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Research Article

On the Stability of an m-Variables Functional Equation in Random Normed Spaces via Fixed Point Method

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At first we find the solution of the functional equation $D_f(x_1, \dots, x_m) := \sum_{k=2}^{m}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{m-k+1}=i_{m-k}+1}^{m}) f(\sum_{i=1, i\neq i_1,\dots,i_{m-k+1}}^{m} x_i - \sum_{r=1}^{m-k+1} x_{i_r}) + f(\sum_{i=1}^{m} x_i) - 2^{m-1} f(x_1) = 0$, where $m \geq 2$ is an integer number. Then, we obtain the generalized Hyers-Ulam-Rassias stability in random normed spaces via the fixed point method for the above functional equation.

1. Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: "when is it true that a function that approximately satisfies a functional equation must be close to an exact solution of the equation?"

If the problem accepts a solution, we say the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively solved by Hyers [2]. The result of Hyers was generalized by Aoki [3] for approximate additive function and by Rassias [4] for approximate linear functions by allowing the difference Cauchy equation $\|f(x+y) - f(x) - f(y)\|$ to be controlled by $\varepsilon(\|x\|^p + \|y\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers, and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the Hyers-Ulam-Rassias stability. In 1994, a generalization of Rassias theorem was obtained by Găvruţa [5], who replaced $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$ (see also [6–24]).

In the sequel we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [25–29]. Throughout this paper, let Δ^+ be the space of distribution functions, that is,

$$\begin{aligned} \Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \longrightarrow [0,1] : F \text{ is left-continuous,} \\ \text{nondecreasing on } \mathbb{R}, F(0) = 0 \text{ and } F(+\infty) = 1\} \end{aligned} \tag{1.1}$$

and the subset $D^+ \subseteq \Delta^+$ is the set

$$D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}, \tag{1.2}$$

where $l^- f(x)$ denotes the left limit of function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{1.3}$$

Definition 1.1 (see [28]). A mapping $T : [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:

(a) T is commutative and associative,

(b) T is continuous,

(c) $T(a, 1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$, and $T_L(a,b) = \max(a+b-1, 0)$ (the Łukasiewicz t-norm).

Recall (see [30, 31]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by

$$T_{i=1}^n x_i = \begin{cases} x_1 & \text{if } n = 1, \\ T\left(T_{i=1}^{n-1} x_i, x_n\right) & \text{if } n \ge 2. \end{cases} \tag{1.4}$$

$T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$.

It is known [31] that for the Łukasiewicz t-norm the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty. \tag{1.5}$$

Definition 1.2 (see [29]). A random normed space (briefly, RN space) is a triple (X, Λ, T), where X is a vector space, T is a continuous t-norm, and Λ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(RN2) $\Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$,

(RN3) $\Lambda_{x+y}(t+s) \geq T(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 1.3. Let (X, Λ, T) be an RN space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\Lambda_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\Lambda_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) An RN space (X, Λ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X. A complete RN space is said to be a random Banach space.

Theorem 1.4 (see [28]). *If (X, Λ, T) is an RN space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \Lambda_{x_n}(t) = \Lambda_x(t)$ *almost everywhere.*

Theorem 1.5 (see [32, 33]). *Let (S, d) be a complete generalized metric space, and let $J : S \to S$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for each given element $x \in S$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.6}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all* $n \geq n_0$,

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J,*

(3) *y^* is the unique fixed point of J in the set $\Omega = \{y \in S \mid d(J^{n_0} x, y) < \infty\}$,*

(4) $d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ *for all* $y \in \Omega$.

The theory of random normed spaces (RN spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The notion of an RN space corresponds to the situations when we do not know exactly the norm of point and we know only probabilities of possible values of this norm. The RN spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces (RN spaces) and fuzzy normed spaces has been recently studied in, Alsina [34], Mirmostafaee et al. [35–38], Miheţ and Radu [26, 27, 39, 40], Miheţ et al. [41, 42], Baktash et al. [43] and Saadati et al. [44].

In this paper, we consider the m-dimensional additive functional equation

$$\sum_{k=2}^{m}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{m-k+1}=i_{m-k}+1}^{m}\right)f\left(\sum_{i=1,i\neq i_1,\ldots,i_{m-k+1}}^{m}x_i-\sum_{r=1}^{m-k+1}x_{i_r}\right)+f\left(\sum_{i=1}^{m}x_i\right)=2^{m-1}f(x_1), \tag{1.7}$$

where $m\geq 2$ is an integer number. It is easy to see that the function $f(x)=ax$ is a solution of the functional equation (1.7).

As a special case, if $m=2$ in (1.7), then the functional equation (1.7) reduces to

$$f(x_1+x_2)+f(x_1-x_2)=2f(x_1). \tag{1.8}$$

Also by putting $m=3$ in (1.7), we obtain

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3}f\left(\sum_{i=1,i\neq i_1,i_2}^{3}x_i-\sum_{r=1}^{2}x_{i_r}\right)+\sum_{i_1=2}^{3}f\left(\sum_{i=1,i\neq i_1}^{3}x_i-x_{i_1}\right)+f\left(\sum_{i=1}^{3}x_i\right)=2^2f(x_1), \tag{1.9}$$

that is,

$$f(x_1-x_2-x_3)+f(x_1-x_2+x_3)+f(x_1+x_2-x_3)+f(x_1+x_2+x_3)=4f(x_1). \tag{1.10}$$

The main purpose of this paper is to prove the stability of (1.7) in random normed spaces via the fixed point method.

2. Results in RN spaces via Fixed Point Method

Lemma 2.1. *Let X and Y be real vector spaces. A function $f:X\to Y$ with $f(0)=0$ satisfies* (1.7) *if and only if $f:X\to Y$ is additive.*

Proof. Let f satisfy the functional equation (1.7). Hence, according to (1.7), we get

$$\begin{aligned}&\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{m-1}=i_{m-2}+1}^{m}f\left(\sum_{i=1,i\neq i_1,\ldots,i_{m-1}}^{m}x_i-\sum_{r=1}^{m-1}x_{i_r}\right)\\&\quad+\sum_{i_1=2}^{3}\sum_{i_2=i_1+1}^{4}\cdots\sum_{i_{m-2}=i_{m-3}+1}^{m}f\left(\sum_{i=1,i\neq i_1,\ldots,i_{m-2}}^{m}x_i-\sum_{r=1}^{m-2}x_{i_r}\right)+\cdots+\sum_{i_1=2}^{m}f\left(\sum_{i=1,i\neq i_1}^{m}x_i-x_{i_1}\right)\\&\quad+f\left(\sum_{i=1}^{m}x_i\right)=2^{m-1}f(x_1)\end{aligned} \tag{2.1}$$

for all $x_1, \ldots, x_m \in X$. Setting $x_i = 0$ $(i = 2, \ldots, m-1)$ in (2.1), we have

$$\begin{aligned} f(x_1 - x_m) &+ \left(\binom{m-2}{1} f(x_1 - x_m) + \binom{m-2}{m-2} f(x_1 + x_m) \right) \\ &+ \cdots + \left(\binom{m-2}{m-3} f(x_1 - x_m) + \binom{m-2}{2} f(x_1 + x_m) \right) \\ &+ \left(\binom{m-2}{m-2} f(x_1 - x_m) + \binom{m-2}{1} f(x_1 + x_m) \right) + f(x_1 + x_m) = 2^{m-1} f(x_1), \end{aligned} \tag{2.2}$$

that is,

$$\left(1 + \sum_{\ell=1}^{m-2} \binom{m-2}{\ell} \right) \left(f(x_1 + x_m) + f(x_1 - x_m) \right) = 2^{m-1} f(x_1) \tag{2.3}$$

for all $x_1, x_m \in X$. On the other hand, we have the relation

$$1 + \sum_{\ell=1}^{m-j} \binom{m-j}{\ell} = \sum_{\ell=0}^{m-j} \binom{m-j}{\ell} = 2^{m-j} \tag{2.4}$$

for all $m > J$. Hence, we obtain from (2.3) and (2.4) that

$$f(x_1 + x_m) + f(x_1 - x_m) = 2f(x_1) \tag{2.5}$$

for all $x_1, x_m \in X$. Setting $x_m = x_1$ in (2.5) we get $f(2x_1) = 2f(x_1)$ for all $x_1 \in X$. Replacing x_1 and x_m by $x_1 + x_m$ and $x_1 - x_m$ in (2.5), respectively, and then using $f(2x_1) = 2f(x_1)$, we obtain that

$$f(x_1 + x_m) = f(x_1) + f(x_m) \tag{2.6}$$

for all $x_1, x_m \in X$, which implies that f is additive.

Conversely, suppose that f is additive, and thus f satisfies the equation $f(x_1 + x_2) = f(x_1) + f(x_2)$. Hence we have $f(0) = 0$ and $f(2x_1) = 2f(x_1)$ for all $x_1 \in X$. Replacing x_1 and x_2 by $x_1 + x_2$ and $x_1 - x_2$ in the additive equation and then using $f(2x_1) = 2f(x_1)$ lead to

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) \tag{2.7}$$

for all $x_1, x_2 \in X$.

Now, we are going to prove our assumption by induction on $m \geq 2$. It holds for $m = 2$; see (2.7). Assume that (1.7) holds for the case, where $m = p$; that is, we have

$$\sum_{k=2}^{p} \left(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{p-k+1}=i_{p-k}+1}^{p} \right) f\left(\sum_{i=1, i \neq i_1, \ldots, i_{p-k+1}}^{p} x_i - \sum_{r=1}^{p-k+1} x_{i_r} \right) + f\left(\sum_{i=1}^{p} x_i \right) = 2^{p-1} f(x_1) \tag{2.8}$$

for all $x_1, \ldots, x_p \in X$. Replacing x_1 by $x_1 + x_{p+1}$ in (2.8), we obtain

$$\sum_{k=2}^{p}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{p-k+1}=i_{p-k}+1}^{p}\right)f\left(x_1 + x_{p+1} + \sum_{i=2, i\neq i_1,\ldots,i_{p-k+1}}^{p} x_i - \sum_{r=1}^{p-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{p+1} x_i\right) = 2^{p-1} f\left(x_1 + x_{p+1}\right) \tag{2.9}$$

for all $x_1, \ldots, x_p \in X$. Replacing x_{p+1} by $-x_{p+1}$ in (2.9), we obtain

$$\sum_{k=2}^{p}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{p-k+1}=i_{p-k}+1}^{p}\right)f\left(x_1 - x_{p+1} + \sum_{i=2, i\neq i_1,\ldots,i_{p-k+1}}^{p} x_i - \sum_{r=1}^{p-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{p} x_i - x_{p+1}\right) = 2^{p-1} f\left(x_1 - x_{p+1}\right) \tag{2.10}$$

for all $x_1, \ldots, x_{p+1} \in X$. Adding (2.9) to (2.10), one gets

$$\sum_{k=2}^{p+1}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{p-k+2}=i_{p-k+1}+1}^{p+1}\right)f\left(\sum_{i=1, i\neq i_1,\ldots,i_{p-k+2}}^{p+1} x_i - \sum_{r=1}^{p-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{p+1} x_i\right) = 2^{p-1}\left(f\left(x_1 + x_{p+1}\right) + f\left(x_1 - x_{p+1}\right)\right) \tag{2.11}$$

for all $x_1, \ldots, x_{p+1} \in X$. Therefore, it follows from (2.7) and (2.11) that (1.7) holds for $m = p+1$. This completes the proof of the theorem. □

From now on, let X be a linear space and (Y, Λ, T_M) a complete RN space. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$D_f(x_1, \ldots, x_m) = \sum_{k=2}^{m}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{m-k+1}=i_{m-k}+1}^{m}\right)f\left(\sum_{i=1, i\neq i_1,\ldots,i_{m-k+1}}^{m} x_i - \sum_{r=1}^{m-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{m} x_i\right) - 2^{m-1} f(x_1) \tag{2.12}$$

for all $x_1, \ldots, x_m \in X$, where $m \geq 2$ is an integer number.

Theorem 2.2. *Let* $\Phi : \underbrace{X \times X \times \cdots \times X}_{m\text{-terms}} \to D^+$ *be a function* ($\Phi(x_1, \ldots, x_m)$ *is denoted by* $\Phi_{x_1,\ldots,x_m}$) *such that, for some* $0 < \alpha < 2$,

$$\Phi_{2x_1,\ldots,2x_m}(\alpha t) \geq \Phi_{x_1,\ldots,x_m}(t) \tag{2.13}$$

for all $x_1, \ldots, x_m \in X$ *and all* $t > 0$*. Suppose that a function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\Lambda_{D_f(x_1,\ldots,x_m)}(t) \geq \Phi_{x_1,\ldots,x_m}(t) \tag{2.14}$$

for all $x_1, \ldots, x_m \in X$ *and all* $t > 0$*. Then, there exists a unique additive function* $A : X \to Y$ *such that*

$$\Lambda_{f(x)-A(x)}(t) \geq \Phi_{x,x,\underbrace{0,\ldots,0}_{m-2}}\left(2^{m-2}(2-\alpha)t\right) \tag{2.15}$$

for all $x \in X$ *and all* $t > 0$*.*

Proof. Letting $x_i = 0$ $(i = 3, \ldots, m)$ in (2.14), we get

$$\Lambda_{(1+\sum_{\ell=1}^{m-2}\binom{m-2}{\ell})(f(x_1+x_2)+f(x_1-x_2))-2^{m-1}f(x_1)}(t) \geq \Phi_{x_1,x_2,0,\ldots,0}(t) \tag{2.16}$$

for all $x_1, x_2 \in X$ and all $t > 0$. Setting $x_1 = x_2 = x$ in (2.16), we obtain from (2.4) and $f(0) = 0$ that

$$\Lambda_{2^{m-2}f(2x)-2^{m-1}f(x)}(t) \geq \Phi_{x,x,0,\ldots,0}(t) \tag{2.17}$$

for all $x \in X$ and all $t > 0$, or

$$\Lambda_{f(2x)/2-f(x)}(t) \geq \Phi_{x,x,0,\ldots,0}\left(2^{m-1}t\right) \tag{2.18}$$

for all $x \in X$ and all $t > 0$. Let S be the set of all functions $h : X \to Y$ with $h(0) = 0$ and introduce a generalized metric on S as follows:

$$d(h,k) = \inf\left\{u \in \mathbb{R}^+ : \Lambda_{h(x)-k(x)}(ut) \geq \Phi_{x,x,0,\ldots,0}(t),\ \forall x \in X,\ \forall t > 0\right\}, \tag{2.19}$$

where, as usual, $\inf\emptyset = +\infty$. It is easy to show that (S, d) is a generalized complete metric space [26, 45].

Now we consider the function $J : S \to S$ defined by

$$Jh(x) := \frac{h(2x)}{2} \tag{2.20}$$

for all $h \in S$ and $x \in X$.

Now let $g, f \in S$ such that $d(f, g) < \varepsilon$. Then,

$$\Lambda_{Jg(x)-Jf(x)}\left(\frac{\alpha\varepsilon}{2}t\right) = \Lambda_{g(2x)-f(2x)}(\alpha\varepsilon t) \geq \Phi_{2x,2x,0,\ldots,0}(\alpha t) \geq \Phi_{x,x,0,\ldots,0}(t), \tag{2.21}$$

that is, if $d(f,g) < \varepsilon$, we have $d(Jf, Jg) < (\alpha/2)\varepsilon$. This means that

$$d(Jf, Jg) \leq \frac{\alpha}{2} d(f,g) \tag{2.22}$$

for all $f, g \in S$, that is, J is a strictly contractive self-function on S with the Lipschitz constant $\alpha/2$.

It follows from (2.18) that

$$\Lambda_{Jf(x)-f(x)}\left(\frac{t}{2^{m-1}}\right) \geq \Phi_{x,x,0,\ldots,0}(t) \tag{2.23}$$

for all $x \in X$ and all $t > 0$, which implies that $d(Jf, f) \leq 1/2^{m-1}$.

Due to Theorem 1.5, there exists a function $A : X \to Y$ such that A is a fixed point of J, that is, $A(2x) = 2A(x)$ for all $x \in X$.

Also, $d(J^n f, A) \to 0$ as $n \to \infty$, implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x) \tag{2.24}$$

for all $x \in X$. If we replace $x_1, \ldots, x_m$ with $2^n x_1, \ldots, 2^n x_m$ in (2.14), respectively, and divide by 2^n, then it follows from (2.13) that

$$\Lambda_{D_f(2^n x_1,\ldots,2^n x_m)/2^n}(t) \geq \Phi_{2^n x_1,\ldots,2^n x_m}(2^n t) = \Phi_{2^n x_1,\ldots,2^n x_m}\left(\alpha^n \left(\frac{2}{\alpha}\right)^n t\right) \geq \Phi_{x_1,\ldots,x_m}\left(\left(\frac{2}{\alpha}\right)^n t\right) \tag{2.25}$$

for all $x_1, \ldots, x_m \in X$ and all $t > 0$. By letting $n \to \infty$ in (2.25), we find that $\Lambda_{D_A(x_1,\ldots,x_m)}(t) = 1$ for all $t > 0$, which implies $D_A(x_1, \ldots, x_m) = 0$, and thus A satisfies (1.7). Hence by Lemma 2.1, the function $A : X \to Y$ is additive.

According to the fixed point alterative, since A is the unique fixed point of J in the set $\Omega = \{g \in S : d(f,g) < \infty\}$, A is the unique function such that

$$\Lambda_{f(x)-A(x)}(ut) \geq \Phi_{x,x,0,\ldots,0}(t) \tag{2.26}$$

for all $x \in X$ and all $t > 0$. Again using the fixed point alterative gives

$$d(f, A) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{2^{m-1}(1-L)} = \frac{1}{2^{m-1}(1-\alpha/2)}, \tag{2.27}$$

which implies the inequality

$$\Lambda_{f(x)-A(x)}\left(\frac{t}{2^{m-2}(2-\alpha)}\right) \geq \Phi_{x,x,0,\ldots,0}(t) \tag{2.28}$$

for all $x \in X$ and all $t > 0$. So,

$$\Lambda_{f(x)-A(x)}(t) \geq \Phi_{x,x,0,\ldots,0}\left(2^{m-2}(2-\alpha)t\right) \tag{2.29}$$

for all $x \in X$ and all $t > 0$. This completes the proof. □

Now, we present a corollary that is an application of the last theorem in the classical case.

Corollary 2.3. *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$, *normed linear spaces, define*

$$\Lambda_x(t) = \frac{t}{t + \|x\|} \tag{2.30}$$

for $x \in X$ *and* $t > 0$. *Define*

$$\Phi_{x_1,\ldots,x_m}(t) = \frac{t}{t + \sum_{i=1}^{m} \|x_i\|^p} \tag{2.31}$$

for all $x_1, \ldots, x_m \in X$ *and all* $t > 0$ *in which* $p < 1$. *Now, for* $\alpha = 2^p$, (2.13) *holds for all* $x_1, \ldots, x_m \in X$ *and all* $t > 0$. *Suppose that an odd function* $f : X \to Y$ *satisfies* (2.14) *for all* $x_1, \ldots, x_n \in X$ *and all* $t > 0$. *Then, by the last theorem there exists a unique additive function* $A : X \to Y$ *such that*

$$\frac{t}{t + \left\|f(x) - A(x)\right\|} \geq \frac{t}{t + 1/2^{m-2}\left(1 - 2^{p-1}\right)\|x\|^p} \tag{2.32}$$

for all $x \in X$ *and all* $t > 0$. *Hence,*

$$\left\|f(x) - A(x)\right\| \leq \frac{1}{2^{m-2}\left(1 - 2^{p-1}\right)}\|x\|^p \tag{2.33}$$

for all $x \in X$.

Theorem 2.4. *Let* $\Phi : \underbrace{X \times X \times \cdots \times X}_{m\text{-terms}} \to D^+$ *be a function such that, for some* $0 < \alpha < 3$,

$$\Phi_{3x_1,\ldots,3x_m}(\alpha t) \geq \Phi_{x_1,\ldots,x_m}(t) \tag{2.34}$$

for all $x_1, \ldots, x_m \in X$ *and all* $t > 0$. *Suppose that an odd function* $f : X \to Y$ *satisfies* (2.14) *for all* $x_1, \ldots, x_n \in X$ *and all* $t > 0$. *Then, there exists a unique additive function* $A : X \to Y$ *such that*

$$\Lambda_{f(x)-A(x)}(t) \geq \Phi_{x,x,x,\underbrace{0,\ldots,0}_{m-3}}\left(2^{m-3}(3-\alpha)t\right) \tag{2.35}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Letting $x_i = 0$ $(i = 4, \ldots, m)$ in (2.14), we get

$$\Lambda_{(1+\sum_{\ell=1}^{m-3}\binom{n-3}{\ell})(f(x_1+x_2+x_3)+f(x_1-x_2+x_3)+f(x_1+x_2-x_3)+f(x_1-x_2-x_3))-2^{m-1}f(x_1)}(t) \geq \Phi_{x_1,x_2,x_3,0,\ldots,0}(t) \tag{2.36}$$

for all $x_1, x_2, x_3 \in X$ and all $t > 0$. Setting $x_1 = x_2 = x_3 = x$ in the last inequality, we obtain by using oddness of f and (2.4) that

$$\Lambda_{2^{m-3}f(3x)+f(x)-2^{m-1}f(x)}(t) \geq \Phi_{x,x,x,0,\ldots,0}(t) \tag{2.37}$$

for all $x \in X$ and all $t > 0$, or

$$\Lambda_{(f(3x)/3)-f(x)}(t) \geq \Phi_{x,x,x,0,\ldots,0}\left(3.2^{m-3}t\right) \tag{2.38}$$

for all $x \in X$ and all $t > 0$. Let S be the set of all odd functions $h : X \to Y$, and introduce a generalized metric on S as follows:

$$d(h,k) = \inf\left\{u \in \mathbb{R}^+ : \Lambda_{h(x)-k(x)}(ut) \geq \Phi_{x,x,x,0,\ldots,0}(t),\ \forall x \in X, \forall t > 0\right\}. \tag{2.39}$$

It is easy to show that (S, d) is a generalized complete metric space [26, 45]. Let $J : S \to S$ be the function defined by

$$Jh(x) := \frac{h(3x)}{3} \tag{2.40}$$

for all $h \in S$ and $x \in X$. One can show that

$$d(Jf, Jg) \leq \frac{\alpha}{3} d(f, g) \tag{2.41}$$

for all $f, g \in S$, that is, J is a strictly contractive self-function on S with the Lipschitz constant $\alpha/3$.

It follows from (2.38) that

$$\Lambda_{Jf(x)-f(x)}\left(\frac{t}{3.2^{m-3}}\right) \geq \Phi_{x,x,x,0,\ldots,0}(t) \tag{2.42}$$

for all $x \in X$ and all $t > 0$, which implies that $d(Jf, f) \leq (1/3.2^{m-3})$.

Due to Theorem 1.5, the sequence $\{J^n\}$ converges to a fixed point A of J, that is,

$$A : X \longrightarrow Y, \qquad A(x) = \lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}, \tag{2.43}$$

and $A(3x) = 3A(x)$ for all $x \in X$.

Also, A is the unique fixed point of J in the set $\Omega = \{g \in S : d(f,g) < \infty\}$, and

$$d(f,A) \leq \frac{1}{1-L} d(f,Jf) \leq \frac{1}{3.2^{m-3}(1-L)} = \frac{1}{3.2^{m-3}(1-\alpha/3)} \tag{2.44}$$

implies the inequality

$$\Lambda_{f(x)-A(x)}\left(\frac{t}{2^{m-3}(3-\alpha)}\right) \geq \Phi_{x,x,x,0,\ldots,0}(t) \tag{2.45}$$

for all $x \in X$ and all $t > 0$. This implies that inequality (2.35) holds. Furthermore, we can obtain that the function $A : X \to Y$ satisfies (1.7). Hence by Lemma 2.1, we get that the function $A : X \to Y$ is additive. □

Now, we present a corollary that is an application of the last theorem in the classical case.

Corollary 2.5. *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$*, normed linear spaces, define*

$$\Lambda_x(t) = \frac{t}{t + \|x\|} \tag{2.46}$$

for $x \in X$ *and* $t > 0$*. Define*

$$\Phi_{x_1,\ldots,x_m}(t) = \frac{t}{t + \sum_{i=1}^{m} \|x_i\|^p} \tag{2.47}$$

for all $x_1, \ldots, x_m \in X$ *and all* $t > 0$ *in which* $p < 1$*. Now, for* $\alpha = 3^p$, (2.34) *holds for all* $x_1, \ldots, x_m \in X$ *and all* $t > 0$*. Suppose that an odd function* $f : X \to Y$ *satisfies* (2.14) *for all* $x_1, \ldots, x_n \in X$ *and all* $t > 0$*. Then, by the last theorem there exists a unique additive function* $A : X \to Y$ *such that*

$$\frac{t}{t + \|f(x) - A(x)\|} \geq \frac{t}{t + 1/2^{m-3}(1 - 3^{p-1})\|x\|^p} \tag{2.48}$$

for all $x \in X$ *and all* $t > 0$*. Hence,*

$$\|f(x) - A(x)\| \leq \frac{1}{2^{m-3}(1 - 3^{p-1})} \|x\|^p \tag{2.49}$$

for all $x \in X$*.*

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