 x_1(t) - b x_1(t),$$

$$\frac{dx_2}{dt} = bx_1(t) - r_2x_2(t) - b_1x_2^2(t) - \frac{a_1x_2^2(t)y(t)}{1+mx_2^2(t)},$$
$$\frac{dy}{dt} = \frac{a_2x_2^2(t-\tau)y(t-\tau)}{1+mx_2^2(t-\tau)} - ry(t). \tag{1.3}$$

Motivated by the work of Xu [12] and F. Li and H. W. Li [14] and considering the intraspecific competition of the immature prey population, we consider the following system:

$$\begin{aligned}\frac{dx_1}{dt} &= ax_2(t) - r_1x_1(t) - bx_1(t) - cx_1^2(t),\\ \frac{dx_2}{dt} &= bx_1(t) - r_2x_2(t) - b_1x_2(t)x_2(t-\tau_1) - \frac{a_1x_2^2(t)y(t)}{1+mx_2^2(t)},\\ \frac{dy}{dt} &= \frac{a_2x_2^2(t-\tau_2)y(t-\tau_2)}{1+mx_2^2(t-\tau_2)} - ry(t),\end{aligned} \tag{1.4}$$

where $x_1(t)$ and $x_2(t)$ denote the population densities of the immature prey and the mature prey at time t, respectively. $y(t)$ denotes the population density of the predator at time t. The parameters a, a_1, a_2, b, b_1, r, r_1, r_2, and m are defined as in system (1.3). c is the intraspecific competition of the immature prey, $\tau_1$is the feedback delay of the mature prey, and τ_2 is the time delay due to the gestation of the predator.

The organization of this paper is as follows. In Section 2, by analyzing the corresponding characteristic equations, the local stability of the positive equilibrium of system (1.4) is discussed, and the existence of Hopf bifurcation at the positive equilibrium is established. In Section 3, we determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions by using the normal form theory and center manifold theorem in [20]. And numerical simulations are carried out in Section 4 to illustrate the main theoretical results. Finally, main conclusions are included.

2. Local Stability and Hopf Bifurcation

From the viewpoint of biology, we are only interested in the positive equilibrium of system (1.4). It is not difficult to verify that system (1.4) has a positive equilibrium $E^0(x_1^0, x_2^0, y^0)$, where

$$\begin{aligned}x_1^0 &= \frac{-(b+r_1)+\sqrt{(b+r_1)^2+4acx_2^0}}{2c},\\ x_2^0 &= \sqrt{\frac{r}{a_2-mr}},\\ y^0 &= \frac{\left(bx_1^0 - r_2x_2^0 - b_1\left(x_2^0\right)^2\right)\left(1+m\left(x_2^0\right)^2\right)}{a_1\left(x_2^0\right)^2},\end{aligned} \tag{2.1}$$

if the following conditions hold: $H_1 : a_2 > mr$, $H_2 : bx_1^0 > (r_2 + b_1x_2^0)x_2^0$.

Let $x_1(t) = z_1(t) + x_1^0$, $x_2(t) = z_2(t) + x_2^0$, $y(t) = z_3(t) + y^0$, and we still denote $z_1(t)$, $z_2(t)$, and $z_3(t)$ by $x_1(t)$, $x_2(t)$, and $y(t)$. Then system (1.4) can be transformed to the following form:

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha_{11}x_1(t) + \alpha_{12}x_2(t) + \sum_{i+j\geq 2} f_1^{(ij)} x_1^i x_2^j, \\
\frac{dx_2}{dt} &= \alpha_{21}x_1(t) + \alpha_{22}x_2(t) + \alpha_{23}y(t) + \beta_{22}x_2(t-\tau_1) \\
&\quad + \sum_{i+j+k+l\geq 2} f_2^{(ijkl)} x_1^i x_2^j y^k x_2^l(t-\tau_1), \\
\frac{dy}{dt} &= \alpha_{33}y(t) + \gamma_{32}x_2(t-\tau_2) + \gamma_{33}y(t-\tau_2) \\
&\quad + \sum_{i+j+k\geq 2} f_3^{(ijk)} y^i x_2^j(t-\tau_2) y^k(t-\tau_2),
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\alpha_{11} &= -r_1 - b - 2cx_1^0, \qquad \alpha_{12} = a, \qquad \alpha_{21} = b, \\
\alpha_{22} &= -r_2 - b_1x_2^0 - \frac{2a_1x_2^0y^0}{\left(1+m\left(x_2^0\right)^2\right)^2}, \qquad \alpha_{23} = -\frac{a_1\left(x_2^0\right)^2}{1+m\left(x_2^0\right)^2}, \\
\alpha_{33} &= -r, \qquad \beta_{22} = -b_1x_2^0, \qquad \gamma_{32} = \frac{2a_2x_2^0y^0}{\left(1+m\left(x_2^0\right)^2\right)^2}, \qquad \gamma_{33} = r, \\
f_1^{(ij)} &= \frac{1}{i!j!}\frac{\partial^{i+j} f_1}{\partial x_1^i(t)\partial x_2^j(t)} \mid \left(x_1^0, x_2^0, y^0\right), \\
f_2^{(ijkl)} &= \frac{1}{i!j!k!l!}\frac{\partial^{i+j+k+l} f_2}{\partial x_1^i(t)\partial x_2^j(t)\partial y^k(t)\partial x_2^l(t-\tau_1)} \mid \left(x_1^0, x_2^0, y^0\right), \\
f_3^{(ijk)} &= \frac{1}{i!j!k!}\frac{\partial^{i+j+k} f_3}{\partial^i(t)\partial x_2^j(t-\tau_2)\partial y^k(t-\tau_2)} \mid \left(x_1^0, x_2^0, y^0\right), \\
f_1 &= ax_2(t) - r_1x_1(t) - bx_1(t) - cx_1^2(t), \\
f_2 &= bx_1(t) - r_2x_2(t) - b_1x_2(t)x_2(t-\tau_1) - \frac{a_1x_2^2(t)y(t)}{1+mx_2^2(t)}, \\
f_3 &= \frac{a_2x_2^2(t-\tau_2)y(t-\tau_2)}{1+mx_2^2(t-\tau_2)} - ry(t).
\end{aligned} \tag{2.3}$$

Then we can get the linearized system of system (2.2)

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha_{11}x_1(t) + \alpha_{12}x_2(t),\\ \frac{dx_2}{dt} &= \alpha_{21}x_1(t) + \alpha_{22}x_2(t) + \alpha_{23}y(t) + \beta_{22}x_2(t-\tau_1),\\ \frac{dy}{dt} &= \alpha_{33}y(t) + \gamma_{32}x_2(t-\tau_2) + \gamma_{33}y(t-\tau_2).\end{aligned} \tag{2.4}$$

Therefore, the corresponding characteristic equation of system (2.4) is

$$\begin{aligned}&\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + \left(n_2\lambda^2 + n_1\lambda + n_0\right)e^{-\lambda\tau_1}\\ &\quad + \left(p_2\lambda^2 + p_1\lambda + p_0\right)e^{-\lambda\tau_2}\\ &\quad + \left(q_1\lambda + q_0\right)e^{-\lambda(\tau_1+\tau_2)} = 0,\end{aligned} \tag{2.5}$$

where $m_0 = (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})\alpha_{33}$, $m_1 = \alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{33} + \alpha_{22}\alpha_{33} - \alpha_{12}\alpha_{21}$, $m_2 = -(\alpha_{11} + \alpha_{22} + \alpha_{33})$, $n_0 = -\alpha_{11}\alpha_{33}\beta_{22}$, $n_1 = (\alpha_{11} + \alpha_{33})\beta_{22}$, $n_2 = -\beta_{22}$, $p_0 = \alpha_{11}\alpha_{23}\gamma_{32} + \alpha_{12}\alpha_{21}\gamma_{33} - \alpha_{11}\alpha_{22}\gamma_{33}$, $p_1 = \alpha_{11}\gamma_{33} + \alpha_{22}\gamma_{33} - \alpha_{23}\gamma_{32}$, $p_2 = -\gamma_{33}$, $q_0 = -\alpha_{11}\beta_{22}\gamma_{33}$, $q_1 = \beta_{22}\gamma_{33}$.

Next, we consider the local stability of the positive equilibrium $E^0(x_1^0, x_2^0, y^0)$ and the Hopf bifurcation of system (1.4) for the different combination of τ_1 and τ_2.

Case 1. ($\tau_1 = \tau_2 = 0$). The characteristic equation (2.5) becomes

$$\lambda^3 + m_{12}\lambda^2 + m_{11}\lambda + m_{10} = 0, \tag{2.6}$$

where $m_{12} = m_2 + n_2 + p_2$, $m_{11} = m_1 + n_1 + p_1 + q_1$, $m_{10} = m_0 + n_0 + p_0 + q_0$.

It is not difficult to verify that $m_{12} > 0$ and $m_{10} > 0$. Thus, all the roots of (2.6) must have negative real parts, if the following condition holds: $H_{11} : m_{12}m_{11} > m_{10}$. Namely, the positive equilibrium $E^0(x_1^0, x_2^0, y^0)$ is locally stable in the absence of time delay, if H_{11} holds.

Case 2. ($\tau_1 > 0, \tau_2 = 0$). On substituting $\tau_2 = 0$, (2.5) becomes

$$\begin{aligned}&\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20}\\ &\quad + \left(n_{22}\lambda^2 + n_{21}\lambda + n_{20}\right)e^{-\lambda\tau_1} = 0,\end{aligned} \tag{2.7}$$

where $m_{22} = m_2 + p_2$, $m_{21} = m_1 + p_1$, $m_{20} = m_0 + p_0$, $n_{22} = n_2$, $n_{21} = n_1 + q_1$, $n_{20} = n_0 + q_0$.

Let $\lambda = i\omega_1 (\omega_1 > 0)$ be a root of (2.7). Then, we have

$$\begin{aligned}n_{21}\omega_1 \sin\tau_1\omega_1 + \left(n_{20} - n_{22}\omega_1^2\right)\cos\tau_1\omega_1 &= m_{22}\omega_1^2 - m_{20},\\ n_{21}\omega_1 \cos\tau_1\omega_1 - \left(n_{20} - n_{22}\omega_1^2\right)\sin\tau_1\omega_1 &= \omega_1^3 - m_{21}\omega_1.\end{aligned} \tag{2.8}$$

Squaring both sides and adding them up, we get the following sixth-degree polynomial equation:

$$\omega_1^6 + e_{22}\omega_1^4 + e_{21}\omega_1^2 + e_{20} = 0, \tag{2.9}$$

where $e_{22} = m_{22}^2 - n_{22}^2 - 2m_{21}$, $e_{21} = m_{21}^2 - 2m_{20}m_{22} + 2n_{20}n_{22} - n_{21}^2$, $e_{20} = m_{20}^2 - n_{20}^2$.

Let $\omega_1^2 = v_1$, then (2.9) becomes

$$v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20} = 0. \tag{2.10}$$

Define

$$f_1(v_1) = v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20}. \tag{2.11}$$

If $e_{20} < 0$, it is easy to know that (2.10) has at least one positive root. On the other hand, if $e_{20} \geq 0$, according to Lemma 2.2 in [25], (2.10) has positive roots if $e_{22}^2 - 3e_{21} > 0$ and $v_1^* = (-e_{22} + \sqrt{e_{22}^2 - 3e_{21}})/3 > 0$, $f_1(v_1^*) \leq 0$ hold. Therefore, we give the following assumption.

H_{21}: equation (2.10) has at least one positive root.

Without loss of generality, we assume that it has three positive roots which are denoted as v_{11}, v_{12}, and v_{13}. Thus, (2.9) has three positive roots $\omega_{1k} = \sqrt{v_{1k}}$, $k = 1,2,3$. The corresponding critical value of time delay $\tau_{1k}^{(j)}$ is

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}}\arccos\left\{\frac{A_{24}\omega_{1k}^4 + A_{22}\omega_{1k}^2 + A_{20}}{B_{24}\omega_{1k}^4 + B_{22}\omega_{1k}^2 + B_{20}}\right\} + \frac{2j\pi}{\omega_{1k}}, \quad k = 1,2,3, \quad j = 0,1,2,\ldots, \tag{2.12}$$

where $A_{24} = n_{21} - m_{22}n_{22}$, $A_{22} = m_{20}n_{22} + m_{22}n_{20} - m_{21}n_{21}$, $A_{20} = -m_{20}n_{20}$, $B_{24} = n_{22}^2$, $B_{22} = n_{21}^2 - 2n_{20}n_{22}$, $B_{20} = n_{20}^2$.

Let $\tau_{10} = \min\{\tau_{1k}^{(0)}\}$, $k \in \{1,2,3\}$, $\omega_{10} = \omega_{1k_0}$.

To verify the transversality condition of Hopf bifurcation, differentiating the two sides of (2.7) with respect to τ_1, and noticing that λ is a function of τ_1, we can obtain

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = -\frac{3\lambda^2 + 2m_{22}\lambda + m_{21}}{\lambda(\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20})} + \frac{2n_{22}\lambda + n_{21}}{\lambda(n_{22}\lambda^2 + n_{21}\lambda + n_{20})} - \frac{\tau_1}{\lambda}. \tag{2.13}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \operatorname{Re}\left[-\frac{3\lambda^2+2m_{22}\lambda+m_{21}}{\lambda(\lambda^3+m_{22}\lambda^2+m_{21}\lambda+m_{20})}\right] + \operatorname{Re}\left[\frac{2n_{22}\lambda+n_{21}}{\lambda(n_{22}\lambda^2+n_{21}\lambda+n_{20})}\right]. \tag{2.14}$$

Therefore,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} = \frac{3\omega_{10}^4+2\left(m_{22}^2-n_{22}^2-2m_{21}\right)\omega_{10}^2+m_{21}^2-2m_{20}m_{22}}{\left(\omega_{10}^3-m_{21}\omega_{10}\right)^2+\left(m_{20}-m_{22}\omega_{10}^2\right)^2} - \frac{2n_{22}^2\omega_{10}^2+n_{21}^2-2n_{20}n_{22}}{\left(n_{22}\omega_{10}^2-n_{20}\right)^2+n_{21}^2\omega_{10}^2}. \tag{2.15}$$

From (2.9), we can get

$$\left(\omega_{10}^3-m_{21}\omega_{10}\right)^2+\left(m_{20}-m_{22}\omega_{10}^2\right)^2=\left(n_{22}\omega_{10}^2-n_{20}\right)^2+n_{21}^2\omega_{10}^2. \tag{2.16}$$

Then, we have

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} = \frac{3v_{1*}^2+2e_{22}v_{1*}+e_{21}}{\left(n_{22}\omega_{10}^2-n_{20}\right)^2+n_{21}^2\omega_{10}^2} = \frac{f_1'(v_{1*})}{\left(n_{22}\omega_{10}^2-n_{20}\right)^2+n_{21}^2\omega_{10}^2}, \tag{2.17}$$

where $v_{1*}=\omega_{10}^2\in\{v_{11},v_{12},v_{13}\}$.

Therefore, $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\lambda=i\omega_{10}}\neq 0$ if $H_{22}: f_1'(v_{1*})\neq 0$ holds. Notice that $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\lambda=i\omega_{10}}$ and $[d\operatorname{Re}(\lambda)/d\tau_1]_{\lambda=i\omega_{10}}$ have the same sign. Then we have $[d\operatorname{Re}(\lambda)/d\tau_1]_{\lambda=i\omega_{10}}\neq 0$ if H_{22} holds. In conclusion, we have the following results.

Theorem 2.1. *Suppose that the conditions H_{21} and H_{22} hold. The positive equilibrium E^0 of system* (1.4) *is asymptotically stable for $\tau_1\in[0,\tau_{10})$ and unstable when $\tau_1>\tau_{10}$. Further, system* (1.4) *undergoes a Hopf bifurcation when $\tau_1=\tau_{10}$.*

Case 3. ($\tau_2>0,\tau_1=0$). On substituting $\tau_2=0$, (2.5) becomes

$$\lambda^3+m_{32}\lambda^2+m_{31}\lambda+m_{30}+\left(p_{32}\lambda^2+p_{31}\lambda+p_{30}\right)e^{-\lambda\tau_2}=0, \tag{2.18}$$

where $m_{32}=m_2+n_2$, $m_{31}=m_1+n_1$, $m_{30}=m_0+n_0$, $p_{32}=p_2$, $p_{31}=p_1+q_1$, $p_{30}=p_0+q_0$.

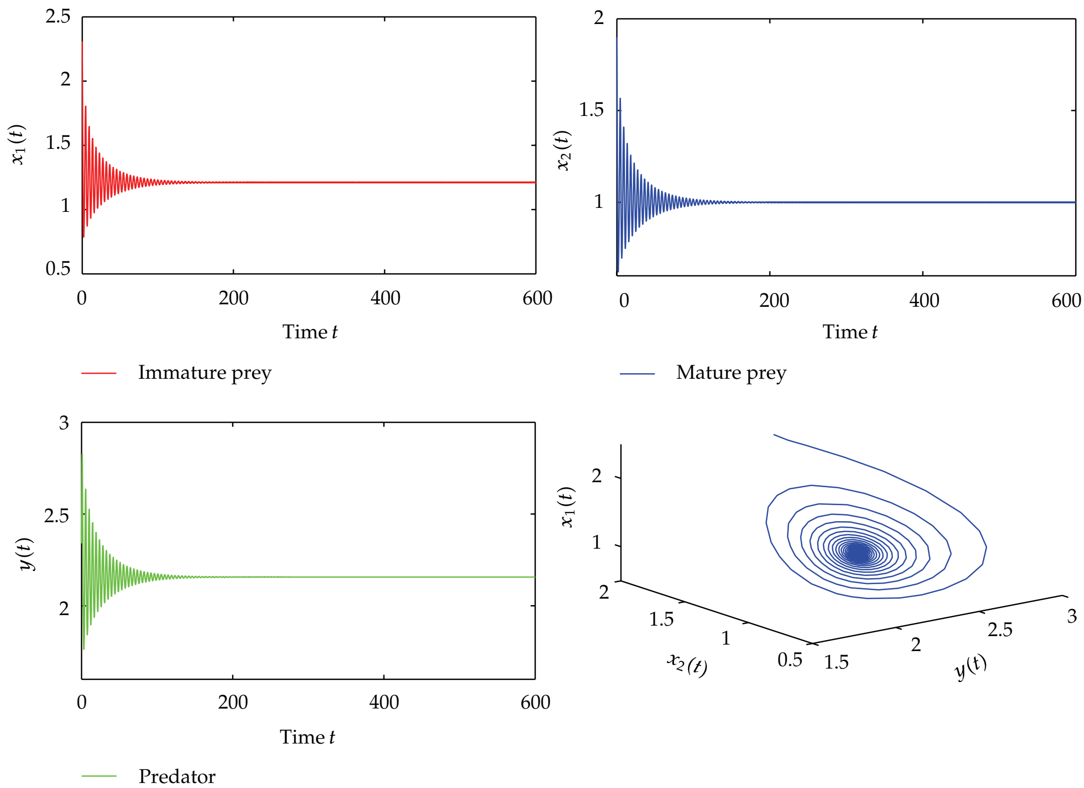

Figure 1: E^0 is locally asymptotically stable for $\tau_1 = 0.8500 < \tau_{10} = 0.9032$ with initial value "2.31; 1.9; 2.34."

Let $\lambda = i\omega_2(\omega_2 > 0)$ be a root of (2.18). Then, we get

$$p_{31}\omega_2 \sin \tau_2\omega_2 + \left(p_{30} - p_{32}\omega_2^2\right) \cos \tau_2\omega_2 = m_{32}\omega_2^2 - m_{30},$$
$$p_{31}\omega_2 \cos \tau_2\omega_2 - \left(p_{30} - p_{32}\omega_2^2\right) \sin \tau_2\omega_2 = \omega_2^3 - m_{31}\omega_2, \tag{2.19}$$

which follows that

$$\omega_2^6 + e_{32}\omega_2^4 + e_{31}\omega_2^2 + e_{30} = 0, \tag{2.20}$$

where $e_{32} = m_{32}^2 - p_{32}^2 - 2m_{31}$, $e_{31} = m_{31}^2 - 2m_{30}m_{32} + 2p_{30}p_{32} - p_{31}^2$, $e_{30} = m_{30}^2 - p_{30}^2$.

Let $\omega_2^2 = v_2$, then (2.20) becomes

$$v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30} = 0. \tag{2.21}$$

Define

$$f_2(v_2) = v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30}. \tag{2.22}$$

Similar as in case (2), we give the following assumption.

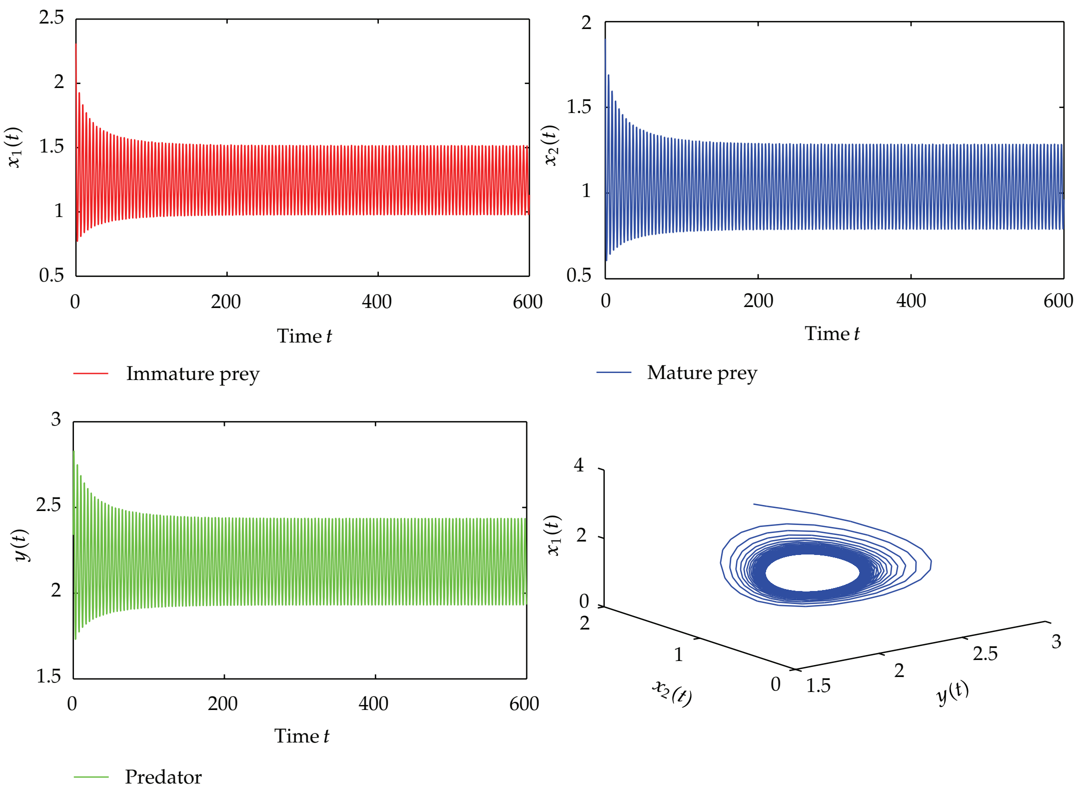

Figure 2: E^0 is unstable for $\tau_1 = 0.9200 > \tau_{10} = 0.9032$ with initial value "2.31; 1.9; 2.34."

H_{31}: equation (2.21) has at least one positive root.

Without loss of generality, we assume that it has three positive roots denoted by v_{21}, v_{22}, and v_{23}. Thus, (2.20) has three positive roots $\omega_{2k} = \sqrt{v_{2k}}, k = 1, 2, 3$.

The corresponding critical value of time delay $\tau_{2k}^{(j)}$ is

$$\tau_{2k}^{(j)} = \frac{1}{\omega_{2k}} \arccos\left\{ \frac{A_{34}\omega_{2k}^4 + A_{32}\omega_{2k}^2 + A_{30}}{B_{34}\omega_{2k}^4 + B_{32}\omega_{2k}^2 + B_{30}} \right\} + \frac{2j\pi}{\omega_{2k}}, \quad k = 1, 2, 3, \quad j = 0, 1, 2, \ldots, \tag{2.23}$$

where $A_{34} = p_{31} - m_{32}p_{32}$, $A_{32} = m_{30}p_{32} + m_{32}p_{30} - m_{31}p_{31}$, $A_{30} = -m_{30}p_{30}$, $B_{34} = p_{32}^2$, $B_{32} = p_{31}^2 - 2p_{30}p_{32}$, $B_{30} = p_{30}^2$.

Let $\tau_{20} = \min\{\tau_{2k}^{(0)}\}, k \in \{1, 2, 3\}, \omega_{20} = \omega_{2k_0}$.

Similar as in case (1), next, we suppose that the condition $H_{32} : f_2'(v_{2*}) \neq 0$ holds, where $v_{2*} = \omega_{20}^2 \in \{v_{21}, v_{22}, v_{23}\}$. Then we have $[d\,\mathrm{Re}(\lambda)/d\tau_2]_{\lambda = i\omega_{20}} \neq 0$. By the above analysis, we have the following results.

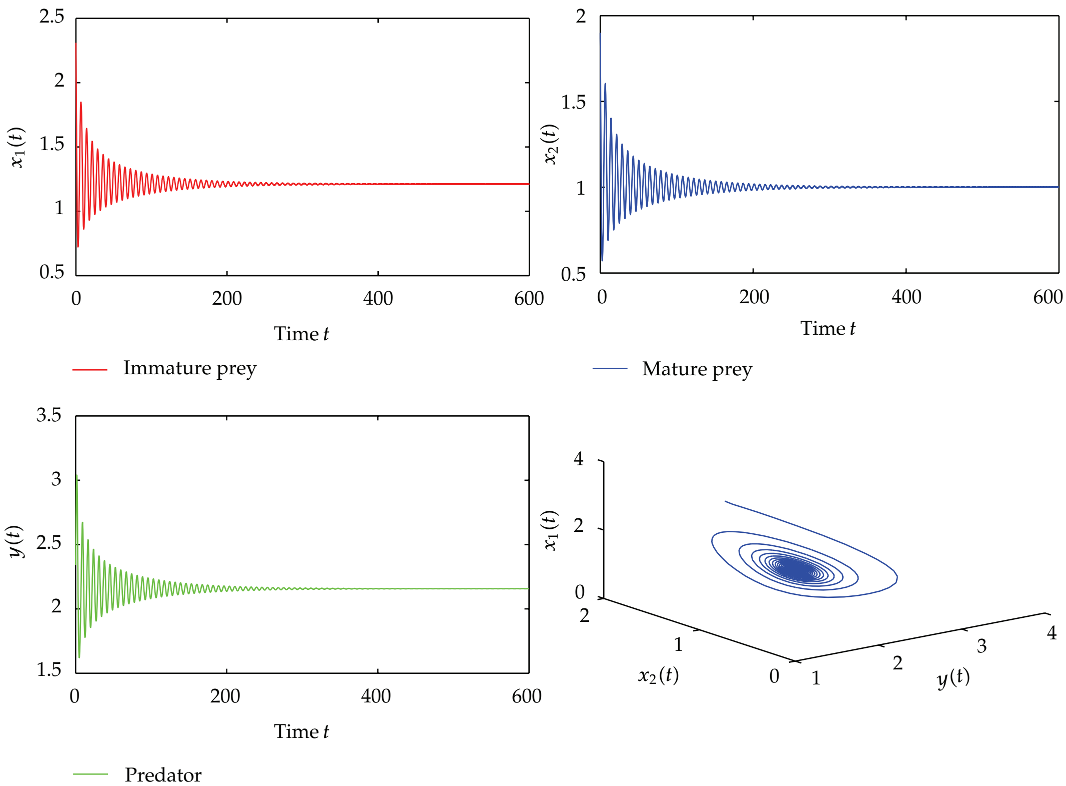

Figure 3: E^0 is locally asymptotically stable for $\tau_2 = 0.4400 < \tau_{20} = 0.5124$ with initial value "2.31; 1.9; 2.34."

Theorem 2.2. *Suppose that the conditions H_{31} and H_{32} hold. The positive equilibrium E^0 of system* (1.4) *is asymptotically stable for $\tau_2 \in [0, \tau_{20})$ and unstable when $\tau_2 > \tau_{20}$. Further, system* (1.4) *undergoes a Hopf bifurcation when $\tau_2 = \tau_{20}$.*

Case 4. ($\tau_1 = \tau_2 = \tau > 0$).

For $\tau_1 = \tau_2 = \tau > 0$, (2.5) can be rewritten in the following form:

$$\begin{aligned} &\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40} \\ &\quad + \left(n_{42}\lambda^2 + n_{41}\lambda + n_{40}\right)e^{-\lambda\tau} \\ &\quad + \left(q_{41} + q_{40}\right)e^{-2\lambda\tau} = 0, \end{aligned} \tag{2.24}$$

where $m_{42} = m_2$, $m_{41} = m_1$, $m_{40} = m_0$, $n_{42} = n_2 + p_2$, $n_{41} = n_1 + p_1$, $n_{40} = n_0 + p_0$, $q_{41} = q_0$, $q_{40} = q_0$.

Multiplying $e^{\lambda\tau}$ on both sides of (2.24), it is obvious to get

$$n_{42}\lambda^2 + n_{41}\lambda + n_{40} + \left(\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40}\right)e^{\lambda\tau} + \left(q_{41}\lambda + q_{40}\right)e^{-\lambda\tau} = 0. \tag{2.25}$$

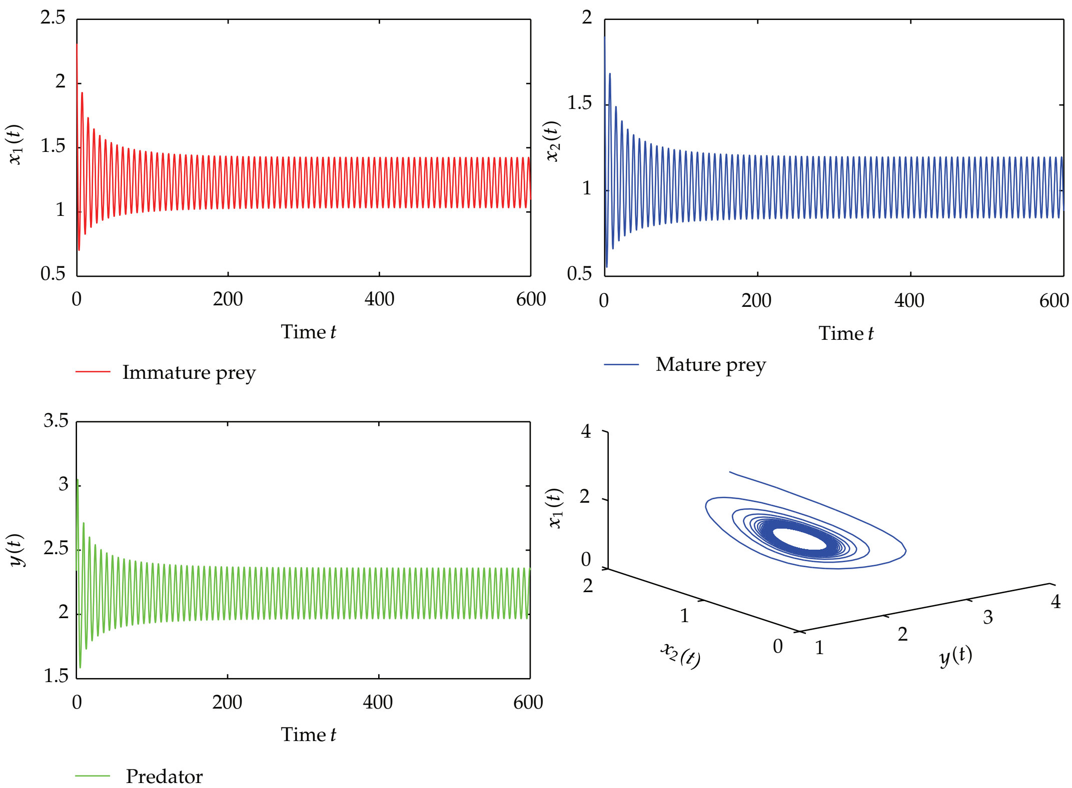

Figure 4: E^0 is unstable for $\tau_2 = 0.5500 > \tau_{20} = 0.5124$ with initial value "2.31; 1.9; 2.34."

Let $\lambda = i\omega(\omega > 0)$ be the root of (2.25). Then, we have

$$\Delta_{41} \sin \tau\omega + \Delta_{42} \cos \tau\omega = \Delta_{45},$$
$$\Delta_{43} \cos \tau\omega + \Delta_{44} \sin \tau\omega = \Delta_{46}, \tag{2.26}$$

where $\Delta_{41} = q_{41}\omega - m_{41}\omega + \omega^3$, $\Delta_{42} = m_{40} - m_{42}\omega^2 + q_{40}$, $\Delta_{43} = q_{41}\omega + m_{41}\omega - \omega^3$, $\Delta_{44} = m_{40} - m_{42}\omega^2 - q_{40}$, $\Delta_{45} = n_{42}\omega^2 - n_{40}$, $\Delta_{46} = -n_{41}\omega$.

It follows that

$$\sin \tau\omega = \frac{A_5\omega^5 + A_3\omega^3 + A_1\omega}{\omega^6 + B_4\omega^4 + B_2\omega^2 + B_0},$$
$$\cos \tau\omega = \frac{A_4\omega^4 + A_2\omega^2 + A_0}{\omega^6 + B_4\omega^4 + B_2\omega^2 + B_0}, \tag{2.27}$$

where $A_0 = (q_{40} - m_{40})n_{40}$, $A_1 = (m_{41} + q_{41})n_{40} - (m_{40} + q_{40})n_{41}$, $A_2 = (m_{40} - q_{40})n_{42} + (q_{41} - m_{41})n_{41} - m_{42}n_{40}$, $A_3 = m_{42}n_{41} - n_{40} - (m_{41} + q_{41})n_{42}$, $A_4 = n_{41} - m_{42}n_{42}$, $A_5 = n_{42}$, $B_0 = m_{40}^2 - q_{40}^2$, $B_2 = m_{41}^2 - q_{41}^2 - 2m_{40}m_{42}$, $B_4 = m_{42}^2 - 2m_{41}$.

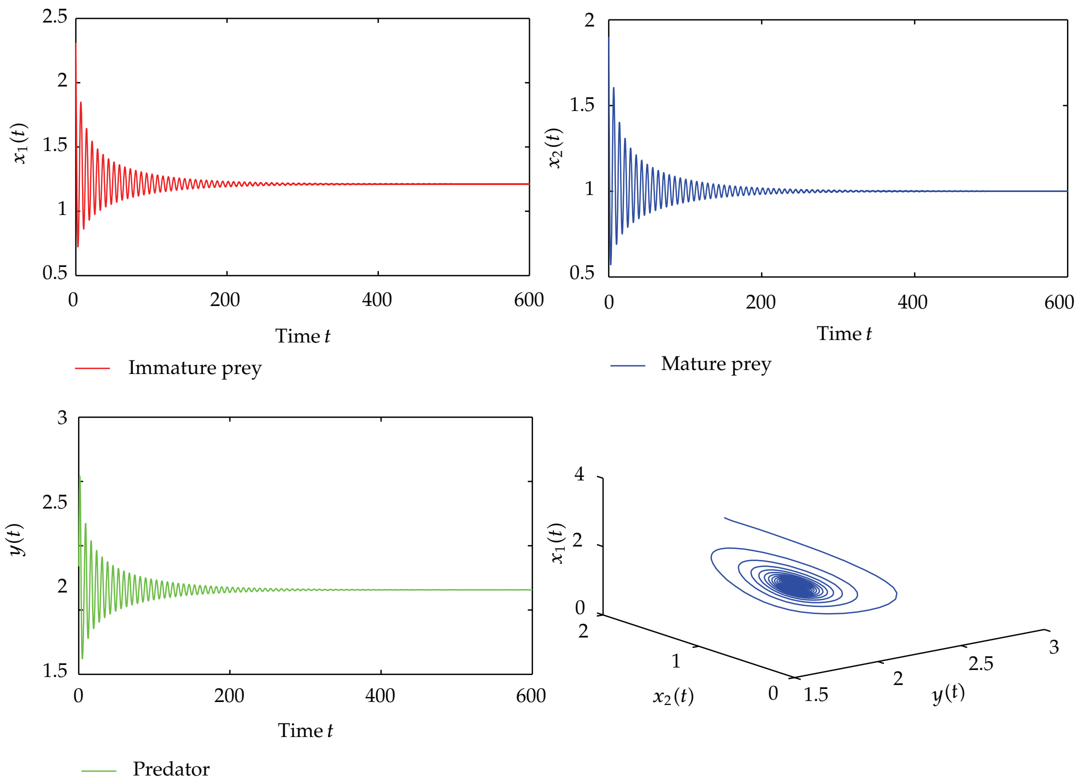

Figure 5: E^0 is locally asymptotically stable for $\tau = 0.3900 < \tau_0 = 0.4178$ with initial value "2.31; 1.9; 2.34."

From (2.27), we can get

$$\omega^{12} + e_{45}\omega^{10} + e_{44}\omega^8 + e_{43}\omega^6 + e_{42}\omega^4 + e_{41}\omega^2 + e_{40} = 0, \tag{2.28}$$

where $e_{45} = 2B_4 - A_5^2$, $e_{44} = B_4^2 + 2B_2 - A_4^2 - 2A_3A_5$, $e_{43} = 2B_0 + 2B_2B_4 - A_3^2 - 2A_1A_5 - 2A_2A_4$, $e_{42} = B_2^2 + 2B_0B_4 - A_2^2 - 2A_0A_4 - 2A_1A_3$, $e_{41} = 2B_0B_2 - A_1^2 - 2A_0A_2$, $e_{40} = B_0^2 - A_0^2$.

Let $\omega^2 = v_3$, then (2.28) becomes

$$v_3^6 + e_{45}v_3^5 + e_{44}v_3^4 + e_{43}v_3^3 + e_{42}v_3^2 + e_{41}v_3 + e_{40} = 0. \tag{2.29}$$

Suppose that (2.29) has at least one positive root, and, without loss of generality, we assume that it has six positive roots which are denoted as v_{31}, v_{32}, v_{33}, v_{34}, v_{35}, and v_{36}. Then, (2.28) has six positive roots $\omega_k = \sqrt{v_{3k}}, k = 1,2,3,4,5,6$.

The corresponding critical value of time delay $\tau_k^{(j)}$ is

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\left\{\frac{A_4\omega_k^4 + A_2\omega_k^2 + A_0}{\omega_k^6 + B_4\omega_k^4 + B_2\omega_{2k}^2 + B_0}\right\} + \frac{2j\pi}{\omega_k}, \quad k = 1,2,3,4,5,6, \quad j = 0,1,2,\ldots. \tag{2.30}$$

Let $\tau_0 = \min\{\tau_k^{(0)}\}$, $k \in \{1,2,3,4,5,6\}$, $\omega_0 = \omega_{k_0}$.

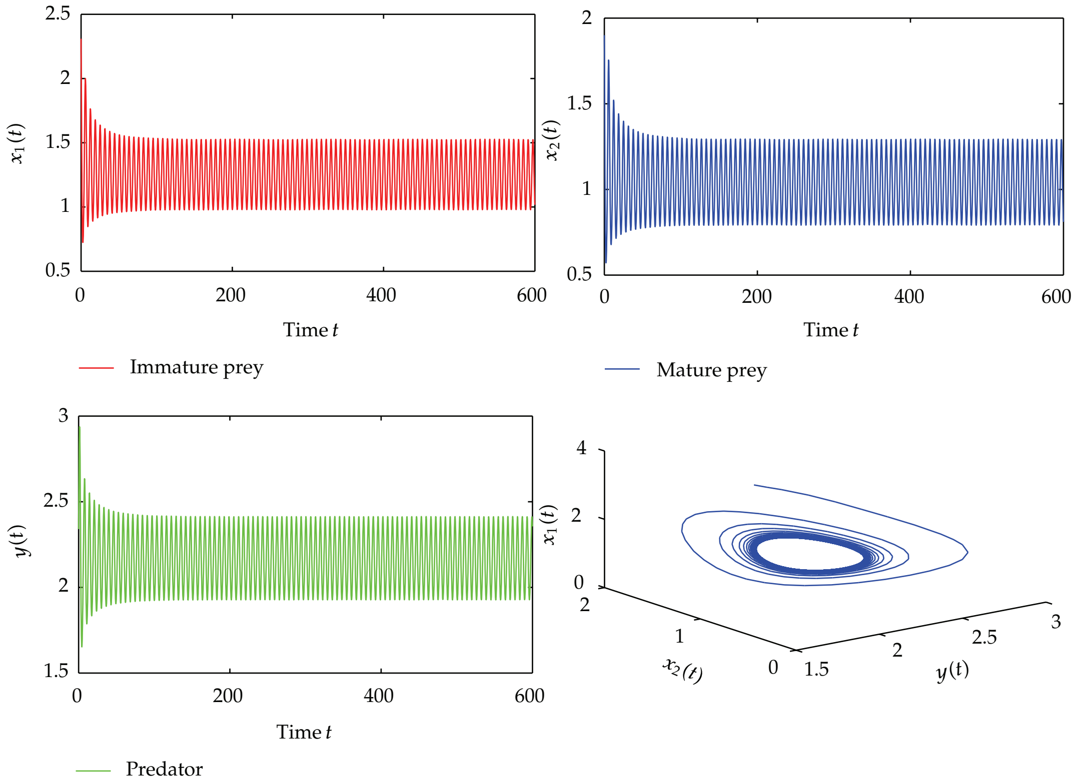

Figure 6: E^0 is unstable for $\tau = 0.4600 > \tau_0 = 0.4178$ with initial value "2.31; 1.9; 2.34."

Next, we verify the transversality condition. Differentiating (2.25) regarding τ and substituting $\tau = \tau_0$, we get

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \operatorname{Re}\left[\frac{A+Bi}{C+Di}\right] = \frac{AC+BD}{C^2+D^2}, \tag{2.31}$$

where

$$\begin{aligned} A &= \left(m_{41} - 3\omega_0^2\right)\cos\tau_0\omega_0 - 2m_{42}\omega_0 \sin\tau_0\omega_0 \\ &\quad + q_{41}\cos\tau_0\omega_0 + n_{41}, \\ B &= \left(m_{41} - 3\omega_0^2\right)\sin\tau_0\omega_0 + 2m_{42}\omega_0\cos\tau_0\omega_0 \\ &\quad - q_{41}\sin\tau_0\omega_0 + 2n_{42}\omega_0, \\ C &= \left(m_{41} - q_{41} - \omega_0^2\right)\omega_0^2\cos\tau_0\omega_0 \\ &\quad + \left(q_{40} + m_{40} - m_{42}\omega_0^2\right)\omega_0\sin\tau_0\omega_0, \end{aligned}$$

$$D = \left(m_{41} + q_{41} - \omega_0^2\right)\omega_0^2 \sin\tau_0\omega_0 + \left(q_{40} - m_{40} + m_{42}\omega_0^2\right)\omega_0 \cos\tau_0\omega_0. \tag{2.32}$$

Thus, if the condition $(H_{42}) : AC + BD \neq 0$ holds, the transversality condition is satisfied.

Theorem 2.3. *Suppose that the conditions H_{41} and H_{42} hold. The positive equilibrium E^0 of system* (1.4) *is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Further, system* (1.4) *undergoes a Hopf bifurcation when $\tau = \tau_0$.*

Case 5. ($\tau_1 > 0$ and $\tau_2 \in [0, \tau_{20})$).

We consider (2.5) with τ_2 in its stable interval, and τ_1 is considered as a parameter. Let $\lambda = i\omega_{1*}(\omega_{1*} > 0)$ be the root of (2.5). Then, we have

$$\begin{aligned} &\Delta_{51} \sin\tau_1\omega_{1*} + \Delta_{52} \cos\tau_1\omega_{1*} = \Delta_{53}, \\ &\Delta_{51} \cos\tau_1\omega_{1*} - \Delta_{52} \sin\tau_1\omega_{1*} = \Delta_{54}, \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} \Delta_{51} &= n_1\omega_{1*} - q_0 \sin\tau_2\omega_{1*} + q_1\omega_{1*} \cos\tau_2\omega_{1*}, \\ \Delta_{52} &= n_0 - n_2\omega_{1*}^2 + q_0 \cos\tau_2\omega_{1*} + q_1\omega_{1*} \sin\tau_2\omega_{1*}, \\ \Delta_{53} &= m_2\omega_{1*}^2 - m_0 - p_1\omega_{1*} \sin\tau_2\omega_{1*} \\ &\quad + \left(p_2\omega_{1*}^2 - p_0\right) \cos\tau_2\omega_{1*}, \\ \Delta_{54} &= \omega_{1*}^3 - m_1\omega_{1*} - p_1\omega_{1*} \cos\tau_2\omega_{1*} \\ &\quad - \left(p_2\omega_{1*}^2 - p_0\right) \sin\tau_2\omega_{1*}. \end{aligned} \tag{2.34}$$

From (2.33), we can get the following transcendental equation:

$$\begin{aligned} &\omega_{1*}^6 + e_{52}\omega_{1*}^4 + e_{51}\omega_{1*}^2 + e_{50} \\ &\quad + \left(c_{54}\omega_{1*}^4 + c_{52}\omega_{1*}^2 + c_{50}\right) \cos\tau_2\omega_{1*} \\ &\quad + \left(c_{55}\omega_{1*}^5 + c_{53}\omega_{1*}^3 + c_{51}\omega_{1*}\right) \sin\tau_2\omega_{1*} = 0, \end{aligned} \tag{2.35}$$

where $e_{50} = m_0^2 + p_0^2 - n_0^2 - q_0^2$, $e_{51} = m_1^2 + p_1^2 - n_1^2 - q_1^2 + 2n_0n_2 - 2m_0m_2 - 2p_0p_2$, $e_{52} = m_2^2 + p_2^2 - n_2^2 - 2m_1$, $c_{50} = 2m_0p_0 - 2n_0q_0$, $c_{51} = 2m_0p_1 + 2n_1q_0 - 2m_1p_0 - 2n_0q_1$, $c_{52} = 2m_1p_1 + 2n_2q_0 - 2m_2p_0 - 2m_0p_2 - 2n_1q_1$, $c_{53} = 2m_1p_2 + 2n_2q_1 + 2p_0 - 2m_2p_1$, $c_{54} = 2m_2p_2 - 2p_1$, $c_{55} = -2p_2$.

In order to give the main results, we suppose that (2.35) has finite positive root. We denote the positive roots of (2.35) as $\omega_{51}, \omega_{52} \cdots \omega_{5k}$. For every ω_{5i} $(i = 1, 2, \ldots, k)$, the corresponding critical value of time delay $\tau_{1_i}^{(j)} | j = 1, 2, \ldots$ is

$$\tau_{1_i}^{(j)} = \frac{1}{\omega_{1*}} \arccos\left\{ \frac{\Delta_{51}\Delta_{54} + \Delta_{52}\Delta_{53}}{\Delta_{51}^2 + \Delta_{52}^2} + 2j\pi \right\}_{\omega_{1*}=\omega_{5i}}, \qquad i = 1, 2, \ldots, k, \quad j = 0, 1, 2, \ldots. \tag{2.36}$$

Let $\tau_{1*}' = \min\{\tau_{1_i}^{(0)} | i = 1, 2, \ldots, k\}$, and ω_{1*}' is the corresponding root of (2.35) with τ_{1*}'.

In the following, we differentiate the two sides of (2.5) with respect to τ_1 to verify the transversality condition.

Taking the derivative of λ with respect to τ_1 in (2.5) and substituting $\tau_1 = \tau_{1*}'$, we get

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau_1=\tau_{1*}}^{-1} = \operatorname{Re}\left[\frac{P_R + P_I i}{Q_R + Q_I i}\right] = \frac{P_R Q_R + P_I Q_I}{Q_R^2 + Q_I^2}, \tag{2.37}$$

where

$$\begin{aligned}
P_R = {} & m_1 - 3(\omega_{1*}')^2 + 2n_2\omega_{1*}' \sin\tau_{1*}'\omega_{1*}' + n_1 \cos\tau_{1*}'\omega_{1*}' \\
& + \sin\tau_2\omega_{1*}' (2p_2\omega_{1*}' - p_1\tau_2\omega_{1*}' - q_1 \sin\tau_{1*}'\omega_{1*}') \\
& + \cos\tau_2\omega_{1*}' \left(p_2\tau_2(\omega_{1*}')^2 + p_1 - p_0\tau_2 + q_1\cos\tau_{1*}'\omega_{1*}'\right), \\
P_I = {} & 2m_2\omega_{1*}' - n_1 \sin\tau_{1*}'\omega_{1*}' + 2n_2\omega_{1*}' \cos\tau_{1*}'\omega_{1*}' \\
& + \sin\tau_2\omega_{1*}' \left(p_0\tau_2 - p_1 - p_2\tau_2(\omega_{1*}')^2 - q_1\cos\tau_{1*}'\omega_{1*}'\right) \\
& + \cos\tau_2\omega_{1*}' (2p_2\omega_{1*}' - p_1\tau_2\omega_{1*}' - q_1 \sin\tau_{1*}'\omega_{1*}'), \\
Q_R = {} & \left(n_0\omega_{1*}' - n_2(\omega_{1*}')^3\right) \sin\tau_{1*}'\omega_{1*}' - n_1(\omega_{1*}')^2 \cos\tau_{1*}'\omega_{1*}' \\
& + \sin\tau_2\omega_{1*}' \left(q_0\omega_{1*}' \cos\tau_{1*}'\omega_{1*}' + q_1(\omega_{1*}')^2 \sin\tau_{1*}'\omega_{1*}'\right) \\
& + \cos\tau_2\omega_{1*}' \left(q_0\omega_{1*}' \sin\tau_{1*}'\omega_{1*}' - q_1(\omega_{1*}')^2 \cos\tau_{1*}'\omega_{1*}'\right), \\
Q_I = {} & \left(n_0\omega_{1*}' - n_2(\omega_{1*}')^3\right) \cos\tau_{1*}'\omega_{1*}' + n_1(\omega_{1*}')^2 \sin\tau_{1*}'\omega_{1*}' \\
& + \sin\tau_2\omega_{1*}' \left(q_1(\omega_{1*}')^2 \cos\tau_{1*}'\omega_{1*}' - q_0\omega_{1*}' \sin\tau_{1*}'\omega_{1*}'\right) \\
& + \cos\tau_2\omega_{1*}' \left(q_1(\omega_{1*}')^2 \sin\tau_{1*}'\omega_{1*}' + q_0\omega_{1*}' \cos\tau_{1*}'\omega_{1*}'\right).
\end{aligned} \tag{2.38}$$

Obviously, if the condition $H_{52} : P_R Q_R + P_I Q_I \neq 0$ holds, the transversality condition is satisfied. Through the above analysis, we have the following results.

Theorem 2.4. *Suppose that the conditions H_{51} and H_{52} hold and $\tau_2 \in [0, \tau_{20})$. The positive equilibrium E^0 of system* (1.4) *is asymptotically stable for $\tau_1 \in [0, \tau'_{1*})$ and unstable when $\tau_1 > \tau'_{1*}$. Further, system* (1.4) *undergoes a Hopf bifurcation when $\tau_1 = \tau'_{1*}$.*

Case 6. ($\tau_2 > 0$ and $\tau_1 \in [0, \tau_{10})$).

We consider (2.5) with τ_1 in its stable interval, and τ_2 is considered as a parameter.

Substitute $\lambda = i\omega_{2*}$ ($\omega_{2*} > 0$) into (2.5). Then, we get

$$\begin{aligned}
&\omega_{2*}^6 + e_{62}\omega_{2*}^4 + e_{61}\omega_{1*}^2 + e_{60} \\
&\quad + \left(c_{64}\omega_{2*}^4 + c_{62}\omega_{2*}^2 + c_{60}\right)\cos\tau_1\omega_{2*} \\
&\quad + \left(c_{65}\omega_{2*}^5 + c_{63}\omega_{2*}^3 + c_{61}\omega_{2*}\right)\sin\tau_1\omega_{2*},
\end{aligned} \tag{2.39}$$

where $e_{60} = m_0^2+n_0^2-p_0^2-q_0^2$, $e_{61} = m_1^2+n_1^2-p_1^2-q_1^2-2n_0n_2-2m_0m_2+2p_0p_2$, $e_{62} = m_2^2+n_2^2-p_2^2-2m_1$, $c_{60} = 2m_0n_0-2p_0q_0$, $c_{61} = 2m_0n_1+2p_1q_0-2m_1n_0-2p_0q_1$, $c_{62} = 2m_1n_1+2p_2q_0-2m_2n_0-2m_0n_2-2p_1q_1$, $c_{63} = 2m_1n_2+2p_2q_1+2n_0-2m_2n_1$, $c_{64} = 2m_2n_2-2n_1$, $c_{65} = -2n_2$.

Similar as in case (5), we give the following assumption. H_{61}: (2.39) has finite positive root.

The positive roots of (2.39) are denoted as $\omega_{61}, \omega_{62}, \ldots, \omega_{6k}$. For every ω_{6i} ($i = 1, 2, \ldots, k$), the corresponding critical value of time delay $\tau_{2_i}^{(j)} \mid j = 1, 2, \ldots$ is

$$\begin{gathered}
\tau_{2_i}^{(j)} = \frac{1}{\omega_{2*}}\arccos\left\{\frac{\Delta_{61}\Delta_{64}+\Delta_{62}\Delta_{63}}{\Delta_{61}^2+\Delta_{62}^2} + 2j\pi\right\}_{\omega_{2*}=\omega_{6i}}, \\
k = 1,2,3,4,5,6, \quad j = 0,1,2,\ldots,
\end{gathered} \tag{2.40}$$

where

$$\Delta_{61} = p_1\omega_{2*} - q_0\sin\tau_1\omega_{2*} + q_1\omega_{2*}\cos\tau_1\omega_{2*}, \tag{2.41}$$

$$\Delta_{62} = p_0 - p_2\omega_{2*}^2 + q_0\cos\tau_1\omega_{2*} + q_1\omega_{2*}\sin\tau_1\omega_{2*}, \tag{2.42}$$

$$\begin{aligned}
\Delta_{63} = {} & m_2\omega_{1*}^2 - m_0 - n_1\omega_{1*}\sin\tau_1\omega_{2*} \\
& + \left(n_2\omega_{2*}^2 - n_0\right)\cos\tau_1\omega_{2*},
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
\Delta_{64} = {} & \omega_{2*}^3 - m_1\omega_{2*} - n_1\omega_{2*}\cos\tau_1\omega_{2*} \\
& - \left(n_2\omega_{2*}^2 - n_0\right)\sin\tau_1\omega_{2*}.
\end{aligned} \tag{2.44}$$

Let $\tau'_{2*} = \min\{\tau_{2_i}^{(0)} \mid i = 1, 2, \ldots, k\}$, and ω'_{2*} is the corresponding root of (2.39) with τ'_{2*}.

Then, we suppose that $H_{62} : [d\,\mathrm{Re}(\lambda)/d\tau_2]_{\lambda=i\omega'_{2*}}$ holds. By the general Hopf bifurcation theorem for FDEs in Hale [26], we have the following results.

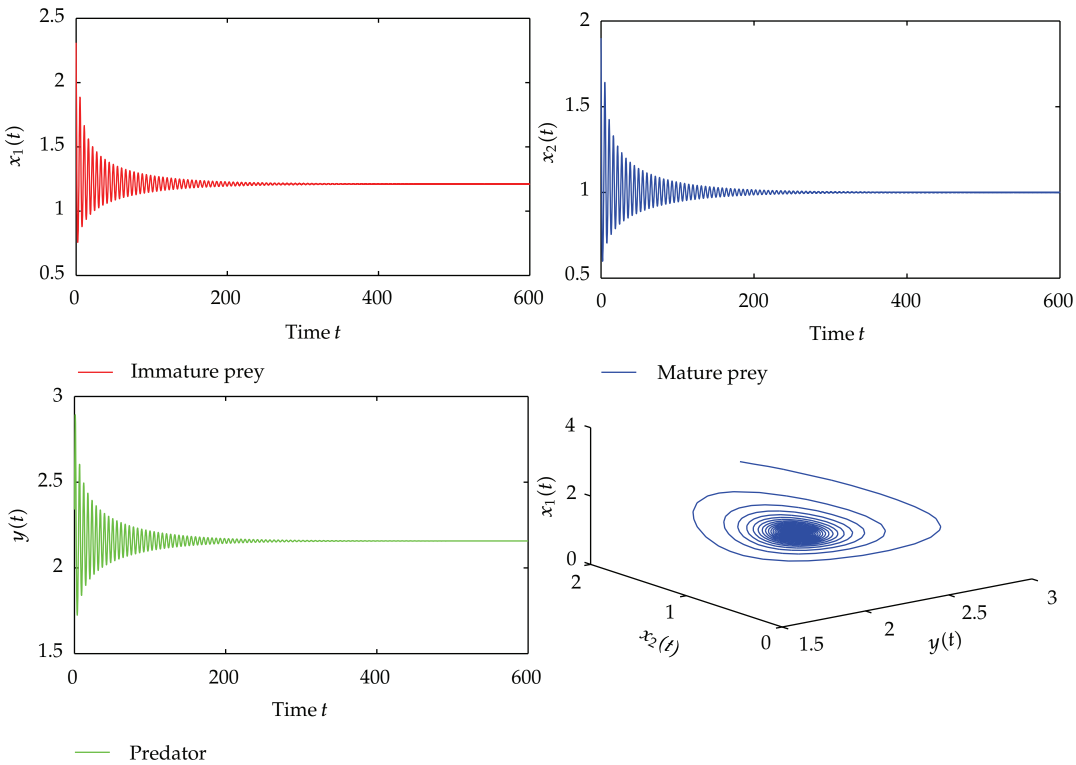

Figure 7: E^0 is locally asymptotically stable for $\tau_1 = 0.6000 < \tau'_{1*} = 0.6125$ and $\tau'_{2*} = 0.25$ with initial value "2.31; 1.9; 2.34."

Theorem 2.5. *Suppose that the conditions H_{61} and H_{62} hold and $\tau_1 \in [0, \tau_{10})$. The positive equilibrium E^0 of system* (1.4) *is asymptotically stable for $\tau_2 \in [0, \tau'_{2*})$ and unstable when $\tau_2 > \tau'_{2*}$. Further, system* (1.4) *undergoes a Hopf bifurcation at when $\tau_2 = \tau'_{2*}$.*

3. Direction and Stability of Bifurcated Periodic Solutions

In Section 2, we have obtained the conditions under which a family of periodic solutions bifurcate from the positive equilibrium of system (1.4) when the delay crosses through the critical value. In this section, we will determine the direction of Hopf bifurcation and stability of bifurcating periodic solutions of system (1.4) with respect to τ_1 for $\tau_2 \in (0, \tau_{20})$ by using the normal form method and center manifold theorem introduced by Hassard et al. [20]. It is considered that system (1.4) undergoes Hopf bifurcation at $\tau_1 = \tau'_{1*}, \tau_2 \in (0, \tau_{20})$. Without loss of generality, we assume that $\tau'_{1*} > \tau'_{2*}$, where $\tau'_{1*} \in (0, \tau_{20})$.

Let $\tau_1 = \tau'_{1*} + \mu, \mu \in R, t = s\tau_1, x_1(s\tau_1) = z_1(s), x_2(s\tau_1) = z_2(s), y(s\tau_1) = z_3(s)$. We still denote s by t. Then, system (1.4) can be transformed into the following system:

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in C = C([-1,0], R^3)$ and $L_\mu : C \to R^3$, $F : R \times C \to R^3$ are given, respectively, by

$$L_\mu \phi = \left(\tau'_{1*}\right)\left(A'\phi(0) + B'\phi\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) + C'\phi(-1)\right),$$
$$F(\mu,\phi) = (\tau'_{1*} + \mu)(F_1, F_2, F_3)^T, \tag{3.2}$$

where

$$\phi(\theta) = \left(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)\right)^T \in C,$$

$$A' = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \gamma_{32} & \gamma_{33} \end{pmatrix},$$

$$C' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F_1 = a_{11}\phi_1^2(0) + \cdots,$$

$$F_2 = a_{21}\phi_2^2(0) + a_{22}\phi_2(0)\phi_2(-1) + a_{23}\phi_2(0)\phi_3(0) + a_{24}\phi_2^3(0) + a_{25}\phi_2^2(0)\phi_3(0) + \cdots,$$

$$F_3 = a_{31}\phi_2^2\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) + a_{32}\phi_2\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\phi_3\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) + a_{33}\phi_2^3\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) + a_{34}\phi_2^2\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\phi_3\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) + \cdots,$$

$$a_{11} = -c, a_{21} = \frac{3ma_1\left(x_2^0\right)^2 y^0 - a_1 y^0}{\left(1 + m\left(x_2^0\right)^2\right)^3},$$

$$a_{22} = -b_1, a_{23} = -\frac{2a_1 x_2^0}{\left(1 + m\left(x_2^0\right)^2\right)^2},$$

$$a_{24} = \frac{4ma_1 x_2^0 y^0\left(1 - m\left(x_2^0\right)^2\right)}{\left(1 + m\left(x_2^0\right)^2\right)^4},$$

$$
\begin{aligned}
a_{25} &= \frac{3ma_1\left(x_2^0\right)^2 - a_1}{\left(1+m\left(x_2^0\right)^2\right)^3},\\
a_{31} &= \frac{a_2y^0 - 3ma_2\left(x_2^0\right)^2 y^0}{\left(1+m\left(x_2^0\right)^2\right)^3},\\
a_{32} &= \frac{2a_2x_2^0}{\left(1+m\left(x_2^0\right)^2\right)^2},\\
a_{33} &= \frac{4ma_2x_2^0y^0\left(m\left(x_2^0\right)^2 - 1\right)}{\left(1+m\left(x_2^0\right)^2\right)^4},\\
a_{34} &= \frac{a_2 - 3ma_2\left(x_2^0\right)^2}{\left(1+m\left(x_2^0\right)^2\right)^3}.
\end{aligned}
\tag{3.3}
$$

Hence, by the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta,\mu)$: $[-1,0] \to R^3$ whose elements are of bounded variation such that

$$
L_\mu \phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \quad \phi \in C\left([-1,0], R^3\right). \tag{3.4}
$$

In fact, we choose

$$
\eta(\theta,\mu) = \begin{cases}
\left(\tau'_{1*+\mu}\right)(A' + B' + C'), & \theta = 0,\\
\left(\tau'_{1*+\mu}\right)(B' + C'), & \theta \in \left[-\dfrac{\tau_{2*}}{\tau_1}, 0\right),\\
\left(\tau'_{1*+\mu}\right)C', & \theta \in \left(-1, -\dfrac{\tau_{2*}}{\tau_1}\right),\\
0, & \theta = -1.
\end{cases}
\tag{3.5}
$$

For $\phi \in C([-1,0], R^3)$, we define

$$
A(\mu)\phi = \begin{cases}
\dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0,\\
\displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0,
\end{cases}
\tag{3.6}
$$

and

$$
R(\mu)\phi = \begin{cases}
0, & -1 \le \theta < 0,\\
F(\mu,\phi), & \theta = 0.
\end{cases}
\tag{3.7}
$$

Then, system (3.1) can be transformed into the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{3.8}$$

For $\varphi \in C'([0,1],(R^3)^*)$, where $(R^3)^*$ is the 3-dimensional space of row vectors, we define the adjoint operator A^* of $A(0)$:

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d^T\eta(\xi,0)\varphi(-\xi), & s = 0. \end{cases} \tag{3.9}$$

For $\phi \in C([-1,0],R^3)$ and $\phi \in C'([0,1],(R^3)^*)$, we define a bilinear inner product:

$$\langle \varphi(s), \phi(\theta) \rangle = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\varphi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.10}$$

where $\eta(\theta) = \eta(\theta,0)$.

By the discussion in Section 2, we know that $\pm\, i\omega'_{1*}\tau'_{1*}$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*.

Suppose that $q(\theta) = (1,q_2,q_3)^T e^{i\omega'_{1*}\tau'_{1*}\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega'_{1*}\tau'_{1*}$ and $q^*(s) = (1/\rho)(1,q_2^*,q_3^*)e^{i\omega'_{1*}\tau'_{1*}s}$ is the eigenvector of corresponding to $-i\omega'_{1*}\tau'_{1*}$. By direction computation, we can get

$$\begin{aligned} q_2 &= \frac{i\omega'_{1*} - \alpha_{11}}{\alpha_{12}}, \\ q_3 &= \frac{\gamma_{32}\left(i\omega'_{1*} - \alpha_{11}\right)e^{-i\tau'_{2*}\omega'_{1*}}}{\alpha_{12}\left(i\omega'_{1*} - \alpha_{33} - \gamma_{33}e^{-i\tau'_{2*}\omega'_{1*}}\right)}, \\ q_2^* &= -\frac{\alpha_{11} + i\omega'_{1*}}{\alpha_{21}}, \\ q_3^* &= \frac{\alpha_{23}\left(\alpha_{11} - i\omega'_{1*}\right)}{\alpha_{12}\left(\alpha_{33} + \gamma_{33}e^{-i\tau'_{2*}\omega'_{1*}} + i\omega'_{1*}\right)}. \end{aligned} \tag{3.11}$$

Then, from (3.10), we can get

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \overline{q}^*(0)q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi \\ &= \frac{1}{\overline{\rho}}\left[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* - \int_{-1}^{0}\left(1,\overline{q}_2^*,\overline{q}_3^*\right)\theta e^{i\tau'_{1*}\omega'_{1*}\theta}d\eta(\theta)\left(1,q_2,q_3\right)^T\right] \\ &= \frac{1}{\overline{\rho}}\left[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + \tau'_{1*}\beta_{22}q_2\overline{q}_2^* e^{-i\tau'_{1*}\omega'_{1*}} + \tau'_{2*}e^{-i\tau'_{2*}\omega'_{1*}}\left(\gamma_{32}q_2 + \gamma_{33}q_3\right)\overline{q}_3^*\right]. \end{aligned} \tag{3.12}$$

Therefore, we can choose

$$\overline{\rho} = 1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + \tau'_{1*}\beta_{22}q_2\overline{q}_2^* e^{-i\tau'_{1*}\omega'_{1*}} + \tau'_{2*}e^{-i\tau'_{2*}\omega'_{1*}}(\gamma_{32}q_2 + \gamma_{33}q_3)\overline{q}_3^*, \tag{3.13}$$

such that $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$.

In the remainder of this section, following the algorithms given in [20] and using similar computation process in [27], we can get the coefficients that can be used to determine the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions:

$$\begin{aligned}
g_{20} = \frac{2\tau'_{1*}}{\overline{\rho}} \Bigg[& a_{11}\left(q^{(1)}(0)\right)^2 \\
& + \overline{q}_2^*\left(a_{21}\left(q^{(2)}(0)\right)^2 + a_{22}q^{(2)}(0)q^{(2)}(-1) + a_{23}q^{(2)}(0)q^{(3)}(0)\right) \\
& + \overline{q}_3^*\left(a_{31}\left(q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right)^2 + a_{32}q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right)\Bigg], \\
g_{11} = \frac{\tau'_{1*}}{\overline{\rho}} \Bigg[& 2a_{11}q^{(1)}(0)\overline{q}^{(1)}(0) + \overline{q}_2^*\Big(2a_{21}q^{(2)}(0)\overline{q}^{(2)}(0) \\
& + a_{22}\left(q^{(2)}(0)\overline{q}^{(2)}(-1) + \overline{q}^{(2)}(0)q^{(2)}(-1)\right) \\
& + a_{23}\left(q^{(2)}(0)\overline{q}^{(3)}(0) + \overline{q}^{(2)}(0)q^{(3)}(0)\right)\Big) \\
& + \overline{q}_3^*\Bigg(2a_{31}q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& + a_{32}\Bigg(q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& + \overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\Bigg)\Bigg)\Bigg], \\
g_{02} = \frac{2\tau'_{1*}}{\overline{\rho}} \Bigg[& a_{11}\left(\overline{q}^{(1)}(0)\right)^2 + \overline{q}_2^*\Big(a_{21}\left(\overline{q}^{(2)}(0)\right)^2 \\
& + a_{22}\overline{q}^{(2)}(0)\overline{q}^{(2)}(-1) + a_{23}\overline{q}^{(2)}(0)\overline{q}^{(3)}(0)\Big) \\
& + \overline{q}_3^*\Bigg(a_{31}\left(\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right)^2 \\
& + a_{32}\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\Bigg)\Bigg],
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
g_{21} = \frac{2\tau'_{1*}}{\overline{\rho}} \Bigg[& a_{11}\left(2W_{11}^{(1)}(0)q^{(1)}(0) + W_{20}^{(1)}(0)\overline{q}^{(1)}(0)\right) \\
& + \overline{q}_2^{*}\Bigg(a_{21}\left(2W_{11}^{(2)}(0)q^{(2)}(0) + W_{20}^{(2)}(0)\overline{q}^{(2)}(0)\right) \\
& \quad + a_{22}\Bigg(W_{11}^{(2)}(0)q^{(2)}(-1) + \frac{1}{2}W_{20}^{(2)}(0)\overline{q}^{(2)}(-1) \\
& \qquad + W_{11}^{(2)}(-1)q^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(-1)\overline{q}^{(2)}(0)\Bigg) \\
& \quad + a_{23}\Bigg(W_{11}^{(2)}(0)q^{(3)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\overline{q}^{(3)}(0) \\
& \qquad + W_{11}^{(3)}(0)q^{(2)}(0) + \frac{1}{2}W_{20}^{(3)}(-1)\overline{q}^{(2)}(0)\Bigg) \\
& \quad + 3a_{24}\left(q^{(2)}(0)\right)^2\overline{q}^{(2)}(0) \\
& \qquad + a_{25}\left(\left(q^{(2)}(0)\right)^2\overline{q}^{(3)}(0) + 2q^{(2)}(0)\overline{q}^{(2)}(0)q^{(3)}(0)\right)\Bigg) \\
& + \overline{q}_3^{*}\Bigg(a_{31}\Bigg(2W_{11}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& \qquad + W_{20}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\Bigg) \\
& \quad + a_{32}\Bigg(W_{11}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& \qquad + \frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& \qquad + W_{11}^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& \qquad + \frac{1}{2}W_{20}^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\Bigg) \\
& \quad + 3a_{33}\left(q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right)^2\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& + a_{34}\Bigg(\left(q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right)^2\overline{q}^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
& \quad + 2q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\Bigg)\Bigg)\Bigg],
\end{aligned}
\tag{3.15}
$$

with

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\tau'_{1*}\omega'_{1*}}e^{i\tau'_{1*}\omega'_{1*}\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau'_{1*}\omega'_{1*}}e^{-i\tau'_{1*}\omega'_{1*}\theta} + E_1 e^{2i\tau'_{1*}\omega'_{1*}\theta},$$
$$W_{11}(\theta) = \frac{g_{11}q(0)}{i\tau'_{1*}\omega'_{1*}}e^{i\tau'_{1*}\omega'_{1*}\theta} - \frac{\overline{g}_{11}\overline{q}(0)}{i\tau'_{1*}\omega'_{1*}}e^{-i\tau'_{1*}\omega'_{1*}\theta} + E_2, \tag{3.16}$$

where E_1, E_2 can be determined by the following equations, respectively,

$$\begin{pmatrix} \alpha'_{11} & -\alpha_{12} & 0 \\ -\alpha_{21} & \alpha'_{22} & -\alpha_{23} \\ 0 & -\gamma_{32}e^{-2i\tau'_{2*}\omega'_{1*}} & \alpha'_{33} \end{pmatrix} E_1 = 2\begin{pmatrix} \Delta_1^{(1)} \\ \Delta_1^{(2)} \\ \Delta_1^{(3)} \end{pmatrix},$$
$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22}+\beta_{22} & \alpha_{23} \\ 0 & \gamma_{32} & \alpha_{33}+\gamma_{33} \end{pmatrix} E_2 = -\begin{pmatrix} \Delta_2^{(1)} \\ \Delta_2^{(2)} \\ \Delta_2^{(3)} \end{pmatrix}, \tag{3.17}$$

with

$$\begin{aligned}
\Delta_1^{(1)} &= a_{11}\left(q^{(1)}(0)\right)^2, \\
\Delta_1^{(2)} &= a_{21}\left(q^{(2)}(0)\right)^2 + a_{22}q^{(2)}(0)q^{(2)}(-1) + a_{23}q^{(2)}(0)q^{(3)}(0), \\
\Delta_1^{(3)} &= a_{31}\left(q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right)^2 + a_{32}q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right), \\
\Delta_2^{(1)} &= 2a_{11}q^{(1)}(0)\overline{q}^{(1)}(0), \\
\Delta_2^{(2)} &= 2a_{21}q^{(2)}(0)\overline{q}^{(2)}(0) \\
&\quad + a_{22}\left(q^{(2)}(0)\overline{q}^{(2)}(-1) + \overline{q}^{(2)}(0)q^{(2)}(-1)\right) \\
&\quad + a_{23}\left(q^{(2)}(0)\overline{q}^{(3)}(0) + \overline{q}^{(2)}(0)q^{(3)}(0)\right), \\
\Delta_2^{(3)} &= 2a_{31}q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right) \\
&\quad + a_{32}\left(q^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\overline{q}^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right. \\
&\quad \left. +\overline{q}^{(2)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)q^{(3)}\left(-\frac{\tau'_{2*}}{\tau'_{1*}}\right)\right),
\end{aligned} \tag{3.18}$$

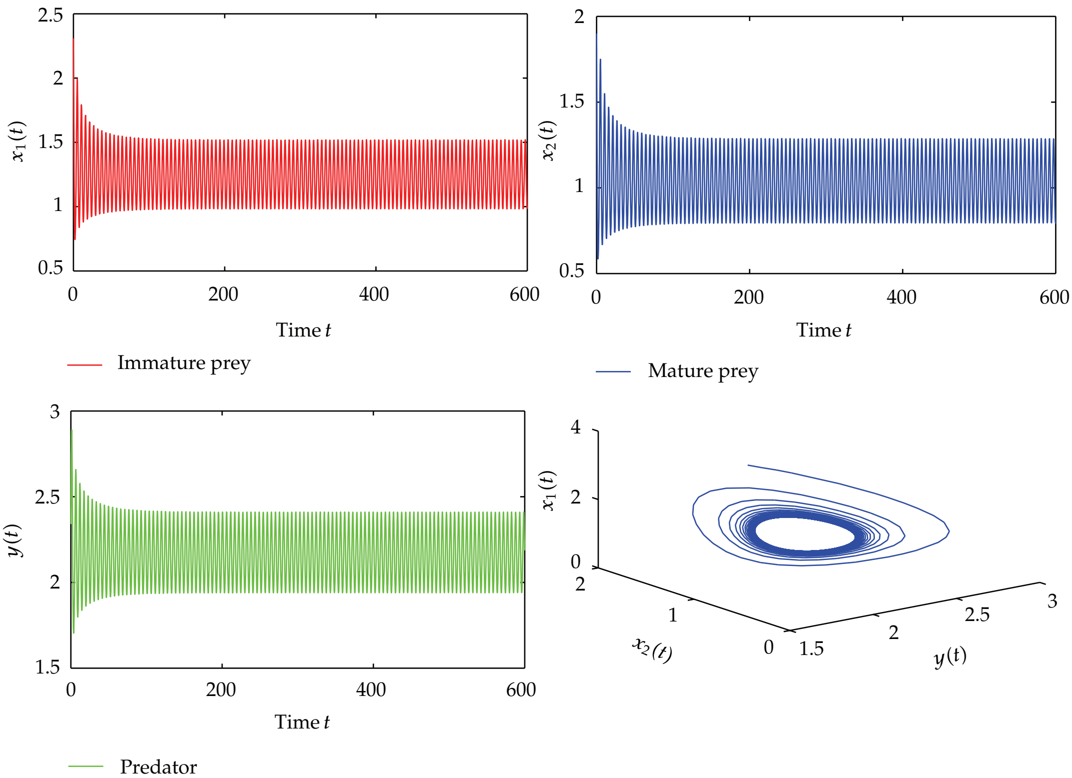

Figure 8: E^0 is unstable for $\tau_1 = 0.6800 > \tau'_{1*} = 0.6125$ and $\tau'_{2*} = 0.25$ with initial value "2.31; 1.9; 2.34."

$$\begin{aligned} \alpha'_{11} &= 2i\omega'_{1*} - \alpha_{11}, \\ \alpha'_{22} &= 2i\omega'_{1*} - \alpha_{22} - \beta_{22}e^{-2i\tau'_{1*}\omega'_{1*}}, \\ \alpha'_{33} &= 2i\omega'_{1*} - \alpha_{33} - \gamma_{33}e^{-2i\tau'_{2*}\omega'_{1*}}. \end{aligned} \tag{3.19}$$

Then, we can calculate the following values:

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau'_{1*}\omega'_{1*}}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \delta &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau'_{1*})\}}, \\ \sigma &= 2\operatorname{Re}\{C_1(0)\}, \\ T &= -\frac{\operatorname{Im}\{C_1(0)\} + \delta\operatorname{Im}\{\lambda'(\tau'_{1*})\}}{\tau'_{1*}\omega'_{1*}}. \end{aligned} \tag{3.20}$$

Based on the above discussion, we can obtain the following results.

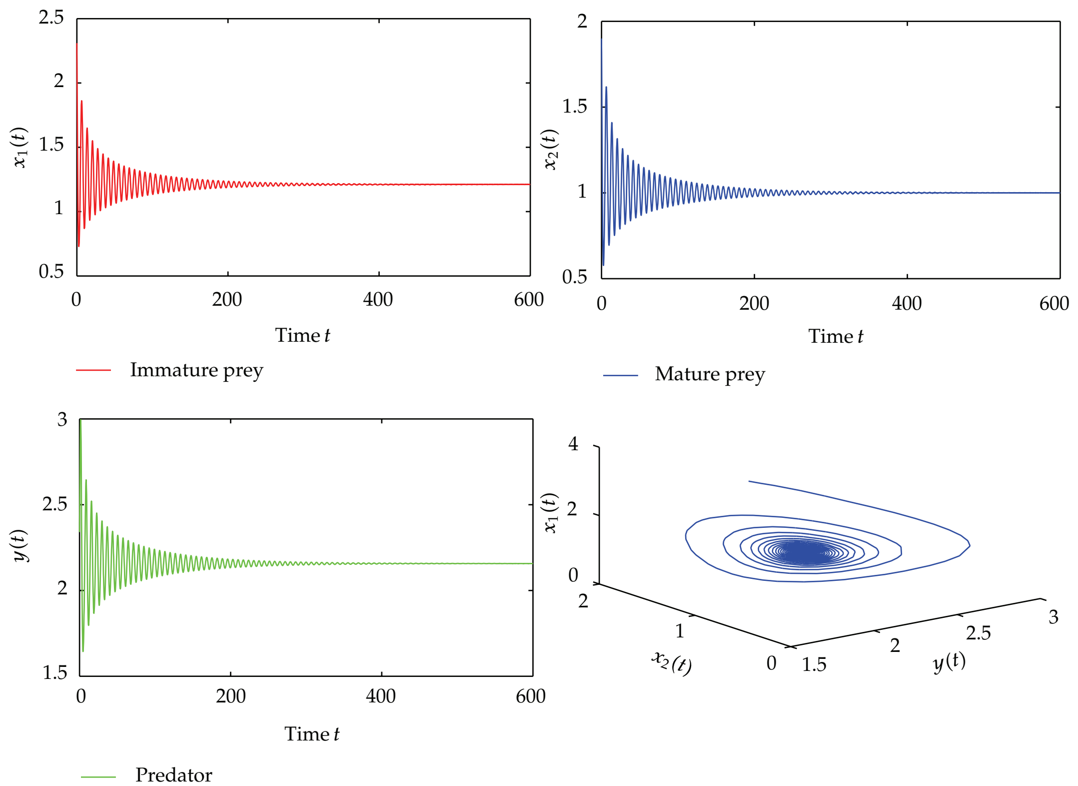

Figure 9: E^0 is locally asymptotically stable for $\tau_2 = 0.4500 < \tau'_{2*} = 0.5094$ and $\tau'_{1*} = 0.15$ with initial value "2.31; 1.9; 2.34."

Theorem 3.1. *From* (3.20) *one has*

(i) *the direction of the Hopf bifurcation is determined by the sign of* δ*: if* $\delta > 0(\delta < 0)$, *then the Hopf bifurcation is supercritical (subcritical);*

(ii) *the stability of bifurcating periodic solutions is determined by the sign of* σ*: if* $\sigma < 0(\sigma > 0)$, *the bifurcating periodic solutions are stable (unstable);*

(iii) *the period of the bifurcating periodic solution is determined by the sign of* T*: if* $T > 0(T < 0)$, *the bifurcating periodic solution increases (decreases).*

4. Numerical Example

In this section, we give some numerical simulations to verify the theoretical analysis in Sections 2 and 3. Let $a = 8$, $a_1 = 4.25$, $a_2 = 3$, $b = 5$, $b_1 = 1$, $c = 0.5$, $m = 2$, $r = 1$, $r_1 = 1$, and $r_2 = 2$. Then, we have the following particular case of system (1.4):

$$\frac{dx_1}{dt} = 8x_2(t) - x_1(t) - 5x_1(t) - 0.5x_1^2(t),$$

$$\frac{dx_2}{dt} = 5x_1(t) - 2x_2(t) - x_2(t)x_2(t-\tau_1) - \frac{4.25x_2^2(t)y(t)}{1+2x_2^2(t)},$$
$$\frac{dy}{dt} = \frac{3x_2^2(t-\tau_2)y(t-\tau_2)}{1+2x_2^2(t-\tau_2)} - y(t). \tag{4.1}$$

It is not difficult to verify that $a_2 > mr$, $bx_1^0 > (r_2 - b_1x_2^0)x_2^0$, namely, the conditions $H_1$and H_2 hold. Therefore, system (4.1) has at least a positive equilibrium. By means of Matlab, we can get that the positive equilibrium of (4.1) is $E_*^0(1.2111, 1.0000, 2.1568)$.

For $\tau_1 > 0$, $\tau_2 = 0$, we can get $\omega_{10} = 1.3881$, $\tau_{10} = 0.9032$. From Theorem 2.2, we know that the positive equilibrium E_*^0 is asymptotically stable when $\tau_1 \in [0, \tau_{10})$. The corresponding waveform and the phase plot are illustrated by Figure 1. When the time delay τ_1 passes through the critical value τ_{10}, the positive equilibrium E_*^0 will lose its stability and a Hopf bifurcation occurs, and a family of periodic solutions bifurcate from the positive equilibrium E_*^0. This property is illustrated by the numerical simulation in Figure 2. Similarly, we have $\omega_{20} = 0.8497$, $\tau_{20} = 0.5124$, when $\tau_2 > 0$, $\tau_1 = 0$. The corresponding waveform and the phase plots are shown in Figures 3 and 4.

For $\tau_1 = \tau_2 = \tau > 0$, we can obtain $\omega_0 = 1.0000$, $\tau_0 = 0.4178$. From Theorem 2.2, we know that, when the time delay τ increases from zero to τ_0, the positive equilibrium E_*^0 is asymptotically stable. Once the time delay τ passes through the critical value τ_0, the positive equilibrium E_*^0 will lose its stability and a Hopf bifurcation occurs. This property is illustrated by the numerical simulation in Figures 5 and 6.

For $\tau_1 > 0$ and $\tau'_{2*} = 0.25 \in [0, \tau_{20})$, we have $\omega'_{1*} = 0.8020$, $\tau'_{1*} = 0.6125$. According to Theorem 2.2, the positive equilibrium E_*^0 is asymptotically stable when $\tau_1 \in [0, \tau'_{1*})$ and unstable when $\tau_1 > \tau'_{1*}$, which can be depicted by the numerical simulation in Figures 7 and 8. In addition, from (3.20), we can obtain $C_1(0) = -1.1949 + 3.2464i$, $\delta = -23.4294$, $\sigma = -2.3898$, $T = 2.4949$. Thus, from Theorem 2.3, we know that the Hopf bifurcation with respect to τ_1 with $\tau'_{2*} = 0.25 \in [0, \tau_{20})$ is subcritical, the bifurcating periodic solutions are stable and increase. Similarly, we have $\omega'_{2*} = 0.8872$, $\tau'_{2*} = 0.5094$, for $\tau_2 > 0$ and $\tau'_{1*} = 0.15 \in [0, \tau_{10})$. The corresponding waveform and the phase plots are shown in Figures 9 and 10.

5. Conclusions

In this paper, a delayed predator-prey system with Holling type III functional response and stage structure for the prey population is investigated. Compared with literature [14], we consider not only the time delay due to the gestation of the predator but also the negative feedback of the mature prey density and the intraspecific competition of the immature prey population. F. Li and H. W. Li [14] has obtained that the species in system (4.1) with only the time delay due to the gestation of the predator could coexist. However, we get that the species could also coexist with some available time delays of the mature prey and the predator. This is valuable from the view of ecology.

The sufficient conditions for the local stability of the positive equilibrium and the existence of local Hopf bifurcation for the possible combinations of two delays are obtained. The main results are given in Theorems 2.1–2.5. By computation, we find that the time delay due to the gestation of the predator is marked because the critical value of τ_2 is smaller than that of τ_1 when we only consider them, respectively. Furthermore, the explicit formulae

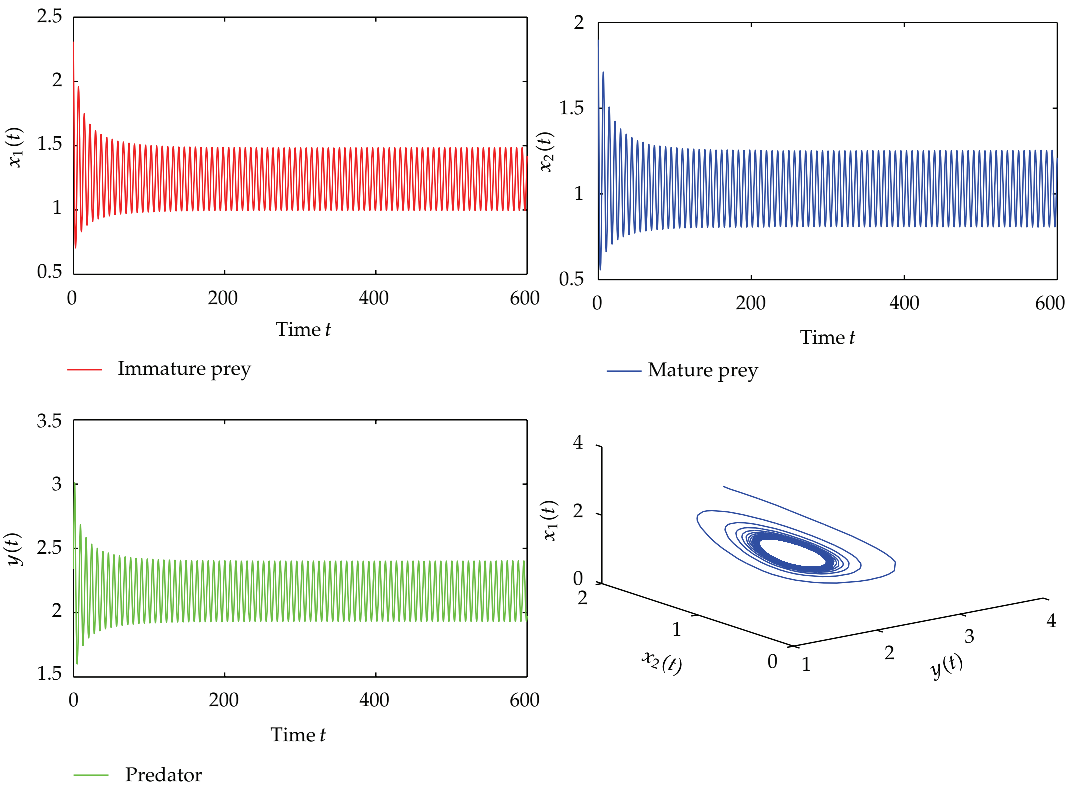

Figure 10: E^0 is unstable for $\tau_2 = 0.5700 > \tau'_{2*} = 0.5094$ and $\tau'_{1*} = 0.15$ with initial value "2.31; 1.9; 2.34."

which determines the direction of the bifurcation and the stability of the bifurcating periodic solutions is established when $\tau > 0$ and $\tau_2 \in [0, \tau_{20})$ by using the normal form theory and center manifold theorem. The main results are given in Theorem 2.3. Finally, numerical simulations are carried out to support the obtained theoretical results.

Acknowledgments

The authors are grateful to the anonymous reviewers for their helpful comments and valuable suggestions on improving the paper. This work was supported by the National Natural Science Foundation of China (61273070), Doctor Candidate Foundation of Jiangnan University (JUDCF12030) and Anhui Provincial Natural Science Foundation under Grant no. 1208085QA11.

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